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# Maximizing a Fractional Linear Programing Problem Subject to a System of Fuzzy Relational Equation Constraints Under Continuous Archimedean Triangular Co-Norm

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***Abstract.*** In this paper an algorithm is proposed to obtain an optimal solution to maximize a fractional linear programming problem whose objective is a fractional function subject to a system of min-T equations with a continuous Archimedean triangular co-norm. This algorithm is illustrated by a numerical example.

***Keywords:*** min-T equation, fuzzy relational equations, fractional objective function.

***AMS Mathematics Subject Classification (2010):*** 90C08

## 1. Introduction

Linear fractional programming problems (LFPP) are a special type of linear programming problem. In this paper, we propose a new method for finding an optimal solution to LFPP subject to a system of fuzzy relation equations constraints. Fuzzy relation equations (FRE) were first introduced by Sanchez [5] and applied to diagnosis problems.

In section 2 we present some basic definitions. In section 3 locking variable is defined and theorems are proved to find optimal solutions. Some rules to reduce the problem size are presented in section 4. A new algorithm is proposed which is illustrated by a numerical example in section 5.

## 2. Preliminaries

In this section we present basic definitions and some properties of min-t conorm.

**Definition 2.1.** (George J.Klir/ Boyuan [3]) A fuzzy union / t-conorm is a binary operation on the unit interval that satisfies the following axioms for all $a, b, d \in [0,1]$.

1. $T(a,0) = a$ (boundary condition)
2. $b \le d$ implies $T(a,b) \le T(a,d)$ (monotonicity)
3. $T(a,b) = T(b,a)$ (commutativity)

4. *T[a,T(b,d)] = T[T(a,b),d]* (associativity)

**LFPP with FRE constraints 2.2**

Maximize $Z(p) = \dfrac{\sum_{i=1}^{n} c_i p_i}{\sum_{i=1}^{n} d_i p_i}$ ……………………………..(1)

subject to $p \circ Q = r$ ………………………………..(2)

where $p \in [0,1]^n$ , $c_i, d_i \in R$ are the co-efficients associated with variable $p_i$, $Q = [q_{ij}]$ is an $n \times m$ matrix with $q_{ij} \leq 1$, $r$ is an $m$ dimensional vector with $0 \leq r_j \leq 1$ and the operation $\circ$ represents the min-T composition operator,where $T$ is a continuous Archimedean triangular co-norm. $[T(p,q) = \max(p,q)]$.

**Definition 2.3.** Let $S(Q,r) = \{p \in [0,1]^n | p \circ Q = r\}$ denote the solution set of (2) and let $I = \{1,2,3, \dots \dots n\}$ and $J = \{1,2,3, \dots \dots m\}$ be two index sets. Then, the solution vectors $p \in [0,1]^n$ of the given problem (2) is obtained by

$$\min_{i \in I} \{T(p_i, q_{ij})\} = r_j, \quad \forall\, j \in J \qquad \text{………………….....(3)}$$

**2.2. Properties of $S(Q,r)$**

(i) Let an element $\underline{p}$ of $S(Q,r)$ be called a minimal solution of (2) if for all $p \in S(Q,r), p \leq \underline{p}$implies $p = \underline{p}$; if for all $p \in S(Q,r), p \geq \underline{p}$ then $\underline{p}$ is the minimal solution of (2).

(ii) Let an element $\overline{p}$ of $S(Q,r)$ be called a maximal solution of (2) if for all $p \in S(Q,r), p \geq \overline{p}$implies $p = \overline{p}$; if for all $p \in S(Q,r), \overline{p} \geq p$, then $\overline{p}$ is the maximum solution.

(iii) the solution set $S(Q,r)$ is not empty it always contains a unique minimal solution $\underline{p}$and it may contain several maximal solutions. Let $\overline{S}(Q,r)$ denote the set of all maximal solutions.

Also $S(Q,r) = \cup_{\overline{p}}\ [\underline{p}, \overline{p}]$ where the union is taken for all $\overline{p} \in \overline{S}(Q,r)$.

**3. Conditions for optimality**

**Definition 3.1.** If $S(Q,r) \neq \emptyset$, the minimal solution $\underline{p} = \{\underline{p_i} / i \in I\}$ of (2) is determined by $\underline{p}_i = \max_{j \in J} \sigma(q_{ij}, r_j)$ …………………………………………………..(4)

where $\sigma(q_{ij}, r_j) = \begin{cases} r_j & if\ q_{ij} < r_j \\ 0 & otherwise \end{cases}$

**Note:** When $\underline{p}$ determined by (4) does not satisfy (2), then $S(Q,r) = \emptyset$. That is, the existence of the minimum solution $\underline{p}$, as determined by (4), is a necessary and sufficient conditions for $S(Q,r) \neq \emptyset$.

**Definition 3.2.** If $S(Q,r) \neq \Phi$ and $p = (p_1, p_2, p_3, \dots \dots p_n)$ be any solution of (2), then $p_i$

is said to be a locking variable if $T(p_i, q_{ij}) = r_j$ for some $j \in J$. The locking set of $p_i$ is denoted by $J(p_i) = \{j \in J | T(p_i, q_{ij}) = r_j\}$. …………………(5)

**Lemma 3.3. [9]** Let $T$ be the continuous Archimedean t-conorm and if $q_{ij} > r_j$ for each $i \in I$ for any equation in (2) then the solution set $S(Q, r) = \emptyset$.

**Lemma 3.4. [9]** Let $T$ be the continuous Archimedean t-conorm and $\underline{p} = (\underline{p_i})_{i \in I}$ be the minimal solution and $p = (p_i)_{i \in I}$ be any solution of (3). If $p_i$ is locking in the j[th] equation, then $\underline{p_i}$ is also locking. If $\underline{p_i}$ is not a locking variable, then for $p_i$ is also non locking.

**Lemma 3.5. [9]** Let $T$ be a continuous t-conorm, and $S(Q, r) \neq \Phi$in(2). If $r_j = 1$ for some j$\in J$, then all variables $p_i, \forall i \in I$ are locking in the $j^{th}$ equation.

**Lemma 3.6. [9]** Let $T$ be a continuous Archemeadian t-conorm and $p = (p_i)_{i \in I}$ be any solution of (2). If $p_i$ is only locking in equations with $r_j = 1$, then $p_i$ can take any value in $[\underline{p_i}, 1]$.

**Lemma 3.7. [9]** Let $T$ be a continuous t-conorm and $\underline{p} = (\underline{p_i})_{i \in I}$ be the minimal solution. If $c_i \leq 0, \forall i \in I$ the cost co efficient in the objective function , then $\underline{p_i}$ is an optimal solution of the given problem.

**Theorem 3.8.** Let $p \circ Q = r$ be a consistent system of min-T equations with a continuous Archimedean t co-norm $T$ and $\underline{p}$ its minimal solution. There exists an optimal solution $p^* = (p_1^*, p_2^*, \dots \dots \dots \dots p_n^*)$ to (2) such that either $p_i^* = \underline{p_i}$ or $p_i^* = 1$ for all $j \in J$.

**Proof:** Suppose that $p^*$ is an optimal solution to (1)-(2) and there exists an index $i \in I$ such that $1 > p_i^* > \underline{p_i}$.

It is clear that $Z(p) = \dfrac{c_k p + \sum_{i \neq k} c_i p_i^*}{d_k p + \sum_{i \neq k} d_i p_i^*}$ is monotone on [0,1].

Accordingly the value of $p_k^*$ can be either increased to 1 or decreased to $\underline{p_k}$ without decreasing the objective value. Therefore, such an optimal solution $p^*$ must exists such that either $p_i^* = \underline{p_i}$ or $p_i^* = 1$ for all $j \in J$.

**Theorem 3.9.** If $J_k(Q) \neq \{\emptyset\}$ for some $k \in I$ and $c_k < d_k$ in the objective function then any optimal solution has $\overline{p}_k = \underline{p_k}$.

**Proof:** Since $J_k(Q) \neq \{\emptyset\}$ for some $k \in I$, also $c_k < d_k$.

But $1 > p_k^* > \underline{p_k}$,

It is clear that $Z(p)=\dfrac{c_k p+\sum_{i\neq k} c_i p_i^*}{d_k p+\sum_{i\neq k} d_i p_i^*}$ is monotone on [0,1].

Now the value of $p_k^*$ can be decreased to $\underline{p}_k$ without decreasing the objective value. Therefore any optimal solution has $\overline{p}_k = \underline{p}_k$.

**Theorem 3.10.** If $J_k(Q) = \{\emptyset\}$ for some $k \in I$ and $c_k > d_k$ in the objective function then any optimal solution has $\overline{p}_k = 1$.

**Proof:** Since $J_k(Q) = \{\emptyset\}$ for some $k \in I$. That implies $p_k$ cannot be a locking in any equation. Also since $c_k > d_k$.
But $1 > p_k^* > \underline{p}_k$,

It is clear that $Z(p)=\dfrac{c_k p+\sum_{i\neq k} c_i p_i^*}{d_k p+\sum_{i\neq k} d_i p_i^*}$ is monotone on [0,1].

Now the value of $p_k^*$ can be increased to 1 without decreasing the objective value. Therefore any optimal solution has $\overline{p}_k = 1$.

**Theorem 3.11.** If $J_k(Q) = \{\emptyset\}$ for some $k \in I$ and $c_k = d_k = 1$ in the objective function then any optimal solution has $\overline{p}_k = 1$ if $\sum_{i\neq k} c_i \overline{p}_i < \sum_{i\neq k} d_i \overline{p}_i$ and $\overline{p}_k = \underline{p}_k$ if $\sum_{i\neq k} c_i \overline{p}_i > \sum_{i\neq k} d_i \overline{p}_i$ .

**Proof: Case (i)**
Since $J_k(Q) = \{\emptyset\}$ for some $k \in I$. That implies $p_k$ cannot be a locking in any equation. Also since $c_k = d_k$.
But $1 > p_k^* > \underline{p}_k$,

It is clear that $Z(p)=\dfrac{c_k p+\sum_{i\neq k} c_i p_i^*}{d_k p+\sum_{i\neq k} d_i p_i^*}$ is monotone on [0,1].

Now the value of $p_k^*$ can be increased to 1 without decreasing the objective value. Therefore any optimal solution has $\overline{p}_k = 1$.

**Case (ii)**
Since $J_k(Q) = \{\emptyset\}$ for some $k \in I$. That implies $p_k$ cannot be a locking in any equation. Also since $c_k = d_k$.
But $1 > p_k^* > \underline{p}_k$,

It is clear that $Z(p)=\dfrac{c_k p+\sum_{i\neq k} c_i p_i^*}{d_k p+\sum_{i\neq k} d_i p_i^*}$ is monotone on [0,1].

Now the value of $p_k^*$ can be decreased to $\underline{p}_k$ without decreasing the objective value.

Therefore any optimal solution has $\overline{p}_k = \underline{p}_k$.

## 4. Rules to reduce the problem.

For the given matrix $Q$, we define, the following index sets.
$J_i(Q) = \{j \in J | p_i \circ q_{ij} = r_j\}, \forall\, i \in I$ and $I_j = \{i \in I | p_i \circ q_{ij} = r_j\}, \forall j \in J$
the index set $j_i(Q)$ is nothing but the locking set $J(\underline{p_i})$ of (2).

**Rule 1:** If $I_j(Q)$ is singleton set for some $j \in J$, ie., $I_j(Q) = \{t\}$, then $\overline{p}_i = \underline{p}_t$ where $\overline{p}_t \in (\overline{p}_i)_{i\in I}$ .

**Proof:** Since $I_j(Q) = \{t\}$.

∴the J$^{th}$equation can be satisfied by the variable $p_t$.

If $p = (p_i)_{i\in I}$ be any solution of (2). Then the t$^{th}$ component of this solution must be locking in the j$^{th}$ equation. By lemma 3.5.

If $r_j = 1$ for some $j \in J$, then all variables $p_i \forall i \in I$ are locking in the j$^{th}$ equation.

=> $I_j(Q)$is not a singleton set.

If $r_j < 1$, then by theorem 3.8 we have $\overline{p}_t = \underline{p}_t$.

If we apply rule 2, the j$^{th}$ column of $Q$ with $j \in J_t(Q)$ can be deleted. The row corresponding to $p_t$ can also be deleted from matrix $Q$.

**Rule 2:** If $I_p(Q) \supseteq I_q(Q)$ for some $p, q \in J$ in the matrix $Q$, then the q$^{th}$ column of $Q$ can be deleted.

## 5. Algorithm

**Step 1:** Find the minimal solution $\underline{p} = \left(\underline{p_i}\right)_{i\in I}$ of the given problem by (4 )

**Step 2:** If $\underline{p} \circ Q = r$, then go to step 3, otherwise stop the process. The given problem is inconsistent $[S(Q, r) = \emptyset]$.

**Step 3:** Compute index sets $J_i$ and $I_j$ for the given matrix Q. Apply rules 1-2 and theorems 3.8-3.11 to determine the values of decision variables as many as possible. If all decision variables have been set, then go to step 4. Otherwise repeat step 3.

**Step 4:** Obtain optimal solution to the given problem.

### 5.1. Numerical example

Consider the following LFPP with continuous Archimedean t-conorm fuzzy relational equations constraint.

$$Maximize\ Z(p) = \frac{2p_1 - 2p_2 + p_3 + p_4 + p_5 + 5p_6 + p_7 + 8p_8 + p_9}{p_1 + 7p_2 + 3p_3 + p_4 + p_5 - p_6 + 4p_7 + 2p_8 + p_9}$$

subject to $p \circ Q = r$
where

$$Q = \begin{pmatrix} 0.25 & 0.75 & 0.45 & 0.70 & 0.68 & 0.43 & 0.80 & 0.35 & 0.48 \\ 0.58 & 0.75 & 0.57 & 0.75 & 0.76 & 0.42 & 0.60 & 0.70 & 0.49 \\ 0.72 & 0.76 & 0.59 & 0.46 & 0.78 & 0.45 & 0.65 & 0.45 & 0.50 \\ 0.65 & 0.33 & 0.60 & 0.55 & 0.63 & 0.30 & 0.75 & 0.16 & 0.55 \\ 0.54 & 0.90 & 0.80 & 0.73 & 0.84 & 0.49 & 0.80 & 0.50 & 0.39 \\ 0.95 & 0.60 & 0.58 & 0.90 & 0.76 & 0.82 & 0.64 & 0.55 & 0.45 \\ 0.53 & 0.78 & 0.82 & 0.36 & 0.84 & 0.65 & 0.56 & 0.45 & 0.44 \\ 0.52 & 0.80 & 0.60 & 0.54 & 0.80 & 0.55 & 0.65 & 0.28 & 0.35 \\ 0.60 & 0.75 & 0.68 & 0.55 & 0.94 & 0.33 & 0.66 & 0.50 & 0.48 \end{pmatrix}$$
$$b = (0.54 \quad 0.75 \quad 0.60 \quad 0.55 \quad 0.76 \quad 0.43 \quad 0.65 \quad 0.50 \quad 0.48)$$

**Step 1:** Find the minimal solution by (4 )

That is $\underline{p} = (0.76, 0.65, 0.60, 0.76, 0.50, 0.75, 0.65, 0.55, 0.43)$

**Step 2:** Since $\underline{p} \circ Q = r$. Then go to step 3.

**Step 3:** Compute index sets $J_i(\underline{P})$ and $I_j$ for the given matrix Q and apply rules 1 – 2 and theorems 3.8-3.11 to determine the values of the decision variables as many as possible. Consider the above matrix Q.

The index sets are

$J\left(\underline{p_1}\right) = \{5\},\ J\left(\underline{p_2}\right) = \{2,5,7\},\ J\left(\underline{p_3}\right) = \{3,7\},\ J\left(\underline{p_4}\right) = \{5\},\ J\left(\underline{p_5}\right) = \{1,8\}, J\left(\underline{p_6}\right) = \{2,5\},\ J\left(\underline{p_7}\right) = \{7\},\ J\left(\underline{p_8}\right) = \{3,4,7\},\ J\left(\underline{p_9}\right) = \{2,4,6,8,9\}$.

$$I_1 = \{5\}, I_2 = \{2,6,9\}, I_3 = \{3,8\}, I_4 = \{8,9\}, I_5 = \{1,2,4,6\}, I_6 = \{9\}, I_7 = \{2,3,7,8\},\ I_8 = \{5,9\}, I_9 = \{9\}.$$

Since $I_1 = \{5\}, I_6 = \{9\}, I_9 = \{9\}$, implies that the variables $p_5$ and $p_9$ are the only locking variable in 1st , 6th and 9th equations. So, by rule 1 any optimal solution has $\overline{p}_5 = \underline{p}_5$and $\overline{p}_9 = \underline{p}_9$. Since $p_5$ and $p_9$ are also locking in equations $\{1,2,4,6,8,9\}$. Hence these columns and rows corresponding to $p_5$ and $p_9$ can be deleted from matrix Q. The reduced matrix Q becomes,

$$Q = \begin{pmatrix} 0.45 & 0.68 & 0.80 \\ 0.57 & 0.76 & 0.60 \\ 0.59 & 0.78 & 0.65 \\ 0.60 & 0.63 & 0.75 \\ 0.58 & 0.76 & 0.64 \\ 0.82 & 0.84 & 0.56 \\ 0.60 & 0.80 & 0.65 \end{pmatrix}$$

The index sets are, $J\left(\underline{p_1}\right) = \{5\},\ J\left(\underline{p_2}\right) = \{5,7\},\ J\left(\underline{p_3}\right) = \{3,7\}, J\left(\underline{p_4}\right) = \{5\}$, $J\left(\underline{p_6}\right) = \{5\},\ J\left(\underline{p_7}\right) = \{7\},\ J\left(\underline{p_8}\right) = \{7\}$.

$I_3 = \{3,8\}, I_5 = \{1,2,4,6\},\ I_7 = \{2,3,7,8\}$.

Since $I_3 \subseteq I_7$ then by rule 2, we delete the colomn 3 of matrix Q.

Since $c_2 < d_2,\ c_3 < d_3,\ c_7 < d_7$, then by theorem (3.9) , we set $\overline{p}_2 = \underline{p}_2,\ \overline{p}_3 = \underline{p}_3$, $\overline{p}_7 = \underline{p}_7,\ p_{2}, p_{3}, p_{7},$ are locking in equations 5,7, deleted the corresponding column of matrix Q.

Now we have to determine the remaining [1,4,6,8] decision variables. But all index set $J\left(\underline{p_i}\right) = \emptyset$.

Since $c_1 > d_1, c_6 > d_6, c_8 > d_8$, by theorem (3.10).
We set $\overline{p}_1 = 1,\ \overline{p}_6 = 1,\ \overline{p}_8 = 1$.
Also, since $c_4 = d_4$. Now, we have to determine $\sum_{i\neq 4} c_i \overline{p}_i$ and $\sum_{i\neq 4} d_i \overline{p}_i$

That implies $\sum_{i\neq 4} c_i \overline{p}_i$ =15.88

$\sum_{i\neq 4} d_i \overline{p}_i$ =11.88

Since $\sum_{i\neq 4} c_i \overline{p}_i > \sum_{i\neq 4} d_i \overline{p}_i$

By theorem (3.11) we set $\overline{p}_4 = \underline{p}_4$. Since all the decision variables have been set, then go to step 4.

**Step 4:** Find the optimal solution to the problem (1) – (2)

$$Z(p) = \frac{\sum_{i=1}^{9} c_i p_i}{\sum_{i=1}^{9} d_i p_i} = 1.31646\,.$$

## REFERENCES


1. G.Birkhoff, Lattice Theory, 3rd edition, American Mathematical Society Colloquium Publications, vol.xxv, Providence, R.I., (1967).
2. J.G.Brown, A note on fuzzy sets, information and control 18 (1971) 32 – 39.
3. J.G.Klir and B.Yuan: fuzzy sets and fuzzy logic, theory and applications, 64 (1995) 61 – 64.
4. J.A.Goguen, L – fuzzy sets, *J.Math.Anal.,* 18 (1967) 145-174.
5. E.Sanchez, Resolution of composite fuzzy relation equation and control, 30 (1976) 38 – 48.
6. L.A.Zadeh, Outline of a new approach to the analysis of complex systems and decision processes, IEEE Trans. On systems, Man, and cybernetics, SMC-3 (1973) 28 – 44.
7. L.A.Zadeh, The concept of a linguistic variable and its application to approximate reasoning I, II, III, *Information Sciences,* 8 (1975) 199 - 257,.
8. L.A.Zadeh, Fuzzy sets, *Information and Control,* 8 (1965) 338 – 353.
9. T.Beaula and K.Saraswathi, Maximizing a linear objective function subject to a system of min-t equations with a continuous Archimedean t-conorm, *Annals of Fuzzy Mathematics and Informatics.*