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A New Approach on Orderings of Triangular Fuzzy Random Variables

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Abstract. In this paper, a new approach on the concept of stochastic orderings, hazard rate orderings and likelihood ratio orderings of triangular fuzzy random variables are presented. Based on these orderings, some propositions of triangular fuzzy likelihood ratio order, triangular fuzzy hazard rate order and triangular fuzzy stochastic order and various properties of these orderings are established.

Keywords: Fuzzy random variables, triangular fuzzy numbers, fuzzy likelihood ratio order, fuzzy hazard rate order and fuzzy stochastic order.

AMS Mathematics Subject Classification (2010): 60A86

1. Introduction

Stochastic ordering of fuzzy random variables, similar to stochastic ordering of random variables has wide and very important applicability in reliability theory, epidemic models and so on. Stochastic orderings of fuzzyrandom variables can effectively be thought of in a population which is characterized by the parameter whose value is not numerical but linguistic.

In this paper, we discuss about the stochastic orderings of fuzzy random variables based on Kwakernaak's [3] fuzzy random variables and that we follow Piriyakumar and Renganathan [3], which provides some interesting properties. We consider continuous random variables with parameters mean μ and standard deviation σ. It likes a normal distribution with parameter N ~ (μ,σ^2). We should not use the legal concept of symmetric form about $X=\mu$, when the value of standard deviation (σ) is very minimum. Then the curve becomes very sharp peak in normal distribution. Now we get the ordered triple ($a_1=\mu-n\sigma$, $a_2=\mu$, $a_3 = \mu + n\sigma$), n = 1, 2, 3 of the normal curve it covered 99.73%. The remaining area is 0.0027. It spreads over outside of the curve $| X -\mu| \geq 3\sigma$ on both sides. It is nearly equal to zero, but every α-cut $\geq\in$, $\in> 0$ so that the normal curve can be treated as a triangular fuzzy number with base ($a_1 = \mu-\sigma$, $a_2 = \mu$, $a_3 = \mu+\sigma$). Here the value of standard deviation is minimum, so that the curve become more sharp.

On the idea of Kurtosis, it likes as a leptokurtic for such a curve $\beta_2> 0$ [i.e$\gamma_2> 0$], mesokurtic (normal curve) $\beta_2 = 3$ [i.e. $\gamma_2=0$] and also platykurtic $\beta_2< 3$ [i.e$\gamma_2< 0$], the standard deviation (σ) is very minimum so that the mesokurtic, leptokurtic and

platykurtic may have also equal interval but the maximum heights of these curves are at $\alpha = 1$ and then we construct the α-cut of triangular fuzzy number.
Consider

$$P\{\mu-\sigma< R <\mu + \sigma\} = P\{(R_{\alpha}^{L}-(\alpha-1)\,\sigma-\mu) \geq 0 \vee (R_{\alpha}^{U}+(\alpha-1)\,\sigma-\mu) \leq 0\}$$

Here, the first part gives the membership function of increasing function and then the second one is a membership function of decreasing function, so that the curve as a union of increasing and decreasing function of the triangular fuzzy number.

Now, we verify the relationship between triangular fuzzy likelihood ratio order, triangular fuzzy hazard rate order and triangular fuzzy stochastic orders. The triangular fuzzy likelihood ratio order is more stronger then the triangular fuzzy hazard rate order, the triangular fuzzy hazard rate order is more stronger then the triangular fuzzy stochastic orders and then verify some of its properties.

The organization of the paper is as follows. Section 2 is employed to briefly mention the α-cut concept in triangular fuzzy random variables with parameters mean μ and standard deviation σ, based on Kwakernaak's fuzzy random variables. Piriyakumar and Renganathan et al., [3,4] introduced the fuzzy analogue of stochastic orderings of random variables. We adopt our approach in these properties by applying some new definitions of triangular fuzzy likelihood ratio ordering, triangular fuzzy hazard rate ordering and triangular fuzzy stochastic ordering of fuzzy random variables. and presented.

In section 3, we prove some propositions of triangular fuzzy likelihood ratio order, triangular fuzzy hazard rate order and triangular fuzzy stochastic order.

2. Preliminaries

In this paper, we analyse about the fuzzy stochastic orderings of triangular fuzzy number, fuzzy likelihood ratio order and fuzzy hazard rate ordering of triangular fuzzy number and their properties related to Kwakernaak's fuzzy random variables.

Now, we consider every continuous random variables with parameters mean μ and standard deviation σ. The pattern of triangular fuzzy number has 3-tuples $A(a_1, a_2, a_3)$. There exists an increasing function between a_1 and a_2,and a decreasing function between a_2 to a_3.

Now, we take the intervals of triangular fuzzy number with $A(a_1=\mu-\sigma, a_2=\mu, a_3 = \mu + \sigma)$. That is, the upper and lower limit of triangular fuzzy number have equal distance from $a_2 = \mu$ and they have the maximum height at $\alpha = 1$, in which the probability of any continuous fuzzy random variable R can be written as

$P\{\mu-\sigma< R<\mu+\sigma\} = P\{(R_{\alpha}^{L}-\sigma\,(\alpha-1)-\mu) \geq 0 \vee (R_{\alpha}^{U}+\sigma\,(\alpha-1)-\mu)\leq 0)\}$ and that each with an interval support which we denote by $(R_{\alpha}^{L}, R_{\alpha}^{U})$. Here R_{α}^{L} and R_{α}^{U} may be $-\infty$ and ∞ respectively.

If the interval of this triangular fuzzy number have limited length with respect to $\mu$and σ, then we can perform fuzzy likelihood ratio order, fuzzy hazard rate order and fuzzy stochastic orderings of triangular fuzzy number and their properties.

Finally we prove some properties of triangular fuzzy random variables such as triangular fuzzy stochastic orderings, triangular fuzzy hazard rate orderings and triangular fuzzy likelihood ordering etc, for two triangular fuzzy random variables R and T with means μ_1, μ_2 and standard deviations σ_1, σ_2 respectively with ($s \leq R \leq t$) and ($u \leq T \leq v$).

Here $s = \mu_1 - \sigma_1$, $t = \mu_2 - \sigma_2$, $u = \mu_1 + \sigma_1$ and $v = \mu_2 + \sigma_2$.

Definition 2.1. A fuzzy set A is defined as $A = \{X, \mu_A(x) : x \in A, \mu_A(x) \in [0,1]\}$

Definition 2.2. The support of fuzzy set A is the set of all points x in X such that $\mu_A(x)>0$ (i.e.,) support (A) = $\{X / \mu_A(x) > 0\}$.

Definition 2.3. The α-cut of α-level set of fuzzy set A is a set consisting of those elements of the universe X whose membership values exceed the threshold level α.

(i.e.,) $A_\alpha = \{X / \mu_A(x) \geq \alpha\}$

Definition 2.4. A fuzzy set A on R must possess at least the following three properties to qualify as a fuzzy number.

i. A must be a normal fuzzy set
ii. A_α must be closed interval for every $\alpha \in [0,1]$
iii. The support of A, ${}^{0+}A$ must be bounded.

Among the various shapes of fuzzy number, triangular fuzzy number (TFN) is the most popular one.

As per the above definition a triangular fuzzy number is basically a fuzzy set, so that it satisfies all the concepts of fuzzy set. But we have the membership function arise within the interval only, denoted by $A = (a_1, a_2, a_3)$.

Definition 2.5. The triangular fuzzy number is a fuzzy number represented with 3-tuples as follows : $A = (a_1,a_2,a_3)$

This representation is interpreted as membership function and holds the following conditions.

(i) a_1 to a_2 is increasing function
(ii) a_2 to a_3 is decreasing function
(iii) $a_1 \leq a_2 \leq a_3$

$$\mu_A(x) = \begin{cases} 0 & \text{for } x < a_1 \\ \dfrac{x - a_1}{a_2 - a_1} & \text{for } a_1 \leq x < a_2 \\ \dfrac{a_3 - x}{a_3 - a_2} & \text{for } a_2 \leq x < a_3 \\ 0 & \text{for } x \geq a_3 \end{cases}$$

so that the α-cut of triangular fuzzy number is

${}^{\alpha}A = [a_1^{\alpha}, a_3^{\alpha}]$

Here $a_1^{\alpha} = (a_2 - a_1)\,\alpha + a_1$

$a_3^{\alpha} = -(a_3 - a_2)\alpha + a_3$

Now, it is possible for applying α-cut of fuzzy sets in triangular fuzzy number.

$\therefore \dfrac{a_1^{\alpha} - a_1}{a_2 - a_1} \geq \alpha \qquad \dfrac{a_3 - a_3^{\alpha}}{a_3 - a_2} \geq \alpha$

$(a_1^{\alpha} - (a_2 - a_1)\alpha - a_1) \geq 0 \qquad (a_3^{\alpha} + (a_3 - a_2)\alpha - a_3) \leq 0$ (2.1)

Now, we apply our limits of triangular fuzzy number

$$a_1 = \mu - \sigma \quad a_2 = \frac{(\mu-\sigma)+(\mu+\sigma)}{2} = \mu \quad a_3 = \mu + \sigma$$

$$\therefore A(a_1, a_2, a_3) = A(\mu-\sigma, \mu, \mu+\sigma)$$

from equation (2.1),

$$a_1^{\alpha} - (a_2 - a_1)\alpha - a_1 = (A_{\alpha}^{L} - (\alpha - 1)\sigma - \mu) \geq 0 \tag{2.2}$$

$$a_3^{\alpha} + (a_3 - a_2)\alpha - a_3 = (A_{\alpha}^{U} + (\alpha - 1)\sigma - \mu) \leq 0 \tag{2.3}$$

The membership function of triangular fuzzy number is

$A_{\alpha} = \{ (A_{\alpha}^{L} - (\alpha-1)\sigma - \mu) \geq 0 \vee (A_{\alpha}^{U} + (\alpha-1)\sigma - \mu) \leq 0\}$

Definition 2.6. If R and T are triangular fuzzy random variables with means μ_1, μ_2 and standard deviations σ_1, σ_2 respectively. The R is said to be triangular fuzzy likelihood ratio order (TFLRO) which is less then (or) equal to T, if whenever s≤t and u ≤ v

Here $s = \mu_1 - \sigma_1 \qquad t = \mu_2 - \sigma_2$

$u = \mu_1 + \sigma_1 \qquad v = \mu_2 + \sigma_2$

$P\{(R_{\alpha}^{L} - (\alpha-1)\sigma_2 - \mu_2) \geq 0 \vee (R_{\alpha}^{U} + (\alpha-1)\sigma_2 - \mu_2) \leq 0\}$

$P\{(T_{\alpha}^{L} - (\alpha-1)\sigma_1 - \mu_1) \geq 0 \vee (T_{\alpha}^{U} + (\alpha-1)\sigma_1 - \mu_1) \leq 0\}$

$\leq P\{(R_{\alpha}^{L} - (\alpha-1)\sigma_1 - \mu_1) \geq 0 \vee (R_{\alpha}^{U} + (\alpha-1)\sigma_1 - \mu_1) \leq 0\}$

$P\{(T_{\alpha}^{L} - (\alpha-1)\sigma_2 - \mu_2) \geq 0 \vee (T_{\alpha}^{U} + (\alpha-1)\sigma_2 - \mu_2) \leq 0\}$

We will write $R \leq^{TFLRO} T$

Definition 2.7. If R and T are triangular fuzzy random variables with means μ_1, μ_2 and standard derivation σ_1, σ_2 respectively and that R and T are two triangular fuzzy number $R(a_1, a_2, a_3)$ and $T(a_1', a_2', a_3')$ respectively. Then R is said to be triangular fuzzy hazard rate ordering (TFHRO) which is less then(or)equal to T, if whenever s ≤ t and u ≤ v and if $u = \mu_1 + \sigma_1 \to \infty$ and $v = \mu_2 + \sigma_2 \to \infty$, then

$P\{(R_{\alpha}^{L} - (\alpha-1)\sigma_2 - \mu_2) \geq 0\}\ P\{(T_{\alpha}^{L} - (\alpha-1)\sigma_1 - \mu_1) \geq 0\}$

$\leq P\{(T_{\alpha}^{L} - (\alpha-1)\sigma_2 - \mu_2) \geq 0\}\ P\{(R_{\alpha}^{L} - (\alpha-1)\sigma_1 - \mu_1) \geq 0\}$

We will write $R \leq^{TFHRO} T$.

Definition 2.8. If R and T are triangular fuzzy random variables with means μ_1, μ_2 and standard deviations σ_1, σ_2 respectively and that R and T are two triangular fuzzy number R(a_1, a_2, a_3) and T(a'_1, a'_2, a'_3) respectively. Then R is said to be triangular fuzzy stochastic ordering (TFSO) which is less then(or)equal to T, if whenever $s \leq t$ and $u \leq v$ and if $t = \mu_2 - \sigma_2 \to -\infty$, $s = \mu_1 - \sigma_1 \to -\infty$

$$P\{(R_\alpha^U + (\alpha-1)\,\sigma_2 - \mu_2) \leq 0\}\ P\,\{(T_\alpha^U + (\alpha-1)\,\sigma_1 - \mu_1) \leq 0\}$$
$$\leq P\{(R_\alpha^U + (\alpha-1)\sigma_1 - \mu_1) \leq 0\}\ P\{(T_\alpha^U + (\alpha-1)\sigma_2 - \mu_2) \leq 0\}$$

This implies that $R \leq^{TFSO} T$.

Note 2.9. The number C(constant) in the fuzzy random variable can be written $C_\alpha^L = C_\alpha^U = C$ because the fuzzy random variable can be associated with the real number on the support of fuzzy random variable.

3. Prepositions on triangular fuzzy stochastic order (FSO), triangular fuzzy likelihood ratio order (TFLRO) and triangular fuzzy hazard rate order (TFHRO)

Proposition 3.1. Suppose C is a number

(i) If $C \leq T$, then $C \leq^{TFLRO} T$
(ii) If $T \leq C$, then $T \leq^{TFLRO} C$

Proof:

(i) If $C \leq T$ then

$$P\,\{(C_\alpha^L - (\alpha-1)\sigma_2 - \mu_2) \geq 0 \vee (C_\alpha^U + (\alpha-1)\sigma_2 - \mu_2) \leq 0\}$$
$$P\{(T_\alpha^L - (\alpha-1)\,\sigma_1 - \mu_1) \geq 0 \vee (T_\alpha^U + (\alpha-1)\,\sigma_1 - \mu_1) \leq 0\}$$
$$\leq P\,\{(C_\alpha^L - (\alpha-1)\sigma_1 - \mu_1) \geq 0 \vee (C_\alpha^U + (\alpha-1)\sigma_1 - \mu_1) \leq 0\}$$
$$P\{(T_\alpha^L - (\alpha-1)\,\sigma_2 - \mu_2) \geq 0 \vee (T_\alpha^U + (\alpha-1)\,\sigma_2 - \mu_2) \leq 0\} \qquad (3.1)$$

Whenever

$$P(s<C<u) = P\{(C_\alpha^L - (\alpha-1)\sigma_1 - \mu_1) \geq 0 \vee (C_\alpha^U + (\alpha-1)\sigma_1 - \mu_1) \leq 0\} = 1 \qquad (3.2)$$

If $C \leq s$ and $u \leq C$ then

$$P\{(C_\alpha^L - (\alpha-1)\sigma_1 - \mu_1) \geq 0 \vee (C_\alpha^U + (\alpha-1)\sigma_1 - \mu_1) \leq 0\} = 0 \qquad (3.3)$$

$$P(t < C < v) = P\{(C_\alpha^L - (\alpha-1)\sigma_2 - \mu_2) \geq 0 \vee (C_\alpha^U + (\alpha-1)\sigma_2 - \mu_2) \leq 0\} = 1 \qquad (3.4)$$

If $C \leq t$ and $v \leq C$ then

$$P\,\{(C_\alpha^L - (\alpha-1)\sigma_2 - \mu_2) \geq 0 \vee (C_\alpha^U + (\alpha-1)\sigma_2 - \mu_2) \leq 0\} = 0 \qquad (3.5)$$

Using equation (3.4) in equation (3.1), we must show

$$P\,\{(T_\alpha^L - (\alpha-1)\sigma_1 - \mu_1) \geq 0 \vee (T_\alpha^U + (\alpha-1)\sigma_1 - \mu_1) \leq 0\}$$
$$\leq P\,\{(C_\alpha^L - (\alpha-1)\sigma_1 - \mu_1) \geq 0 \vee (C_\alpha^U + (\alpha-1)\sigma_1 - \mu_1) \leq 0\}$$
$$P\,\{(T_\alpha^L - (\alpha-1)\sigma_2 - \mu_2) \geq 0 \vee (T_\alpha^U + (\alpha-1)\sigma_2 - \mu_2) \leq 0\} \qquad (3.6)$$

Usingequation(3.5) in equation (3.1), we get

$0 \leq$ P $\{(C_{\alpha}^{L} - (\alpha-1)\sigma_1 - \mu_1) \geq 0 \vee (C_{\alpha}^{U} + (\alpha-1)\sigma_1 - \mu_1) \leq 0\}$

P $\{(T_{\alpha}^{L} - (\alpha-1)\sigma_2 - \mu_2) \geq 0 \vee (T_{\alpha}^{U} + (\alpha-1)\sigma_2 - \mu_2) \leq 0\}$

$$P\{s < T < u\} = P\{(T_{\alpha}^{L} - (\alpha-1)\sigma_1 - \mu_1) \geq 0 \vee (T_{\alpha}^{U} + (\alpha-1)\sigma_1 - \mu_1) \leq 0\} = 0 \quad (3.7)$$

Using equation (3.7) in equation (3.6), we get

$0 \leq$ P $\{(C_{\alpha}^{L} - (\alpha-1)\sigma_1 - \mu_1) \geq 0 \vee (C_{\alpha}^{U} + (\alpha-1)\sigma_1 - \mu_1) \leq 0\}$

P $\{(T_{\alpha}^{L} - (\alpha-1)\sigma_2 - \mu_2) \geq 0 \vee (T_{\alpha}^{U} + (\alpha-1)\sigma_2 - \mu_2) \leq 0\}$ (3.8)

It is true.

Usingequation(3.2) in equation (3.6), we get

P $\{(T_{\alpha}^{L} - (\alpha-1)\sigma_1 - \mu_1) \geq 0 \vee (T_{\alpha}^{U} + (\alpha-1)\sigma_1 - \mu_1) \leq 0\}$

$\leq$ P $\{(T_{\alpha}^{L} - (\alpha-1)\sigma_2 - \mu_2) \geq 0 \vee (T_{\alpha}^{U} + (\alpha-1)\sigma_2 - \mu_2) \leq 0\}$ (3.9)

Since $s = \mu_1 - \sigma_1 < C$ and $t = \mu_2 - \sigma_2 < C$

Given $C \leq T$ Then equation (3.9) becomes

$P\{(C \vee (T_{\alpha}^{U} + \sigma_1(\alpha-1) - \mu_1)) \leq 0\} \leq$

$P\{(C \vee (T_{\alpha}^{U} + (\alpha-1)\sigma_2 - \mu_2)) \leq 0\}$

However, this is true.

Since $(u = \mu_1 + \sigma_1) \leq (v = \mu_2 + \sigma_2)$

$\therefore$ u = v

This implies that $C \leq^{TFLRO} T$

Similarly,

We get $T \leq^{TFLRO} C$.

Proposition 3.2.

$R \leq^{TFLRO} T \quad \Rightarrow R \leq^{TFHRO} T$

Proof: Since $R \leq^{TFLRO} T$

$\Rightarrow P\{(R_{\alpha}^{L} - (\alpha-1)\sigma_2 - \mu_2) \geq 0 \vee (R_{\alpha}^{U} + (\alpha-1)\sigma_2 - \mu_2) \leq 0\}$

$P\{(T_{\alpha}^{L} - (\alpha-1)\sigma_1 - \mu_1) \geq 0 \vee (T_{\alpha}^{U} + (\alpha-1)\sigma_1 - \mu_1) \leq 0\}$

$\leq P\{(R_{\alpha}^{L} - (\alpha-1)\sigma_1 - \mu_1) \geq 0 \vee (R_{\alpha}^{U} + (\alpha-1)\sigma_1 - \mu_1) \leq 0\}$

$P\{(T_{\alpha}^{L} - (\alpha-1)\sigma_2 - \mu_2) \geq 0 \vee (T_{\alpha}^{U} + (\alpha-1)\sigma_2 - \mu_2) \leq 0\}$

Suppose $v = \mu_2 + \sigma_2 \rightarrow \infty$ and $u = \mu_1 + \sigma_1 \rightarrow \infty$

Then P $\{(R_{\alpha}^{L} - (\alpha-1)\sigma_2 - \mu_2) \geq 0\}$ $P\{(T_{\alpha}^{L} - (\alpha-1)\sigma_1 - \mu_1) \geq 0\}$

$\leq$ P $\{(R_{\alpha}^{L} - (\alpha-1)\sigma_1 - \mu_1) \geq 0\}$ $P\{(T_{\alpha}^{L} - (\alpha-1)\sigma_2 - \mu_2) \geq 0\}$

This implies that $R \leq^{TFHRO} T$.

Proposition 3.3. $R \leq^{TFLRO} T \Rightarrow R \leq^{TFSO} T$

Proof:

Since $R \leq^{TFLRO} T \quad \Rightarrow$

$P\{(R_\alpha^L-(\alpha-1)\sigma_2-\mu_2)\geq 0 \vee (R_\alpha^U+(\alpha-1)\sigma_2-\mu_2)\leq 0\}$

$P\{(T_\alpha^L-(\alpha-1)\sigma_1-\mu_1)\geq 0 \vee (T_\alpha^U+(\alpha-1)\sigma_1-\mu_1)\leq 0\}$

$\leq P\{(R_\alpha^L-(\alpha-1)\sigma_1-\mu_1)\geq 0 \vee (R_\alpha^U+(\alpha-1)\sigma_1-\mu_1)\leq 0\}$

$P\{(T_\alpha^L-(\alpha-1)\sigma_2-\mu_2)\geq 0 \vee (T_\alpha^U+(\alpha-1)\sigma_2-\mu_2)\leq 0\}$

If we let $s=\mu_1-\sigma_1\to-\infty$ $t=\mu_2-\sigma_2\to-\infty$

Then

$P\{(R_\alpha^U+(\alpha-1)\sigma_2-\mu_2)\leq 0\}\ P\{(T_\alpha^U+(\alpha-1)\sigma_1-\mu_1)\leq 0\}$

$\leq P\{(R_\alpha^U+(\alpha-1)\sigma_1-\mu_1)\leq 0\}\ P\{(T_\alpha^U+(\alpha-1)\sigma_2-\mu_2)\leq 0\}$

This implies that

$R \leq^{TFSO} T$.

Proposition 3.4. For any C one has $R \leq^{TFLRO} T$

$\Leftrightarrow R + C \leq^{TFLRO} T + C$

$\Leftrightarrow R - C \leq^{TFLRO} T-C$

Proof: Since $R \leq^{TFLRO} T \Leftrightarrow$

$P\{(R_\alpha^L-(\alpha-1)\sigma_2-\mu_2)\geq 0 \vee (R_\alpha^U+(\alpha-1)\sigma_2-\mu_2)\leq 0\}$

$P\{(T_\alpha^L-(\alpha-1)\sigma_1-\mu_1)\geq 0 \vee (T_\alpha^U+(\alpha-1)\sigma_1-\mu_1)\leq 0\}$

$\leq P\{(R_\alpha^L-(\alpha-1)\sigma_1-\mu_1)\geq 0 \vee (R_\alpha^U+(\alpha-1)\sigma_1-\mu_1)\leq 0\}$

$P\{(T_\alpha^L-(\alpha-1)\sigma_2-\mu_2)\geq 0 \vee (T_\alpha^U+(\alpha-1)\sigma_2-\mu_2)\leq 0\}$

$\Leftrightarrow P\{(R_\alpha^L-(\alpha-1)\sigma_2-\mu_2)\geq 0 \vee (R_\alpha^U+(\alpha-1)\sigma_2-\mu_2)\leq 0\}+C$

$P\{(T_\alpha^L-(\alpha-1)\sigma_1-\mu_1)\geq 0 \vee (T_\alpha^U+(\alpha-1)\sigma_1-\mu_1)\leq 0\}+C$

$\leq P\{(R_\alpha^L-(\alpha-1)\sigma_1-\mu_1)\geq 0 \vee (R_\alpha^U+(\alpha-1)\sigma_1-\mu_1)\leq 0\}+C$

$P\{(T_\alpha^L-(\alpha-1)\sigma_2-\mu_2)\geq 0 \vee (T_\alpha^U+(\alpha-1)\sigma_2-\mu_2)\leq 0\}+C$

$\Leftrightarrow P\{(R_\alpha^L-(\alpha-1)\sigma_2-(\mu_2+C_\alpha^L))\geq 0 \vee (R_\alpha^U+(\alpha-1)\sigma_2-(\mu_2+C_\alpha^U))\leq 0\}$

$P\{(T_\alpha^L-(\alpha-1)\sigma_1-(\mu_1+C_\alpha^L))\geq 0 \vee (T_\alpha^U+(\alpha-1)\sigma_1-(\mu_1+C_\alpha^U))\leq 0\}$

$\leq P\{(R_\alpha^L-(\alpha-1)\sigma_1-(\mu_1+C_\alpha^L))\geq 0 \vee (R_\alpha^U+(\alpha-1)\sigma_1-(\mu_1+C_\alpha^U))\leq 0\}$

$P\{(T_\alpha^L-(\alpha-1)\sigma_2-(\mu_2+C_\alpha^L))\geq 0 \vee (T_\alpha^U+(\alpha-1)\sigma_2-(\mu_2+C_\alpha^U))\leq 0\}$

$\Leftrightarrow P\{((R+C)_\alpha^L-(\alpha-1)\sigma_2-(\mu_2+C_\alpha^L))\geq 0\vee((R+C)_\alpha^U+(\alpha-1)\sigma_2-(\mu_2+C_\alpha^U))\leq 0\}$

$P\{((T+C)_\alpha^L-(\alpha-1)\sigma_1-(\mu_1+C_\alpha^L))\geq 0\vee((T+C)_\alpha^U+(\alpha-1)\sigma_1-(\mu_1+C_\alpha^U))\leq 0\}$

$\leq$P{($(R+C)^L_\alpha$ –(α–1)σ_1–(μ_1+ C^L_α))≥0∨($(R+C)^U_\alpha$ +–(α–1)σ_1–(μ_1+ C^U_α)) ≤ 0}

P{($(T+C)^L_\alpha$ – (α–1)σ_2–(μ_2+ C^L_α))≥0∨($(T+C)^U_\alpha$ +(α–1)σ_2– (μ_2+ C^U_α)) ≤ 0}

⇔R + C $\leq^{TFLRO}$ T + C.

Similarly

R – C $\leq^{TFLRO}$T – C.

Proposition 3.5. If R $\leq^{TFHRO}$ T, then R $\leq^{TFSO}$ T

Proof: Since R $\leq^{TFHRO}$ T

⇒ P {(R^L_α–(α–1)σ_2–μ_2) ≥ 0} P{(T^L_α –(α–1)σ_1–μ_1) ≥ 0}

$\leq$P {(R^L_α–(α–1)σ_1–μ_1) ≥ 0} P{(T^L_α –(α–1)σ_2–μ_2) ≥ 0}

If we let s = μ_1–σ_1→–∞

P {(R^L_α–(α–1)σ_2–μ_2) ≥ 0} $\leq$P{(T^L_α –(α–1)σ_2–μ_2) ≥ 0}

Here P{(T^L_α –(α–1)σ_1–μ_1) ≥ 0} = 1

P {(R^L_α–(α–1)σ_1–μ_1) ≥ 0} = 1

which implies that R $\leq^{TFSO}$ T.

Proposition 3.6. For any C, one has R $\leq^{TFHRO}$ T

⇔ (i) R + C $\leq^{TFHRO}$ T + C

(ii) R – C $\leq^{TFHRO}$ R – C

Proof: Since R $\leq^{TFHRO}$ T

P{(R^L_α–(α–1)σ_2–μ_2) ≥ 0} P{(T^L_α –σ_1 (α–1) –μ_1) ≥ 0}

$\leq$P {(R^L_α–(α–1)σ_1–μ_1) ≥ 0} P{(T^L_α –(α–1)σ_2–μ_2) ≥ 0}

⇔[P{(R^L_α–(α–1)σ_2–μ_2) ≥ 0}+C] [P{(T^L_α –(α–1)σ_1–μ_1) ≥ 0}+C]

$\leq$[P {(R^L_α–(α–1)σ_1–μ_1) ≥ 0}+ C] [P{(T^L_α –(α–1)σ_2–μ_2) ≥ 0} + C]

⇔ P {(R^L_α–(α–1) σ_2– (μ_2+ C^L_α)) ≥ 0} P {(T^L_α – (α–1)σ_1– (μ_1+ C^L_α)) ≥ 0}

$\leq$P {(R^L_α–(α–1)σ_1– (μ_1+ C^L_α)) ≥ 0} P {(T^L_α – (α–1)σ_2– (μ_2+ C^L_α)) ≥ 0}

⇔P {($(R+C)^L_\alpha$ –(α–1)σ_2–(μ_2+ C^L_α))≥0}

P{($(T+C)^L_\alpha$– (α–1)σ_1–(μ_1+ C^L_α)) ≥ 0}

$\leq$P {($(R+C)^L_\alpha$ –(α–1)σ_1–(μ_1+ C^L_α))≥0}

P{($(T+C)^L_\alpha$– (α–1)σ_2–(μ_2+ C^L_α)) ≥ 0}

which implies that R + C $\leq^{TFHRO}$ T + C

Similarly, R – C $\leq^{TFHRO}$ T – C.

Proposition 3.7.

$$R \le^{TFHRO} T \Leftrightarrow \frac{P\{(T_\alpha^L-(\alpha-1)\sigma_2-\mu_2)\ge 0\}}{P\{(R_\alpha^L-\sigma_2(\alpha-1)-\mu_2)\ge 0\}}$$

is non-decreasing for $t < \sup R$.

Proof: Since $R \le^{TFHRO} T$

$P\{(R_\alpha^L-(\alpha-1)\sigma_2-\mu_2)\ge 0\}\ P\{(T_\alpha^L-\sigma_1(\alpha-1)-\mu_1)\ge 0\}$

$\le P\{(R_\alpha^L-(\alpha-1)\sigma_1-\mu_1)\ge 0\}\ P\{(T_\alpha^L-(\alpha-1)\sigma_2-\mu_2)\ge 0\}$

we get

$$\Leftrightarrow \frac{P\{(T_\alpha^L-(\alpha-1)\sigma_1-\mu_1)\ge 0\}}{P\{(R_\alpha^L-(\alpha-1)\sigma_1-\mu_1)\ge 0\}} \le \frac{P\{(T_\alpha^L-(\alpha-1)\sigma_2-\mu_2)\ge 0\}}{P\{(R_\alpha^L-(\alpha-1)\sigma_2-\mu_2)\ge 0\}} \tag{3.10}$$

Here RHS is non-decreasing for $t < \sup R$.
Conversely,

If RHS of equation(3.10) is non-decreasing for $t < \sup R$, then

$\Leftrightarrow P\{(R_\alpha^L-(\alpha-1)\sigma_2-\mu_2)\ge 0\}\ P\{(T_\alpha^L-\sigma_1(\alpha-1)-\mu_1)\ge 0\}$

$\le P\{(R_\alpha^L-\sigma_1(\alpha-1)-\mu_1)\ge 0\}\ P\{(T_\alpha^L-(\alpha-1)\sigma_2-\mu_2)\ge 0\}$

However, it automatically holds for $s \le t < \sup R$

Since $P\{(R_\alpha^L-(\alpha-1)\sigma_2-\mu_2)\ge 0\} = 0$ for $t \ge \sup R$.

Proposition 3.8.

$R \le^{TFHRO} T \Leftrightarrow$

(i) $$\frac{P\{(R_\alpha^L-(\alpha-1)\sigma_2-\mu_2)\ge 0\}}{P\{(R_\alpha^L-(\alpha-1)\sigma_1-\mu_1)\ge 0\}} \le \frac{P\{(T_\alpha^L-(\alpha-1)\sigma_2-\mu_2)\ge 0\}}{P\{(T_\alpha^L-(\alpha-1)\sigma_1-\mu_1)\ge 0\}}$$

for $s = \mu_1-\sigma_1 < \min\{\sup R, \sup T\}$

(ii) $$\frac{P\{(R_\alpha^L-(\alpha-1)\sigma_2-\mu_2)\ge 0\}}{P\{(R_\alpha^L-(\alpha-1)\sigma_1-\mu_1)\ge 0\}} \le \frac{P\{(T_\alpha^L-(\alpha-1)\sigma_2-\mu_2)\ge 0\}}{P\{(T_\alpha^L-(\alpha-1)\sigma_1-\mu_1)\ge 0\}}$$

for $s < t \le \min\{\sup R, \sup T\}$

Proof: By definition of $R \le^{TFHRO} T$, statement (i) exists and statement (ii) also exists from statement (i). Now, we prove statement (ii) implies $R \le^{TFHRO} T$.

Note that

$$\frac{P\{(R_\alpha^L-(\alpha-1)\sigma_2-\mu_2)\ge 0\}}{P\{(R_\alpha^L-(\alpha-1)\sigma_1-\mu_1)\ge 0\}} \le \frac{P\{(T_\alpha^L-(\alpha-1)\sigma_2-\mu_2)\ge 0\}}{P\{(T_\alpha^L-(\alpha-1)\sigma_1-\mu_1)\ge 0\}}$$

for $s < t \le \min\{\sup R, \sup T\}$

This implies

$P\{(R_\alpha^L-(\alpha-1)\sigma_2-\mu_2)\ge 0\}\ P\{(T_\alpha^L-(\alpha-1)\sigma_1-\mu_1)\ge 0\}$

$\le P\{(R_\alpha^L-\sigma_1(\alpha-1)-\mu_1)\ge 0\}\ P\{(T_\alpha^L-(\alpha-1)\sigma_2-\mu_2)\ge 0\}$

for $s < t \le \min\{\sup R, \sup T\}$ (3.11)

If $\sup T < \sup R$. Then we can put $t = \mu_2 - \sigma_2 = \sup T$ in equation (3.11) we get,

$P\{(T_\alpha^L - (\alpha-1)\sigma_2 - \mu_2) \ge 0\} = 0$

$\therefore P\{(R_\alpha^L - (\alpha-1)\sigma_2 - \mu_2) \ge 0\}\, P\{(T_\alpha^L - (\alpha-1)\sigma_1 - \mu_1) \ge 0\} = 0$

Then we pick s, so $P\{(T_\alpha^L - (\alpha-1)\sigma_1 - \mu_1) \ge 0\} > 0$ to get

$P\{(R_\alpha^L - (\alpha-1)\sigma_2 - \mu_2) \ge 0\} = 0$

$\therefore$ we get $t \ge \sup R$, which is a contradiction.

$\therefore \sup R \le \sup T$ we get equation (3.11) holds for $s \le t \le \sup R$.

This implies $\dfrac{P\{(T_\alpha^L - (\alpha-1)\sigma_2 - \mu_2) \ge 0\}}{P\{(R_\alpha^L - (\alpha-1)\sigma_2 - \mu_2) \ge 0\}}$ (by Proposition (3.7))

is non-decreasing for $t < \sup R$.

This implies $R \le^{TFHRO} T$.

Proposition 3.9. $R \le^{TFHRO} T \Leftrightarrow$ for $s = \mu_1 - \sigma_1 < \min\{\sup R, \sup T\}$ and $t = \mu_2 - \sigma_2 > 0$ one has

$$\frac{P\{(R_\alpha^L - (\sigma_1+\sigma_2)(\alpha-1) - (\mu_1+\mu_2)) > 0\}}{P\{(R_\alpha^L - (\alpha-1)\sigma_1 - \mu_1) > 0\}} \le \frac{P\{(T_\alpha^L - (\sigma_1+\sigma_2)(\alpha-1) - (\mu_1+\mu_2)) > 0\}}{P\{(T_\alpha^L - (\alpha-1)\sigma_1 - \mu_1) > 0\}}.$$

$\Leftrightarrow$ for $s < \min\{\sup R, \sup T\}$ and $t > 0$ one has

$$\frac{P\{(R_\alpha^L - (\sigma_1+\sigma_2)(\alpha-1) - (\mu_1+\mu_2)) \ge 0\}}{P\{(R_\alpha^L - (\alpha-1)\sigma_1 - \mu_1) \ge 0\}} \le \frac{P\{(T_\alpha^L - (\sigma_1+\sigma_2)(\alpha-1) - (\mu_1+\mu_2)) \ge 0\}}{P\{(T_\alpha^L - (\alpha-1)\sigma_1 - \mu_1) \ge 0\}}$$

Proof: Let $s = \mu_1 - \sigma_1 < \sup R$, $t = \mu_2 - \sigma_2 > 0$.

$$\Rightarrow \frac{P\{(R_\alpha^L - (\sigma_1+\sigma_2)(\alpha-1) - (\mu_1+\mu_2)) > 0\}}{P\{(R_\alpha^L - (\alpha-1)\sigma_1 - \mu_1) > 0\}}$$

$$= \frac{P\{(R_\alpha^L - (\sigma_1+\sigma_2)(\alpha-1) - (\mu_1+\mu_2)) > 0\}}{P\{(R_\alpha^L - (\alpha-1)\sigma_1 - \mu_1) > 0\}}$$

since $R > s + t = ((\mu_1+\mu_2) - (\sigma_1+\sigma_2)) \Rightarrow R > \mu_1 - \sigma_1 = s$.

Similarly, $\dfrac{P\{(T_\alpha^L - (\sigma_1+\sigma_2)(\alpha-1) - (\mu_1+\mu_2)) > 0\}}{P\{(T_\alpha^L - (\alpha-1)\sigma_1 - \mu_1) > 0\}} \le$

$$\frac{P\{(T_\alpha^L - (\sigma_1+\sigma_2)(\alpha-1) - (\mu_1+\mu_2)) > 0\}}{P\{(T_\alpha^L - \sigma_1(\alpha-1) - \mu_1) > 0\}}$$

Therefore, the first $\Leftrightarrow$ follows from proposition (3.8). The second $\Leftrightarrow$ follows from the first $\Leftrightarrow$ and the fact that

$$P\{(R_\alpha^L - (\sigma_1+\sigma_2)(\alpha-1) - (\mu_1+\mu_2)) \geq 0\}$$
$$= \lim_{n\to\infty} P\{(R_\alpha^L - (\sigma_1+\sigma_2)(\alpha-1) - (\mu_1+\mu_2)+(1/n)) > 0\}$$

and $P\{(R_\alpha^L - (\sigma_1+\sigma_2)(\alpha-1) - (\mu_1+\mu_2)) > 0\}$

$$= \lim_{n\to\infty} P\{(R_\alpha^L - (\sigma_1+\sigma_2)(\alpha-1) - (\mu_1+\mu_2)-(1/n)) \geq 0\}$$

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