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A New Method to Find Critical Path from Multiple Paths in Project Networks

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Abstract. In this paper, we have proposed a method to find the critical path from multiple critical paths where the durations are uncertain. We have formed a linear programming approach for the JKV representation of trapezoidal fuzzy numbers. A real life example is given to illustrate the proposed approach.

Keywords: Trapezoidal fuzzy number, Ranking, Critical Path

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1. Introduction

When the activity times in the project are deterministic and known, critical path method (CPM) has been demonstrated to be a useful tool in managing projects in an efficient manner to meet this challenge. The purpose of CPM is to identify critical activities on the critical path so that resources may be concentrated on these activities in order to reduce the project length time. The successful implementation of CPM requires the availability of clear determined time duration for each activity. However, in practical situations this requirement is usually hard to fulfill, since many of activities will be executed for the first time. To deal with such real life situations, [10] introduced the concept of fuzzy set. Since there is always uncertainty about the time duration of activities in the network planning, due to which fuzzy critical path method (FCPM) was proposed since the late 1970s.

In particular, problems of determining possible values of latest starting times and floats in networks with imprecise activity durations which are represented by fuzzy or interval numbers have attracted many researchers [8]. [3] have provided a complete solution to the problem of finding the maximal float of an activity. [1] proposed an approach based on the extension principle and linear programming formulation to critical path analysis in networks with fuzzy activity durations. [4] developed a fuzzy PERT approach to evaluate plant construction project scheduling risk under uncertain resource capacity. [2] assigns a different level of importance to each activity on a critical path for a randomly chosen set of activities. The determination of critical activities is carried out via the so-called critical path method [5]. [6-8] compares the fuzzy and the stochastic approaches to find the shortest or the longest distance between nodes in a graph with fuzzy arc values. [9] gave a new representation for trapezoidal fuzzy number and found the critical path using the representation.

The paper is organized as follows, the next section deals with the basic definitions, and in section 3, we have discussed the method to calculate the start, finish and float times. We have given the method to formulate the linear programming problem for critical path in section 4. In section 5, we have proposed a method to find a critical path from multiple paths. A numerical example is illustrated in section 6.

2. Preliminaries

Definition 2.1. A fuzzy number with membership function in the form

$$\mu_{\tilde{A}}(x)=\begin{cases}\dfrac{x-a}{b-a}, a\le x<b\\ 1 \quad , b\le x<c\\ \dfrac{d-x}{d-c}, c\le x\le d\\ 0 \quad , otherwise\end{cases}$$

is called a trapezoidal fuzzy number $\tilde{A}$=(a,b,c,d).

Definition 2.2. A trapezoidal fuzzy number $\tilde{A}$=(a,b,c,d) is said to be a zero trapezoidal fuzzy number if and only if a=0,b=0,c=0,d=0.

Definition 2.3. A trapezoidal fuzzy number $\tilde{A}$=(a,b,c,d) is said to be non negative trapezoidal fuzzy number if and only if $a \ge 0$.

Definition 2.4. Two trapezoidal fuzzy numbers $\tilde{A}=(a_1,b_1,c_1,d_1)$ and $\tilde{B}=(a_2,b_2,c_2,d_2)$ are said to be equal i.e., $\tilde{A}=\tilde{B}$ if and only if $a_1=a_2, b_1=b_2, c_1=c_2, d_1=d_2$.

2.5. L-R representation of a trapezoidal fuzzy number

L-R representation of a trapezoidal fuzzy number has the membership function of the form $$\mu_{\tilde{A}}(x)=\begin{cases}L\left(\dfrac{a^L-x}{\alpha}\right), -\infty\le x<a^L\\ 1 \quad , \quad a^L\le x\le a^U\\ R\left(\dfrac{x-a^U}{\beta}\right), a^U\le x<\infty\end{cases}$$

where α *and* β are left and right hand spreads. In the closed interval [a^L,a^U], the membership function is equal to 1. The function L and R are strictly decreasing and upper semi continuous on supp$(\tilde{A})=\{r/\mu_{\tilde{A}}(r)>0\}$, verifying that L(x)=L(-x), R(x)=R(-x) and L(0)=R(0)=1.

2.6. Arithmetic operations

In this section addition and subtraction operations between two trapezoidal fuzzy numbers are given. Arithmetic operations between (a,b,c,d) type trapezoidal fuzzy numbers.

Let $\tilde{A}_1 = (a_1, b_1, c_1, d_1)$ & $\tilde{A}_2 = (a_2, b_2, c_2, d_2)$ be two trapezoidal fuzzy numbers. Then

$$\tilde{A}_1 \oplus \tilde{A}_2 = (a_1 + a_2, b_1 + b_2, c_1 + c_2, d_1 + d_2)$$

$$\tilde{A}_1 \Theta \tilde{A}_2 = (a_1 - d_2, b_1 - c_2, c_1 - b_2, d_1 - a_2)$$

2.7. Arithmetic operations between $(a^L, a^U, \alpha, \beta)$ type trapezoidal fuzzy numbers

Let $\tilde{A}_1 = (a^{L_1}, a^{U_1}, \alpha_1, \beta_1)$ & $\tilde{A}_2 = (a^{L_2}, a^{U_2}, \alpha_2, \beta_2)$ be two trapezoidal fuzzy numbers. Then

$$\tilde{A}_1 \oplus \tilde{A}_2 = (a^{L_1} + a^{L_2}, a^{U_1} + a^{U_2}, \alpha_1 + \alpha_2, \beta_1 + \beta_2)$$

$$\tilde{A}_1 \Theta \tilde{A}_2 = (a^{L_1} - a^{U_2}, a^{U_1} - a^{L_2}, \alpha_1 + \beta_2, \beta_1 + \alpha_2)$$

2.8. Ranking function

An efficient approach for comparing the fuzzy numbers is by the use of a ranking function $\Re : F(R) \to R,$ where F(R) is a set of fuzzy numbers defined on set of real numbers which maps each fuzzy number into the real line. Let (a,b,c,d) be a trapezoidal fuzzy number.

Then $\Re$(a,b,c,d)= $\dfrac{a+b+c+d}{4}$

Definition 2.9. If $\tilde{A} = (a, b, c, d)$ is a trapezoidal fuzzy number, then the mode of $\tilde{A}$ is given by $\operatorname{mod} e\,\tilde{A} = \dfrac{b+c}{2}$

Definition 2.10. If $\tilde{A} = (a, b, c, d)$ is a trapezoidal fuzzy number, then the divergence of $\tilde{A}$ is given by $Div\,\tilde{A} = d - a$

Definition 2.11. If $\tilde{A} = (a, b, c, d)$ is a trapezoidal fuzzy number, then the Left and Right Spreads of $\tilde{A}$ is given by $Left\ Spread\ \ \tilde{A} = b - a$ and $Right\ Spread\ \ \tilde{A} = d - c$

Definition 2.12. Let (a,b,c,d) $= (a^L - \alpha, a^L, a^U, a^U + \beta)$ be a trapezoidal fuzzy number, then its JKV representation is $(x, y, \alpha, \beta)_{JKV}$, where x= $a^L - \alpha$, y= $a^U + \beta$.

Definition 2.13. A trapezoidal fuzzy number $\tilde{A}$= $(x, y, \alpha, \beta)_{JKV}$ is said to be zero trapezoidal fuzzy number iff x=0,y=0,α=0,β=0.

Definition 2.14. A trapezoidal fuzzy number $\tilde{A}$= $(x, y, \alpha, \beta)_{JKV}$ is said to be non negative iff x≥0, y≥0.

Definition 2.15. Two trapezoidal fuzzy numbers $\tilde{A}_1 = (x_1, y_1, \alpha_1, \beta_1)_{JKV}$ and $\tilde{A}_2 = (x_2, y_2, \alpha_2, \beta_2)_{JKV}$ are said to be equal i.e., $\tilde{A}_1 = \tilde{A}_2$ if and only if $x_1=x_2$, $y_1=y_2$, $\alpha_1=\alpha_2$, $\beta_1=\beta_2$.

2.16. Arithmetic operations between JKV trapezoidal fuzzy numbers
Let $(a_{1,},b_1,c_1,d_1)$ and $(a_{2,},b_2,c_2,d_2)$ be two trapezoidal fuzzy numbers and $(x_1, y_1, \alpha_1, \beta_1)_{JKV}$ and $(x_2, y_2, \alpha_2, \beta_2)_{JKV}$ be their JKV representation, then the addition operation is defined as

$$\tilde{A}_1 \oplus \tilde{A}_2 = (x_1 + x_2, y_1 + y_2, \alpha_1 + \alpha_2, \beta_1 + \beta_2)$$

Image of $\tilde{A}_1$ [- $\tilde{A}_1$] is defined as

$$-\tilde{A}_1 = (-y, -x, \beta, \alpha)$$

2.17. Ranking function for JKV trapezoidal fuzzy numbers
The ranking function defined above for (a,b,c,d) is converted for $(x, y, \alpha, \beta)_{JKV}$ as

$$\Re(x, y, \alpha, \beta)_{JKV} = \frac{2x + 2y + \alpha - \beta}{2}$$

Definition 2.18. If $\tilde{A} = (a,b,c,d)$ is a trapezoidal fuzzy number, and $(x, y, \alpha, \beta)_{JKV}$ is its JKV representation, then the mode of $\tilde{A}$ is given by $\operatorname{mod} e\,\tilde{A} = \dfrac{\alpha + x + y - \beta}{2}$

Definition 2.19. If $\tilde{A} = (a,b,c,d)$ is a trapezoidal fuzzy number, and $(x, y, \alpha, \beta)_{JKV}$ is its JKV representation then the divergence of $\tilde{A}$ is given by $Div\,\tilde{A} = y - x$

Definition 2.20. If $\tilde{A} = (a,b,c,d)$ is a trapezoidal fuzzy number, and $(x, y, \alpha, \beta)_{JKV}$ is its JKV representation then the Left and Right Spreads of $\tilde{A}$ is given by $Left\ Spread\ \ \tilde{A} = \alpha$ and $Right\ Spread\ \ \tilde{A} = \beta$

Example 2.21. If (a,b,c,d)=(60,100,150,180) be a trapezoidal fuzzy number, then its L-R representation is (100,150,40,30) and its JKV representation is (60,180,40,30)

3. Calculating fuzzy time values and critical path in a fuzzy project network
A fuzzy project network is an acyclic digraph, where the vertices represent events, and the directed edges represent the activities, to be performed in a project. Formally, a fuzzy project network is represented by N=(V,A,T). Let $V=\{v_1,v_2,\ldots v_n\}$ be a set of vertices, where v_1 and v_n are the start and final events of the project and each v_i belongs to some path from v_1 to v_n. Let A. Let $A \subset VxV$ be a set of directed edge $a_{ij}=(v_i,v_j)$, that represents the activities to be performed in the project. Activity a_{ij} is then represented by one and only one arrow starting with a event v_i and ending with event v_j. For each activity a_{ij}, a

fuzzy number $\tilde{t}_{ij} \in$ T is defined, where $\tilde{t}_{ij}$ is the fuzzy time required for the completion of a_{ij}. A critical path is a longest path from v_1 to v_n and an activity a_{ij} on the critical path is called a critical activity. Let $\tilde{E}_i$ & $\tilde{L}_i$ be the earliest event time for event I, respectively. Let $\tilde{E}_j$ & $\tilde{L}j$ be the earliest event time, and the latest event time for event j, respectively. Let $D_j = \{i / i \in V \text{ and } a_{ij} \in A\}$ be a set of events obtained from event j$\in$ V and i<j. We then obtain E_j using the following equations

$$\tilde{E}_j = \max[\tilde{E}_i \oplus \tilde{t}_{ij}], i \in D_j, where\, \tilde{E}_1 = 0$$

Similarly, let H_i ={j/j$\in$ V and $a_{ij}\in$ A} be a set of events obtained from event i$\in$ V and i<j. We then obtain $\tilde{L}_i$ using the following equations.

$$\tilde{L}_i = \min[\tilde{L}_j \Theta \tilde{t}_{ij}], j \in H_i, where\, \tilde{L}_n = \tilde{E}_n$$

The interval $[\tilde{E}_i, \tilde{L}_j]$ is the time during which a_{ij} must be completed. When the earliest fuzzy event time and latest fuzzy event time have been obtained, we can calculate the total float of each activity. For activity i-j in a fuzzy network, the total float of the activity i-j can be computed as $\tilde{T}_{ij} = \tilde{L}_j \Theta \tilde{E}_i \Theta \tilde{t}_{ij}$

For the JKV representation of trapezoidal fuzzy numbers,
Earliest start is calculated by using the formula

$$\tilde{E}_j = \max[\tilde{E}_i \oplus \tilde{t}_{ij}], i \in D_j, where\, \tilde{E}_1 = 0$$

Latest finish is calculated by

$$\tilde{L}_i = \min[\tilde{L}_j \oplus (-\tilde{t}_{ij})], j \in H_i, where\, \tilde{L}_n = \tilde{E}_n$$

Total float of each activity can be calculated by

$$\tilde{T}_{ij} = \tilde{L}_j \oplus (-\tilde{E}_i) \oplus (-\tilde{t}_{ij})$$

Hence we can obtain the earliest fuzzy event time, latest event time and the total float of every activity by using the above equations.

4. Linear programming approach to find the critical path of a project network

The problem of finding critical path in a project network can be formulated as follows

$$Max \sum_{(i,j)\in A} t_{ij} x_{ij}$$

Subject to

$$\sum_{j:(i,j)\in A} x_{ij} - \sum_{j:(i,j)\in A} x_{ji} = \begin{cases} 1 & if\ i = 1 \\ -1 & if\ i = n \\ 0 & if\ i \neq 1, n \end{cases}$$

5. Proposed Method to find the critical path in a project network

Step 1: Formulate the critical path problem as a linear programming problem.

Step 2: Find the solution of the linear programming. If the solution is unique, then the critical path can be found from the solution.

Step 3: If there are alternate solutions, then the total float of each activity can be found and the critical path can be found such that the sum of total floats of the activities in the path is zero.
Step 4: If there is a unique critical path then the solution is achieved.
Step 5: If not, find the mode of the total float and rank the paths such that the path with maximum mode is the critical path.
Step 6: If the modes of all the paths are equal, then find the divergence of the paths and the path with maximum divergence will be the critical path.
Step 7: If the divergence of the paths are same, then compare the left and right spreads of all the paths and the path with maximum spread will be the critical path.

6. Numerical example: critical path for a product sale

As an example of critical path scheduling, consider the five activities associated with the production of a certain product and its sale as shown in table 6.1 and 6.2, where the durations are uncertain. Figure 6.3 and 6.4 shows the network diagram associated with these five activities.

Production and Sale of a Product			
Activity	Description	Predecessor	Duration
A	Design of product	-	(2,3,5,6)
B	Product model	-	(9,12,14,17)
C	Product testing	A	(7,8,10,11)
D	For Retail Sale	A	(12,18,20,26)
E	For Wholesale	B,C	(6,8,12,14)

Table 6.1: Production and sale of a product

Activity(i-j)	Name of the Activity	Duration
1-2	A	(2,3,5,6)
1-3	B	(9,12,14,17)
2-3	C	(7,8,10,11)
2-4	D	(12,18,20,26)
2-5	E	(6,8,12,14)

Table 6.2: Duration of activities

Then its Linear Programming Formulation is given by

$$Max[(2,6,1,1)_{JKV}\, x_{12} \oplus (9,17,3,3)_{JKV}\, x_{13} \oplus (7,11,1,1)_{JKV}\, x_{23}$$
$$\oplus (6,14,2,2)_{JKV}\, x_{34} \oplus (12,26,6,6)_{JKV}\, x_{24}]$$

Subject to

$$x_{12} + x_{13} = 1$$
$$x_{12} - x_{23} - x_{24} = 0$$
$$x_{13} + x_{23} - x_{34} = 0$$
$$- x_{24} - x_{34} = -1$$
$$x_{12}, x_{13}, x_{23}, x_{24}, x_{34} \geq 0$$

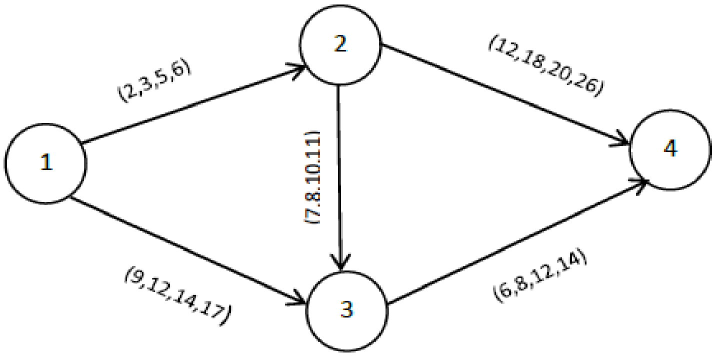

Figure 6.3: Network representation of the project

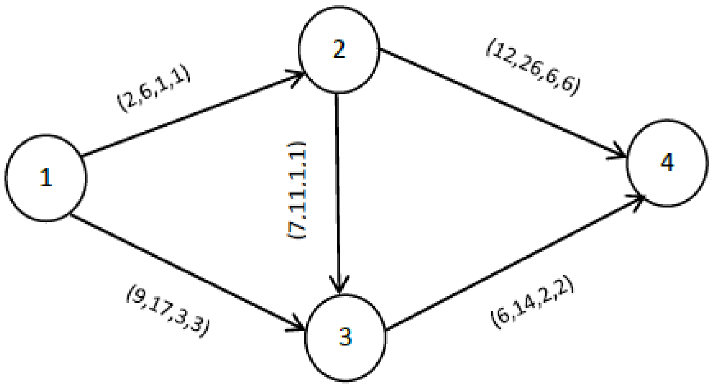

Figure 6.4: Network in JKV Representation.

Using the ranking function, the linear programming formulation will be

$Max[(8x_{12} \oplus 26x_{13} \oplus 18x_{23} \oplus 20_{JKV}\, x_{34} \oplus 38_{JKV}\, x_{24}]$

Subject to

$x_{12} + x_{13} = 1$

$x_{12} - x_{23} - x_{24} = 0$

$x_{13} + x_{23} - x_{34} = 0$

$-x_{24} - x_{34} = -1$

$x_{12}, x_{13}, x_{23}, x_{24}, x_{34} \geq 0$

Using TORA software, it can be found that there are multiple critical paths. The critical paths obtained are

i. $1-2-4$
ii. $1-2-3-4$
iii. $1-3-4$

Since all the paths are critical, we calculate the start, finish and total floats in the following table.

Activity	Duration	Earliest Start	Latest Finish	Total float
1-2	$(2,6,1,1)_{JKV}$	$(0,0,0,0)_{JKV}$	$(-10,18,2,2)_{JKV}$	$(-16,16,1,1)_{JKV}$
1-3	$(9,17,3,3)_{JKV}$	$(0,0,0,0)_{JKV}$	$(1,25,3,3)_{JKV}$	$(-16,16,0,0)_{JKV}$
2-3	$(7,11,1,1)_{JKV}$	$(2,6,1,1)_{JKV}$	$(1,25,3,3)_{JKV}$	$(-16,16,1,1)_{JKV}$
2-4	$(12,26,6,6)_{JKV}$	$(2,6,1,1)_{JKV}$	$(15,31,5,5)_{JKV}$	$(-17,17,-2,-2)_{JKV}$
3-4	$(6,14,2,2)_{JKV}$	$(9,17,3,3)_{JKV}$	$(15,31,5,5)_{JKV}$	$(-16,16,0,0)_{JKV}$

Table 6.5: Start, finish and floats of each activity

Total Floats of the paths is given in the following table.

Paths	Total Float	Rank
1-2-4	$(-33,33,-1,-1)_{JKV}$	0
1-2-3-4	$(-32,32,1,1)_{JKV}$	0
1-3-4	$(-32,32,0,0)_{JKV}$	0

Table 6.6: Total float of paths

Since all the paths are of rank zero, we calculate the mode of each path. All the paths have same mode. Hence we calculate the divergence of the paths, from which the critical path can be calculated, which is given below

Paths	Total Float	Mode	Divergence
1-2-4	$(-33,33,-1,-1)_{JKV}$	0	66
1-2-3-4	$(-32,32,1,1)_{JKV}$	0	64
1-3-4	$(-32,32,0,0)_{JKV}$	0	64

Hence it can be found that the critical path of the network is 1-2-4.

7. Conclusion

We have proposed a new method to find the critical path from multiple critical paths by using the JKV representation of trapezoidal fuzzy numbers. A real life problem is considered in this case to illustrate the method. This paves the way for further research.

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