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Fuzzy Number Fuzzy Measures and Fuzzy Integrals

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Abstract. By using the concepts of fuzzy number fuzzy measures and fuzzy valued functions a theory of fuzzy integrals is investigated. In this paper we have established the fuzzy version of Generalised monotone Convergence theorem and generalised Fatous lemma.

Keywords: Fuzzy number, Fuzzy-valued functions, Fuzzy integral, Fuzzy number fuzzy measure.

AMS Mathematics Subject Classification (2010): 28E10

1. Introduction

In the preceding paper [2], it is introduced that a concept of fuzzy number fuzzy measures, defined the fuzzy integral of a function with respect to a fuzzy number fuzzy measure and shown some properties and generalized convergence theorems. It is well-known that a fuzzy-valued function [3,4] is an extension of a function (point-valued), and the fuzzy integral of fuzzy-valued functions with respect fuzzy measures (point-valued) has been studied [3]; so it is natural to ask whether we can establish a theory about fuzzy integrals of fuzzy valued function with respect to fuzzy number fuzzy measures, the answer is just the paper's purpose. The paper is considered as a subsequent one of our earlier paper is considered as a subsequent one of our earlier work in [2]. In fact, it is also a continued work of [3]. Since what we will discuss in the following is a generalization of works in [2, 3].

Throughout the paper, R^+ will denote the interval$[0,\infty)$, X is an arbitrary fixed set, $\bar{\mathcal{A}}$ is a fuzzy σ-Algebra [1] formed by the fuzzy-subsets of X,$(X,\bar{\mathcal{A}})$ is a fuzzy Measurable space, $\mu:\bar{\mathcal{A}}\rightarrow R^+$ is a fuzzy measure in Sugeno's sense, $\int_{\bar{A}}\bar{f}\,d\mu$ is the resulting fuzzy integral [1].Operation $\in\{+,.,\wedge,\vee\}$, F(x) is the set of all $\bar{\mathcal{A}}$ - measurable functions from x to R^+, M(x) denotes the set of all fuzzy measures, $l(R^+)$ denotes the set of interval-numbers, R^+ denote the set of fuzzy numbers [2,3], $\bar{F}(x)$ denotes the set of all $\bar{\mathcal{A}}$ -measurable interval-valued functions [3]. $\bar{F}(x)$ denotes the set of all $\bar{\mathcal{A}}$ measurable fuzzy valued functions [3]. $\bar{M}(x)$ denotes the set of interval number fuzzy measures [2],

$\bar{M}(x)$ denotes the set of fuzzy Number fuzzy Measures [2], we will adopt the preliminaries in [2-4]. Here we omit them for brevity, for more details see [2-4].

2. Preliminaries

Definition 2.1. Let $\bar{f} \in \bar{F}(x), \bar{A} \in \bar{\mathcal{A}}, \bar{\mu} \in \bar{M}(x)$. Then the fuzzy integral of $\bar{f}$ and $\bar{A}$ with respect to $\bar{\mu}$ is defined as $\int_{\bar{A}} \bar{f} d\mu = [\int_{\bar{A}} f^- d\mu^-, \int_{\bar{A}} f^+ d\mu^+]$ where $\bar{f}^-(x) = \sup \bar{f}^-(x)$ and $\bar{f}^+(x) = \sup \bar{f}^+(x)$, $\mu^-(x) = \inf \mu^-(x)$ and $\mu^+(x) = \sup \mu^+(x)$

Definition: 2.2. Let $\bar{f} \in \bar{F}(x), \bar{A} \in \bar{\mathcal{A}}, \bar{\mu} \in \bar{M}(x)$. Then the fuzzy integral of $\bar{f}$ and $\bar{A}$ with respect to $\bar{\mu}$ is defined as $\int_A \bar{f} d\bar{\mu}(r) = \sup \{\lambda \in (0,1] : r \in \int_A \bar{f}_\lambda d\bar{\mu}_\lambda\}$ where $\bar{f}_\lambda(x) = \{r \in (0,1] : \bar{f}(x)(r) > \lambda\}$ and $\bar{\mu}_{\bar{\lambda}}$ is similar.

3. Main results

Theorem 3.1. Let $\bar{f} \in \bar{F}(x)$, $\bar{A} \in \bar{\mathcal{A}}, \bar{\mu} \in \bar{M}(x)$.Then $\int_A \bar{f}^- d\bar{\mu}^- \in \mathrm{R}^+$ and the following equation holds $\left(\int_A \bar{f} d\bar{\mu}\right)_\lambda = \int_A \bar{f}_\lambda d\mu_\lambda \; for \; \lambda \in (0,1]$ (2.1)

Proof: The condition is sufficient.

To prove that the condition is necessary it is enough to verify equation (2.1)

For a fixed $\lambda \in (0,1]$ let λ_n =(1- 1/n+1) λ then $\lambda_n \uparrow \lambda$.

It is easy to see that

$$\begin{aligned}\bar{f}_\lambda(x) &= \cap_{\lambda' < \lambda} \bar{f}_{\lambda'}(x)\\ &= \cap_{n=1}^{\infty} \bar{f}_{\lambda n}(x)\\ &= \lim_{n\to\infty} \bar{f}_{\lambda n}(x)\end{aligned}$$

Then we have $\bar{f}_{\lambda n}^- \uparrow \bar{f}_\lambda^-$, $\bar{f}_{\lambda n}^+ \uparrow \bar{f}_\lambda^+$

Similarly, $\bar{\mu}_{\lambda n}^- \uparrow \bar{\mu}_{\bar{\lambda}}^-$, $\bar{\mu}_{\lambda n}^+ \uparrow \bar{\mu}_\lambda^+$

We have

$$\int_{\bar{A}} \bar{f}_{\lambda n}^- d\bar{\mu}_{\lambda n}^- \uparrow \int_{\bar{A}} f_\lambda^- d\bar{\mu}_{\bar{\lambda}}^-$$

$$\int_{\bar{A}} \bar{f}_{\lambda n}^+ d\bar{\mu}_{\lambda n}^+ \downarrow \int_{\bar{A}} \bar{f}_\lambda^+ d\bar{\mu}_\lambda^+$$

Hence

$$\begin{aligned}\left(\int_{\bar{A}} \bar{f} d\bar{\mu}\right)_\lambda &= \cap_{n=1}^{\infty} \int_{\bar{A}} \bar{f}_{\lambda n} d\bar{\mu}_n\\ &= \lim_{n\to\infty} \int_{\bar{A}} \bar{f}_{\lambda n} d\bar{\mu}_n\\ &= \int_{\bar{A}} \bar{f}_{\lambda n} d\bar{\mu}_n\end{aligned}$$

Hence the theorem.

Theorem 3.2. Fuzzy integral of fuzzy valued functions with respect to fuzzy number fuzzy measures have the following property, $\bar{f}_1 \leq \bar{f}_2$, $\bar{\mu}_1 \leq \bar{\mu}_2$ => $\int_{\bar{A}} \bar{f}_1 \, d\bar{\mu}_1 \leq \int_A \bar{f}_2 \, d\bar{\mu}_2$

Proof : $\lambda \in (0,1]$. Let λ_n =(1- 1/n+1) λ then $\lambda_n \uparrow \lambda$.

It is easy to see that $\overline{(f_1)}_\lambda(x) = \cap_{\lambda' < \lambda} \bar{f}_{1_{\lambda'}}(x)$

$$= \cap_{n=1}^{\infty} \bar{f}_{1_{\lambda n}}(x)$$

$$= \lim_{n\to\infty} \bar{f}_{1_{\lambda n}}(x)$$

Then we have $\overline{(f_1)}_{\lambda n}^{-} \uparrow \overline{(f_1)}_{\lambda}^{-}$, $\overline{(f_1)}_{\lambda n}^{+} \uparrow \overline{(f_1)}_{\lambda}^{+}$

By generalised monotone convergence theorem

$\int_{\bar{A}} (\bar{f}_1)_{\lambda n}{}^{+} d\bar{\mu}_{1\lambda n}{}^{+} \uparrow \int_{\bar{A}} \bar{f}_{1\lambda}{}^{+} d\bar{\mu}_{1\lambda}{}^{+}$

$\int_{\bar{A}} (\bar{f}_1)_{\lambda n}{}^{-} d\bar{\mu}_{1\lambda n}{}^{-} \downarrow \int_{\bar{A}} \bar{f}_{1\lambda}{}^{-} d\bar{\mu}_{1\lambda}{}^{-}$ Hence

$$\left(\int_{\bar{A}} \bar{f}_1 d\bar{\mu}_1\right)_\lambda = \cap_{n=1}^{\infty} \int_{\bar{A}} \bar{f}_{1\lambda n} d\bar{\mu}_{1\lambda n}$$

$$= \lim_{n\to\infty} \int_{\bar{A}} \bar{f}_{1\lambda n} d\bar{\mu}_{1\lambda n}$$

$$= \int_{\bar{A}} \bar{f}_{1\lambda} d\bar{\mu}_{1\lambda}$$

$$= \int_{\bar{A}} \bar{f}_1 \, d\mu_1$$

$$\leq \int_{\bar{A}} \bar{f}_2 \, d\mu_2$$

Hence the theorem.

Theorem 3.3. Fuzzy integral of fuzzy valued functions with respect to fuzzy number fuzzy measure A⊂ B => $\int_A f \, d\mu \leq \int_B f \, d\mu$

Proof: For a fixed $\lambda \in (0,1]$. let λ_n = (1- 1/n+1) λ then $\lambda_n \uparrow \lambda$.

It is easy to see that $\bar{f}_\lambda(x) = \cap_{\lambda' < \lambda} \bar{f}_{\lambda'}(x)$

$$= \cap_{n=1}^{\infty} \bar{f}_{\lambda n}(x)$$

$$= \lim_{n\to\infty} \bar{f}_{\lambda n}(x)$$

Then we have $\bar{f}_{\lambda n}^{-} \uparrow \bar{f}_{\lambda}^{-}$, $\bar{f}_{\lambda n}^{+} \uparrow \bar{f}_{\lambda}^{+}$

By generalised monotone convergence theorem

$\int_A f_{\lambda n}{}^{-} d\mu_{\lambda n}{}^{-} \downarrow \int_A f_\lambda{}^{-} d\mu_\lambda{}^{-}$

$\int_A f_{\lambda n}{}^{+} d\mu_{\lambda n}{}^{+} \downarrow \int_A f_\lambda{}^{+} d\mu_\lambda{}^{+}$

$$\left(\int_A (f d\mu)\right)_\lambda = \cap_{n=1}^{\infty} \int_A f_{\lambda n} d\,\mu_{\lambda n}$$

$$= \lim_{n\to\infty} \int_A f_{\lambda n} d\,\mu_{\lambda n}$$

$$= \int_A f_\lambda d\,\mu_\lambda$$

$$= \int_A \cup_{\lambda \in (0,1]} \lambda f_\lambda d\,\mu_\lambda$$

$$= \int_A \bar{f} \, d\mu \leq \int_B \bar{f} \, d\mu \qquad (A \subset B)$$

Hence the theorem.

4. Convergence theorems

In this section we canvass the convergence of sequences of fuzzy integrals.

Theorem 4.1. (Generalised Monotone Convergence theorem)

Let $\{\bar{f}_n$ (n≥1), $\bar{f}\}\subset \bar{F}(x)$, $\{\mu_n$ (n≥1), $\mu\}\subset \bar{M}(x)$.

Then

(i) $\bar{f}_n^{-}\uparrow\bar{f}^{-}$ on A, $\bar{\mu}_{\lambda}^{+}\uparrow\bar{\mu}$ => $\int_A \bar{f}_{\lambda n}{}^{-}\, d\mu_{\lambda n}{}^{-} \downarrow \int_A \bar{f}_{\lambda}{}^{-}\, d\mu_{\lambda}{}^{-}$ (3.1)

ii) $\bar{f}_{\lambda}^{+}\downarrow\bar{f}^{+}$ on A, $\bar{\mu}_n^{+}\downarrow\bar{\mu}^{+}$ => $\int_A \bar{f}_n{}^{+} d\bar{\mu}_n \downarrow \int_A \bar{f}^{+}\, d\bar{\mu}^{+}$ (3.2)

Proof: To prove (i) it is sufficient to verify equation(3.1). For λ_k =(1- 1/1+k) λ then $\lambda_k\uparrow\lambda$. By the proof of Theorem 2.1 we obtain

$\bar{f}_{\lambda} = \lim_{n\to\infty}\lim_{k\to\infty}\bar{f}_{n\,\lambda k}$

$\bar{\mu}_{\lambda} = \lim_{n\to\infty}\lim_{k\to\infty}\bar{\mu}_{n\,\lambda k}$

Then

$$(\lim_{n\to\infty}\int_A \bar{f}_n d\,\bar{\mu}_n)_{\lambda k} = \cap_{n=1}^{\infty}\lim_{n\to\infty}(\int_A \bar{f}_n\, d\overline{\mu_n})_{\lambda k}$$
$$= \lim_{k\to\infty}\lim_{n\to\infty}\int_A(\bar{f}_n)_{\lambda k}\, d\,(\bar{\mu}_n)_{\lambda k}$$
$$= \int_A \lim_{k\to\infty}\lim_{n\to\infty}(\bar{f}_n)_{\lambda k}\, d\,(\lim_{k\to\infty}\lim_{n\to\infty}(\overline{\mu_n})_{\lambda k}$$
$$= \int_A \bar{f}_{\lambda} d\,\bar{\mu}_{\lambda} = \int_A(\bar{f}\, d\,\bar{\mu})_{\lambda}$$

This proves (i) and (ii) is similar.

Theorem 4.2. (Generalised Fatous lemma) Let $\{\bar{f}_n$ (n≥1), $\bar{f}\}\subset \bar{F}(x)$, $\{\bar{\mu}_n$ (n≥1), $\underline{\lim}\,\bar{\mu}_n$, $\overline{lim}\,\bar{\mu}_n\}\subset \bar{M}(x)$ then

(i) $\int_A \underline{\lim}\ \bar{f}_n d\underline{\lim}\ \bar{\mu}_n \le \underline{\lim}\int_A \bar{f}\, d\overline{\mu_n}$

(ii) $\overline{lim}\int_A \bar{f} d\bar{\mu}_n \le \int_A(\overline{lim}\bar{f}_n) d\,(\overline{lim}\bar{\mu}_n)$

Proof: To prove (i) For $\lambda\in[0,1]$ let λ_k =(1- 1/1+k) λ then $\lambda_k\uparrow\lambda$.

$\bar{f}_{\lambda} = \lim_{k\to\infty}\lim_{n\to\infty}\bar{f}_{n\,\lambda k}$

$\bar{\mu}_{\lambda} = \lim_{k\to\infty}\lim_{n\to\infty}\bar{\mu}_{n\,\lambda k}$ then

$$(\lim_{n\to\infty}\int_A \bar{f}_n d\,\bar{\mu}_n)_{\lambda} = \cap_{k=1}^{\infty}\lim_{n\to\infty}(\int_A \bar{f}_n\, d\bar{\mu}_n)_{\lambda k}$$
$$= \int_A \lim_{k\to\infty}\lim_{n\to\infty}(\bar{f}_n)_{\lambda k}\, d\lim_{k\to\infty}\lim_{n\to\infty}(\bar{\mu}_n)_{\lambda k}$$
$$= \int_A \lim_{k\to\infty}\lim_{n\to\infty}\inf\,(\bar{f}_n)_{\lambda k}\, d\lim_{k\to\infty}\lim_{n\to\infty}inf\,(\bar{\mu}_n)_{\lambda k}$$
$$= \int_A \lim_{k\to\infty}\lim_{n\to\infty}(\underline{\lim}\ \bar{f}_n)_{\lambda k}\, d\lim_{k\to\infty}\lim_{n\to\infty}\underline{\lim}\,(\bar{\mu}_n)_{\lambda k}$$
$$\le \underline{\lim}\int_A \lim_{k\to\infty}\lim_{n\to\infty}(\bar{f}_n)_{\lambda k}\, d\lim_{k\to\infty}\lim_{n\to\infty}(\bar{\mu}_n)_{\lambda k}$$

$$\leq \underline{\lim} \int_{\bar{A}} \bigcup_{\lambda\epsilon(0,1]} \lambda(f_n)_\lambda d(\bar{\mu}_n)_\lambda$$

$$= \underline{\lim} \int_{\bar{A}} \bar{f}_n d\,\bar{\mu}_n$$

(ii) $(\lim_{n\to\infty} \int_{\bar{A}} \bar{f}_n d\,\bar{\mu}_n)_\lambda = \bigcap_{k=1}^{\infty} \lim_{n\to\infty} (\int_{\bar{A}} \bar{f}_n\, d\bar{\mu}_n)_{\lambda k}$

$$= \lim_{k\to\infty} \lim_{n\to\infty} (\int_{\bar{A}} \bar{f}_n\, d\bar{\mu}_n)_{\lambda k}$$

$$= \lim_{k\to\infty} \lim_{n\to\infty} (sup \int_{\bar{A}} \bar{f}_n\, d\bar{\mu}_n)_{\lambda k}$$

$$= \lim_{k\to\infty} \lim_{n\to\infty} (\overline{lim} \int_{\bar{A}} \bar{f}_n\, d\bar{\mu}_n)_{\lambda k}$$

$$\leq \overline{lim} \int_{\bar{A}} \lim_{k\to\infty} \lim_{n\to\infty} (\bar{f}_n)\, d\bar{\mu}_n)_\lambda$$

$$\leq \overline{lim} \int_{\bar{A}} \lim_{n\to\infty} (\bar{f})_\lambda\, d(\bar{\mu}_n)_\lambda$$

$$= \int_{\bar{A}} \overline{lim}\, (\bar{f})\, d(\overline{lim}\bar{\mu}_n)$$

Hence the theorem.

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