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On Fuzzy 2-Metric Spaces

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Abstract. Fuzzy metric spaces, which are essentially a kind of Quasi-2-metrics in general topology, is extended to form the fuzzy 2-metric spaces. Standard results in Metric spaces such as Cantor's intersection theorem, Baire's category theorem are proved in the realm of fuzzy 2-metric spaces.

Keywords: Fuzzy metric spaces, Fuzzy pseudo-metric, fuzzy open balls, fuzzy closure, fuzzy topology

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1. Introduction

The theory of fuzzy sets was introduced by Zadeh [8] in 1965. Many authors have introduced the concept of fuzzy metric space in their own ways. George and Veeramani [2] modified the notion of fuzzy metric space with the help of t-norms. Since then, many authors studied the existing results in metric spaces to fuzzy metric spaces using t-norm. To mention a few, Sharma [7] introduced the concept of fuzzy 2-metric spaces; Cho [1] proved a common fixed point theorem for four mappings in fuzzy metric spaces and Han [3] extended the above results to fuzzy 2-metric spaces. Further Priyanka and Malviya [6] obtained some common fixed point theorem for occasionally weekly compatible mapping in fuzzy 2-metric spaces.

Ming [4] was the first to study the fuzzy metric space using fuzzy points. In this paper, we extend the study to fuzzy 2-metric spaces using fuzzy points. Also, we introduce the concept of fuzzy 2-metric spaces as an extension of fuzzy metric spaces studied by Ming [4] which are essentially a kind of quasi-2-metrics in general topology. Some results on fuzzy 2-metric spaces are obtained as special cases, some well known results of metric spaces like Cantor's intersection theorem and Baire's category theorem for fuzzy metric spaces are proved.

2. Preliminaries

Let X be a nonempty set, I = [0,1] be the unit interval. The pair (x, α), x$\in$X, $\alpha\in$I is called a fuzzy point in X, denoted by P_x^α or P(x, α), sometimes simply P. Fuzzy point $P_x^{1-\alpha}$ is called the dual point of P_x^α, usually the dual point of P is denoted by P*. The set of all fuzzy points in X is denoted by $IP_*(X) = \{P_x^\alpha : x\in X, \alpha\in[0,1]\}$.

For any fuzzy set $A\in I^X$, the collection of all mappings of X into I, we say that $P_x^\alpha \in A \Leftrightarrow \alpha < A(x)$ or $A(x) = 1$ and $P_x^\alpha \in A \Leftrightarrow \alpha \le A(x)$ and $A(x) \neq 0$.

Let P_x^α, $P_y^\beta \in IP_*(X)$ be fuzzy points, then we say that $P_x^\alpha \le P_y^\beta \Leftrightarrow x = y, \alpha \le \beta$ and $P_x^\alpha < P_y^\beta \Leftrightarrow x = y, \alpha < \beta$. Any two fuzzy points P_x^α, P_y^β are said to be comparable if $P_x^\alpha \le P_y^\beta$ or $P_y^\beta \le P_x^\alpha$.

Definition 2.1. ([5]) Let X be a nonempty set. A real valued function ρ defined on XxXxX satisfying the following:

(i) There are three points a, b, $c \in X$ such that $\rho(a, b, c) \neq 0$.
(ii) $\rho(a, b, c) = 0$ if and only if atleast two of the three points are equal.
(iii) $\rho(a, b, c) = \rho(a, c, b) = \rho(b, c, a) = \ldots$
(iv) $\rho(a, b, c) \le \rho(a, b, d) + \rho(a, d, c) + \rho(d, b, c)$ is called a 2-metric on X and (X, ρ) is called a 2-metric space. ρ is non negative and symmetric about the three variables a, b, c.

3. Fuzzy 2-metric

In this section, first we shall generalize the definition of 2-metric space and fuzzy metric space to fuzzy 2-metric space in the following way.

Definition 3.1. A fuzzy 2-metric for a set X is a mapping $e: IP_*(X) x\ IP_*(X) x\ IP_*(X) \to [0, \infty)$ satisfying the following axioms: For any $P_1, P_2, P_3, P_4 \in IP_*(X)$.

1. $e(P_1, P_2, P_3) = 0$ if atleast 2 of the three fuzzy points are comparable.
2. $e(P_1, P_2, P_3) \le e(P_1, P_2, P_4) + e(P_1, P_4, P_3) + e(P_4, P_2, P_3)$.
3. $e(P_1, P_2, P_3) = e(P_1, P_3^*, P_2^*) = e(P_2^*, P_3^*, P_1^*) = \ldots$
4. $e(P_1, P_2, P_3) > 0$ if no two of the three points are comparable.

We call (X, e) a fuzzy 2-metric space. In the above definition if (4) is omitted, then e is called a fuzzy pseudo-2-metric and (X, e) a fuzzy pseudo-2-metric space, if (3) and (4) are omitted, then e is called a fuzzy quasi-2-metric and (X , e) a fuzzy quasi-2-metric space. Essentially, the fuzzy 2-metric is a kind of special quasi- 2-metrics. Example: 3.2

Let X = {0, 1, 1/2, 1/3, ... }. Define d: XxXxX→[0,∝)

$$\text{by } d(x, y, z) = \begin{cases} 1, \text{ if } x, y, z \text{ are distinct and } \{1/n, 1/n+1\} \subset \{x, y, z\} \\ \qquad \text{for some positive integer } n \\ 0, \text{ otherwise} \end{cases}$$

Then d is a 2- metric on X. Define the fuzzy 2-metric for X as

$$e: IP_*(X) x\ IP_*(X) x\ IP_*(X) \to [0, \infty)$$

given by

$$e(P_x^\alpha, P_y^\beta\ P_z^\gamma) = \text{Max}\ \{d(x, y, z), \gamma - \beta - \alpha\}$$

Definition 3.3. Let e be a fuzzy quasi-2-metric on X and $P_0 \in IP_*(X)$, $\in >0$. The fuzzy set $B(P_0, \in) = \cup\{P_1: P_1 \in IP_*(X)$ and $e(P_0, P_1, P) < \in$, for all $P \in IP_*(X)\}$ is called a fuzzy $\in$-open ball with centre P_0 and radius $\in$.

Lemma 3.4. Let (X, e) be a fuzzy quasi-2-metric space and $B(P_0, \in)$ be a fuzzy $\in$-open ball of P_0, then for any $P_1 \in B(P_0,\in)$ there is a fuzzy δ-open ball $B(P_1, \delta)$ of P_1 such that $B(P_1, \delta) \subset B(P_0, \in)$.

Proof: Define $\delta = 1/2\ [\in - e(P_0, P_1, P)]$ and consider $B(P_1, \delta)$. Then we claim that $B(P_1, \delta) \subset B(P_0, \in)$, for if $P_2 \in B(P_1, \delta)$ then $e(P_1, P_2, P) < \delta$ for all $P \in IP_*(X) \Rightarrow e(P_1, P_2, P_0) < \delta$. Now by axiom (2), $e(P_0, P_2, P) \le e(P_0, P_2, P_1) + e(P_0, P_1, P) + e(P_1, P_2, P) < \delta + e(P_0, P_1, P) + \delta = 2\delta + e(P_0, P_1, P) = \in$ therefore $e(P_0, P_2, P) < \in$. Thus $B(P_1, \delta) \subset B(P_0, \in)$.

As in the case of fuzzy metric space [4] we can show that the collection of all fuzzy open balls will generate a topology for fuzzy-2-metric space. Let us denote the topology induced by the fuzzy pseudo-2-metric by *T*. For $P_0 \in IP_*(X)$, $\in >0$,

$\underline{B}(P_0, \in) = \cup\{P_1: P_1 \in IP_*(X)$ and $e(P_0, P_1, P) \le \in$, $\forall\ P \in IP_*(X)\}$ is called a fuzzy $\in$- closed ball with centre P_0 and radius $\in$. Usually we call $B(P_0, \in)$ the fuzzy $\in$-open ball of P_0 and $\underline{B}(P_0, \in)$ the fuzzy $\in$-closed ball of P_0. In a fuzzy pseudo-2-metric space every fuzzy $\in$-open ball is a fuzzy open set and every fuzzy $\in$-closed ball is a fuzzy closed set.

Definition 3.5. If A is a Fuzzy set, then the closure of A, denoted as $\overline{A}$ is defined as $\{P_x \in IP_*(X)$ such that each fuzzy open set containing P_x intersects $A\}$.

4. Convergence, compactness for fuzzy pseudo 2-metric spaces

In [2], a detailed study on convergence, compactness for fuzzy metric spaces has been made by A.George and P.Veeramani. In this section, we shall generalize these concepts for sequence in fuzzy pseudo- 2-metric spaces.

A sequence $\{P_{Xn}\}$ in a fussy pseudo-2-metric space (X, e) converges to a fuzzy point $P_X \in IP_*(X)$, if and only if, for $\in >0$, there exists $n_0 \in N$ such that $P_{Xn} \in B(P_X, \in)$; for all $n \ge n_0$.

Theorem 4.1. Let (X, e) be a fuzzy pseudo-2-metric space and *T* be the topology induced by the fuzzy pseudo-2-metric, then a sequence $\{P_{Xn}\}$ in $IP_*(X)$ converges to P_X in $IP_*(X)$ if and only if for each $\in >0$, $e(P_{Xn}, P_X, P) \to 0$ as $n \to \propto$ for all $P \in IP_*(X)$.

Proof: Suppose $\{P_{Xn}\} \to P_X$, then for $\in >0$, there exists $n_0 \in N$ such that $P_{Xn} \in B(P_X, \in)$, for all $n \ge n_0$. This implies $e(P_{Xn}, P_X, P) < \in$ for all $n \ge n_0$ and for all $P \in IP_*(X)$. Hence, $e(P_{Xn}, P_X, P) - 0 < \in$. Thus $e(P_{Xn}, P_X, P) \to 0$ as $n \to \propto$, for each $\in >0$ conversely, if $e(P_{Xn}, P_X, P) \to 0$ $n \to \propto$ and for all $P \in IP_*(X)$. Then for each $\in >0$, there exists $n_0 \in N$ such that $e(P_{Xn}, P_X, P) - 0 < \in$ $\forall$ $n \ge n_0$. which implies $e(P_{Xn}, P_X, P) < \in$ for all $n \ge n_0$. Therefore $P_{Xn} \in B(P_X, \in)$, for all $n \ge n_0$. $P_{Xn} \to P_x$ in $IP_*(X)$.

Definition 4.2. A sequence $\{P_{Xn}\}$ in a fuzzy pseudo-2-metric space (X,e) is a cauchy sequence if and only if $\lim_{n\to\infty} e(P_{Xm}, P_{Xn}, P)=0$ for all $P\in IP_*(X)$.

A fuzzy pseudo-2-metric space, in which every cauchy sequence converges is called a complete fuzzy pseudo-2-metric space.

Definition 4.3. A sequence $\{P_{Xn}\}$ in a fuzzy pseudo-2-metric space (X,e) is said to be a F-cauchy sequence if and only if for each $\in >0$, ther exist $n_0\in N$ such that $e(P_{Xn}, P_{Xm}, P)<\in$ for all $n,m \geq n_0$ and for all $P\in IP_*(X)$.

Theorem 4.4. Let $\{P_{Xn}\}$ be a F - cauchy sequence in a fuzzy pseudo-2-metric space (X, e). Then $\{P_{Xn}\}$ is convergent if it has a covergent subsequence.

Proof: Let $\left\{P_{X_{n_k}}\right\}$ be a convergent subsequence of $\{P_{Xn}\}$ and assume that $\left\{P_{X_{n_k}}\right\}$ converges to P_X in $IP_*(X)$ as $k\to\infty$. Let $\in >0$ be given. Since $\{P_{Xn}\}$ is a F-cauchy sequence, there exists $n_0\in N$ such that $e(P_{Xn}, P_{Xm}, P) <\in$ for all $m, n \geq n_0$ and for all $P\in IP_*(X)$. Since, $\left\{P_{X_{n_k}}\right\}$ converges to P_X, there exists $j\in N$ such that $e(P_{X_{n_k}}, P_X, P)<\in$ for all $n_k \geq j$. Therefore, for all n, $n_k\geq\max(n_0, j)$, $e(P_{Xn}, P_X, P)\leq e(P_{Xn}, P_X, P_{X_{n_k}})+e(P_{Xn}, P_{X_{n_k}}, P)+ e(P_{X_{n_k}}, P_X, P) <\in$. Hence $\{P_{Xn}\}$ converges to P_X as $n\to\infty$ Hence, the theorem.

5. Analogue of cantor's intersection theorem, Baire's theorem, uniform boundedness principle

Definition 5.1. Let (X, e) be a fuzzy pseudo-2-metric space. Then a collection of sets $\{F_n\}_{n\in I}$ is said to have fuzzy diameter zero if and only if there exists $n\in I$ such that $e(P_x, P_y, P_z) <\in$ for all $P_x, P_y, \quad P_z \in F_n$.

Definition 5.2. A non empty subset F of fuzzy pseudo-2-metric space (X, e) has fuzzy diameter zero if and only if F is a singleton set.

Theorem 5.3. (Cantor's intersection theorem) A necessary and sufficient condition that a fuzzy pseudo-2-metric space (X, e) be complete is that every nested sequence of nonempty closed sets $\{F_n\}_{n=1}^{\infty}$ with fuzzy diameter zero have nonempty intersection.

Proof: Let (X, e) be the given fuzzy pseudo-2-metric space and suppose that every nested sequence of nonempty closed sets $\{F_n\}_{n=1}^{\infty}$ with fuzzy diameter zero have nonempty intersection.

We have to prove that (X, e) is complete. Let $\{P_{Xn}\}$ be a F - Cauchy sequence in X. Take $A_n = \{P_{Xn}, P_{Xn+1}, P_{Xn+2}, \ldots\}$ and $F_n = \overline{A_n}$, then we claim that

$\{F_n\}$ has fuzzy diameter zero. Since $\{P_{Xn}\}$ is a F - cauchy sequence, there exists $n_0 \in N$, such that $e(P_{Xn}, P_{Xm}, q) <\in$ for all m, n $\geq n_0$ and for all $q\in IP_*(X)$. Therefore,

$$e(P_x, P_y, q) <\in \tag{5.1}$$

for all P_x, P_y and $q\in A_{n_0}$. Let $P_u, P_v\in F_{n_0}$. Then there exist sequences $\{P_{u_n}\}$ and $\{P_{v_n}\}$ in A_{n_0} such that $\{P_{u_n}\}$ converges to P_u and $\{P_{v_n}\}$ converges to P_v. Hence $P_{u_n} \in B(P_u, \in)$ and $P_{v_n} \in B(P_v, \in)$, for sufficiently large n. Hence

$$e(P_{u_n}, P_u, P)<\in,\ e(P_{v_n}, P_v, P)<\in \tag{5.2}$$

for all P. Now, $e(P_u, P_v, P_x) \leq e(P_u, P_v, P_z)+e(P_u, P_z, P_x) +e(P_z, P_v, P_x)$. By repeated application of axiom (2) in the definition of fuzzy 2-metric space and using (5.1) and (5.2). We get, $e(P_u, P_v, P_x)<\in$ for all $P_u, P_v, P_x \in F_{n_0}$. Thus $\{F_n\}$ has fuzzy diameter zero. Hence by hypothesis $\bigcap_{n=1}^{\infty} F_n$ is nonempty. Take $x\in \bigcap_{n=1}^{\infty} F_n$. Then for $\in >0$, there exists $n_1\in N$ such that $e(P_{Xn}, P_X, P)<\in$ for all $n\geq n_1$. Therefore, for each $\in >0$, $e(P_{Xn}, P_X, P)$ converges to zero as $n\to\propto$. Hence $\{P_{Xn}\}$ converges to P_X. Therefore, (X, e) is complete. Conversely, suppose that (X, e) is a complete fuzzy pseudo-2-metric space and $\{F_n\}_{n=1}^{\infty}$ is a nested sequence of nonempty closed sets with fuzzy diameter zero. Let $P_{Xn} \in F_n$, n=1, 2, ... Since $\{F_n\}$ has fuzzy diameter zero, for $\in >0$, there exists $n_0\in N$ such that $e(P_x, P_y, P) <\in$, for all P_x, P_y, $P\in F_{n_0}$. Therefore, $e(P_{Xn}, P_{Xm}, P)<\in$ for all n, m $\geq n_0$. Since, $P_{Xn} \in F_n \subset F_{n_0}$ and $P_{Xm} \in F_m \subset F_{n_0}$, $\{P_{Xn}\}$ is a F-cauchy sequence. But (X,e) is a complete fuzzy metric space and hence $\{P_{Xn}\}$ converges to P_X. Now for each fixed n, $P_{x_k} \in F_n$ for all k $\geq$ n. Therefore, $P_x\in \overline{F_n} = F_n$, for every n, and hence $P_X \in \bigcap_{n=1}^{\infty} F_n$. Hence the theorem.

Remark 5.4. The element $P_X \in \bigcap_{n=1}^{\infty} F_n$ in the above theorem is unique. For if there are two elements $P_x, P_y \in \bigcap_{n=1}^{\infty} F_n$. Since $\{F_n\}_{n=1}^{\infty}$ has fuzzy diameter zero, for each fixed $\in = 1/n>0$, $e(P_x, P_y, P_z) <1/n$ for all $P_x, P_y, P_z \in F_n$ and for each n. This implies $e(P_x, P_y, P_z)\to 0$ as $n\to\propto$ and hence $P_x = P_y$.

Theorem 5.6. (Baire's Theorem) Let (X, e) be a complete fuzzy pseudo-2-metric space. Then the intersection of countable number of dense open sets is dense.

Proof: Let B_o be a nonempty open subset of X and $D_1, D_2, D_3, \ldots$, be dense open sets in X. Since D_1 is dense in X, $B_0 \cap D_1 \neq \phi$. Let $P_1 \in B_0\cap D_1$, since $B_0 \cap D_1$ is open, there exist $\in_1>0$, such that $B(P_1, \in_1) \subset B_0 \cap D_1$. Choose $\in_1^1 <\in_1$ such that

$B(P_1, \in_1^1) \subset B_0 \cap D_1$. Let $B_1 = B(P_1, \in_1^1)$ since D_2 is dense in $IP_*(X)$. $B_1 \cap D_2 \neq \phi$. Let $P_2 \in B_1 \cap D_2$. Since $B_1 \cap D_2$ is open, there exist $\in_2 > 0$ such that $B(P_2, \in_2) \subset B_1 \cap D_2$. Choose $\in_2^1 < \in_2$ such that $B(P_2, \in_2^1) \subset B_1 \cap D_2$.

Let $B_2 = B(P_2, \in_2^1)$ proceeding similarly by induction. We can find a $P_n \in B_{n-1} \cap D_n$. Since $B_{n-1} \cap D_n$ is open, there exist $\in_n > 0$ such that $B(P_n, \in_n) \subset B_{n-1} \cap D_n$. Choose $\in_n^1 < \in_n$ such that $B(P_n, \in_n^1) \subset B_{n-1} \cap D_n$. Let $B_n = B(P_n, \in_n^1)$. Now we claim that $\{P_n\}$ is a F - cauchy sequence. For given $\in > 0$, choose $n_0 \in N$ such that for $n \geq n_0$ and $m \geq n$, we have, $e(P_m, P_n, P) \leq \in$. Thus $\{P_n\}$ is a F-cauchy sequence. Since (X,e) is complete $P_n \to P$ for some P in $IP_*(X)$. But $P_{x_k} \in B(P_n, \in_n^1)$ for all $k \geq n$ and we know that every closed ball is a closed set. Hence, $P \in B(P_n, P_n^1) \subset B_{n-1} \cap D_n$, for all n. Therefore, $B_0 \cap [\bigcap_{n=1}^{\infty} D_n] \neq \phi$. Hence, $\bigcap_{n=1}^{\infty} D_n$ is dense in X. Hence the theorem

Remark 5.7. A complete fuzzy pseudo-2-metrix cannot be represented as the union of a sequence of nowhere dense sets and hence it is not of first category.

Theorem 5.8. (Uniform Boundedness Principle) Let F be a collection of real continuous functions f defined on a complete fuzzy pseudo-2-metric space (X, e) and suppose for each $P_x \in IP_*(X)$ there exists a real no. $K(P_x)$ such that $f(P_x) \leq K(P_x)$, for all $f \in F$. Then there exists an open ball $S = B(P_0, \in)$, $P_0 \in IP_*(X)$, $\in > 0$ and a constant k such that $f(P_x) \leq k$, for all $P \in S$, $f \in F$.

Proof: For each $f \in F$, and each $k \in N$, define $\xi(k, f) = \{P_x : f(P_x) \leq k\}$. Now f is continuous and hence its complement $\{P_x : f(P_x) > k\}$ is open. Therefore, $\xi(k, f)$ is closed. Define $\xi_k = \cap\{\xi(k, f) : f \in F\}$, ξ_k being an intersection of closed sets is closed. Now we claim $X = \cup\{\xi_k, k = 1, 2, 3, \ldots\}$. For if $P_x \in IP_*(X)$. Then, $f(P_x) \leq k(P_x)$, for all $f \in F$, and hence there exists an integer k_{Px}, such that $f(P_x) \leq k_{Px}$, for all f. Hence $P_x \in \xi_{k_{Px}}$. Hence our claim, since (X, e) is a complete fuzzy pseudo-2-metric space, by cantor's intersection theorem it is of second category.

Hence, atleast one of the sets, say ξ_k, is not nowhere dense. Therefore, $\overline{\xi}_k$ contain an open ball $S = B(P_0, \in)$, $P_0 \in IP_*(X)$. $\in > 0$ such that $S \subset \overline{\xi}_k = \xi_k$. Therefore, if $P_x \in S$, then $P_x \in \xi_k$ and hence $f(P_x) \leq k$, for all $f \in F$. Hence the theorem.

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