

Fuzzy bi- ω Languages

D. Cokilavany¹ and R. Venkatesan

Department of Mathematics
Anna University, Chennai - 600 025, India.

¹Corresponding Author, E-mail:cokila@annauniv.edu

Abstract. In this paper, we consider the recognizable device of fuzzy bi- ω language as fuzzy bi- ω automata. Further, we extend the concept of homomorphism of semigroup with infinite product to homomorphism of semigroup with bi infinite product. This concept is proposed in the place of fuzzy bi- ω automata.

Keywords: Fuzzy bi- ω language; Fuzzy bi- ω automata.

AMS Mathematics Subject Classification (2010): 68Q70, 68Q45, 18B20, 03E72

1. Introduction

Bi - infinite or two sided infinite words are the natural extension of (right) infinite words [1,3,4,6]. This extension can be obtained by replacing the set of natural numbers by the set of integers, since the infinite words are the functions of natural numbers into the input alphabets. The theory was first extended in [5] and since then the study has been continued in [2].

Section 2 summarizes some preliminaries on fuzzy bi- ω languages. Section 3 gives the definition for fuzzy bi- ω automaton to recognize fuzzy bi- ω language. Section 4 presents a definition for homomorphism of semigroup with bi-infinite products to accept bi- ω languages and equivalence between fuzzy bi- ω automaton and homomorphism of semigroup with bi-infinite product

2. Fuzzy bi- ω languages

A function ω from the set of integers Z into the set of alphabets A is called a bi-infinite string or bi-infinite word. A bi-infinite word is denoted as $\cdots\omega(-1)\omega(0)\omega(1)\cdots$ or if we write $\omega(i) = a_i$ for each $i \in Z$, then the bi-infinite word ω is also denoted as $\cdots a_{-1}a_0a_1 \cdots$. The set of all bi-infinite words over the alphabet A is denoted by ${}^\omega A^\omega$. Any subset of ${}^\omega A^\omega$ is called the bi-infinite language. Any fuzzy set of ${}^\omega A^\omega$ is called the fuzzy bi-infinite language.

The shift function $\tau: {}^\omega A^\omega \rightarrow {}^\omega A^\omega$ is defined for every $\omega \in {}^\omega A^\omega$ and for every $j \in Z$, $\tau(\omega(j)) = \omega(j-1)$ or $\tau(a_j) = a_{j-1}$. Let L be a bi-infinite of ${}^\omega A^\omega$. Then,

D.Cokilavany and R.Venkatesan

$$\tau(L) = \{\tau(\omega) \mid \omega \in L\}.$$

L is said to be shift invariant if $L = \tau(L)$. Shift invariant means that there is no special role for a_0 .

The set of infinite words A^ω is actually the set of right infinite words. Let ${}^\omega A$ denote the set of left infinite words. We can build a left infinite word u from the right infinite word w in the following way. Let $w \in A^\omega$,

$$u_{-i} = w_i, \quad i \in \mathbb{N},$$

denote \overline{w} the left infinite word defined by the right infinite word. In the similar way, we obtain a set $\overline{X} = \{\overline{w} \mid w \in X\}$, where X is a right infinite language.

Similarly, we can build a bi-infinite word $\omega = uw$. Let $u \in {}^\omega A$ and $w \in A^\omega$, then the bi-infinite word ω is defined by

$$\omega_n = \begin{cases} u_n, & \text{if } n \leq 0 \\ w_{n-1}, & \text{if } n \geq 1 \end{cases}$$

and is denoted by $[u, w]$. Suppose u and w are fuzzy left infinite word and fuzzy right infinite word, then the weight function is calculated as follows:

$$W(\omega) = \vee \{W(u) \wedge W(w) \mid \omega = uw\}, \quad \forall u \in {}^\omega A, w \in A^\omega.$$

Let $X \in F({}^\omega A)$ and $Y \in F(A^\omega)$, then we write

$$[X, Y] = \{[x, y] \in {}^\omega A^\omega \mid x \in {}^\omega A, y \in A^\omega\}$$

and

$$XY = \bigcup_{n \in \mathbb{Z}} \tau^n([X, Y]).$$

3. Recognition of fuzzy bi- ω languages by fuzzy bi- ω automata

In this section, we introduce the concept of fuzzy bi- ω automaton to accept fuzzy bi- ω language. Further, we prove the equivalences of fuzzy bi- ω language recognized by fuzzy bi- ω automaton, fuzzy bi- ω language obtained by fuzzy Buchi recognizable languages and bi- ω rational languages.

Definition 3.1. A fuzzy bi- ω automaton is a 5-tuple $\mathfrak{F} = (S, A, f, \mathcal{I}, \mathfrak{B})$, where

- S is the finite set of states.
- A is the finite set of input alphabets.
- $f: S \times A \times S \rightarrow [0,1]$ is the fuzzy transition function.
- $\mathcal{I} : S \rightarrow [0,1]$ is a fuzzy set of left-infinite repetitive states.
- $\mathfrak{B} : S \rightarrow [0,1]$ is a fuzzy set of right-infinite repetitive states.

Two transitions $f(s_i, \omega_i, s_{i+1}) > 0$ and $f(s_j, \omega_j, s_{j+1}) > 0$ are consecutive if $i = j$. A run of the fuzzy bi- ω automaton \mathfrak{F} on a bi- ω word $\omega = \cdots \omega_{-1} \omega_0 \omega_1 \cdots$ is a bi- ω word if $R = \cdots s_{-1} s_0 s_1 \cdots \in {}^\omega S^\omega$.

Fuzzy bi- ω Languages

The set of right infinitely often states that occurs in the run R is denoted by $\text{inf}(R^+)$ and is defined by

$$\text{inf}(R^+) = \{s \in S \mid \text{there exists infinitely many } i \text{ such that } s_i = s\}$$

A right infinite run in \mathfrak{F} is successful if it visits \mathfrak{B} infinitely often, that is, $\text{inf}(R^+) \cap \mathfrak{B} \neq \emptyset$. A fuzzy right ω -word is called an accepted word by \mathfrak{F} , if the corresponding right infinite run R is successful in \mathfrak{F} . The set of right ω -words recognized by \mathfrak{F} is a set of label of right successful runs in \mathfrak{F} and is denoted by $L^\omega(\mathfrak{F})$

The set of left infinitely often states that occurs in the run R is denoted by $\text{inf}(R^-)$ and is defined as

$$\text{inf}(R^-) = \{s \in S \mid \text{there exists infinitely many } i \leq 0 \text{ such that } s_i = s\}$$

A left infinite run in \mathfrak{F} is successful if it visits \mathfrak{I} infinitely often, that is $\text{inf}(R^-) \cap \mathfrak{I} \neq \emptyset$. A fuzzy left ω -word is called accepted word by \mathfrak{F} , if the corresponding left infinite run R is successful in \mathfrak{F} . The set of left ω -words recognized by \mathfrak{F} is a set of labels of right successful runs in \mathfrak{F} and is denoted by ${}^\omega L(\mathfrak{F})$.

A run in \mathfrak{F} is successful if $\text{inf}(R^-) \cap \mathfrak{I} \neq \emptyset$ and $\text{inf}(R^+) \cap \mathfrak{B} \neq \emptyset$. A fuzzy bi- ω word is called an accepted word by \mathfrak{F} , if the corresponding run R is successful in \mathfrak{F} . The set of fuzzy bi- ω words recognized by \mathfrak{F} is a set of labels of successful runs in \mathfrak{F} and is denoted by ${}^\omega L^\omega(\mathfrak{F})$. A set X of fuzzy bi- ω words is recognizable if there exists a fuzzy bi- ω automaton \mathfrak{F} such that $X = {}^\omega L^\omega(\mathfrak{F})$ or simply $X = L(\mathfrak{F})$.

A fuzzy bi- ω language μ is said to be recognizable if there exists a finite fuzzy bi- ω automaton \mathfrak{F} . A fuzzy set μ of ${}^\omega A^\omega$ is bi- ω rational if μ is a finite union of fuzzy sets of the form ${}^\omega XYZ^\omega$ with X, Y and Z are recognizable fuzzy languages of some fuzzy automaton.

Theorem 3.2. For each fuzzy bi- ω language μ of ${}^\omega A^\omega$, the following conditions are equivalent

1. $\mu \in F\{{}^\omega A^\omega\}$ is recognizable by fuzzy bi- ω automaton,
2. μ is shift invariant and is a finite union of the form $[\overline{X}, Y]$ for $X, Y \in F(A^\omega)$, where X and Y are recognizable by fuzzy Buchi automaton,
3. μ is a finite union of fuzzy sets of the form \overline{XY} for $X, Y \in F(A^\omega)$, where X and Y are recognizable by fuzzy Buchi automaton,
4. μ is bi- ω rational.

Proof: (1) implies (2). Let $\mathfrak{F} = (S, A, f, \mathfrak{I}, \mathfrak{B})$ be a fuzzy bi- ω automaton such that $\mu = {}^\omega L^\omega(\mathfrak{F})$. Construct two fuzzy Buchi automaton \mathfrak{F}_1 and \mathfrak{F}_2 for every $s \in S$, $\mathfrak{F}_1 = (S, A, f, \mathfrak{I}, s)$ and $\mathfrak{F}_2 = (S, A, f, s, \mathfrak{B})$. Let $X_s = L^\omega(\mathfrak{F}_1)$ and $Y_s = L^\omega(\mathfrak{F}_2)$ be

D.Cokilavany and R.Venkatesan

the two fuzzy languages recognized by \mathfrak{F}_1 and \mathfrak{F}_2 respectively. Clearly, by construction

$$\mu = \bigcup_{s \in S} [\overline{X_s}, Y_s].$$

Every fuzzy bi- ω language is shift invariant. That is no special importance for the origin.

Further,(2) implies (3). If $\mu = \bigcup_{j=1}^n [X_j, Y_j]$, then

$$\mu = \bigcup_{i \in Z} \tau^i(\mu) \tag{1}$$

$$= \bigcup_{j=1}^n \left(\bigcup_{i \in Z} \tau^i [X_j, Y_j] \right) \tag{2}$$

$$= \bigcup_{j=1}^n X_j Y_j. \tag{3}$$

Now, (3) implies (4). Since X and Y are recognizable by fuzzy Buchi automaton, by theorem 3.3.2, they are of the form

$$X = \bigcup_{i=1}^n U_i V_i^\omega, \quad Y = \bigcup_{j=1}^n \bigcup_{j=1}^n W_j Z_j^\omega$$

Now

$$\overline{XY} = \bigcup_{i=1}^n \bigcup_{j=1}^n \overline{V_i^\omega U_i W_j Z_j^\omega}$$

where U_i, V_i, W_j and Z_j are fuzzy languages recognized by some fuzzy automata.

Hence μ is bi- ω rational. To prove, (4) implies (1), let μ be bi- ω rational. Then μ is a finite union of fuzzy sets of the form ${}^\omega XYZ^\omega$ where X, Y and Z are recognizable fuzzy languages of some fuzzy automaton. Construct a fuzzy bi- ω automaton $\mathfrak{F} = (S, A, f, \mathfrak{I}, \mathfrak{B})$, where \mathfrak{I} and \mathfrak{B} are singleton fuzzy sets such that X, Y and Z are recognizable fuzzy languages of fuzzy automata $F_1 = (S, A, f, \mathfrak{I}, \mathfrak{I})$, $F_2 = (S, A, f, \mathfrak{I}, \mathfrak{B})$ and $\mathfrak{F}_2 = (S, A, f, \mathfrak{B}, \mathfrak{B})$ respectively. Hence, ${}^\omega XYZ^\omega = {}^\omega L^\omega(\mathfrak{F})$.

4. Recognition of fuzzy bi- ω languages by homomorphism

This section presents the definitions of semigroup with left infinite product, semigroup with bi-infinite product and homomorphism of semigroup with bi-infinite products. Also, the equivalence between fuzzy bi- ω automaton and homomorphism of semigroup with infinite product is proved.

Definition 4.1. Semigroup with left infinite product is a 5-tuple $(M, N, *, \Delta)$, where

- (M, \cdot) is a semigroup.
- $*$: $N \times M \rightarrow N$ is the mixed product satisfying for every $l, m \in M$ and $n \in N$, $(n * m) \cdot l = n * (m \cdot l)$.
- Δ is the left infinite product.

Definition 4.2. Semigroup with bi-infinite product is a 4 - tuple (M, N, L, P) , where

- M is a semigroup.

Fuzzy bi- ω Languages

- (M, N) is a semigroup with infinite product.
- (M, L) is a semigroup with left infinite product.

with a surjective map $\lambda: N \times L \rightarrow P$ such that for $n \in N, l \in L$ and $m \in N$ we have $\lambda(nm, l) = \lambda(n, ml)$.

Definition 4.3. Let $\mathcal{S} = (M_1, N_1, L_1, P_1)$ and $\mathcal{T} = (M_2, N_2, L_2, P_2)$ be two semigroups with bi-infinite product. A homomorphism of semigroups with bi-infinite product is a 4-tuple $h = (h_1, h_2, h_3, h_4)$ such that (h_1, h_2) is a homomorphism of semigroup with infinite product, (h_1, h_3) is a homomorphism of semigroup with left infinite product and $h_4: P_1 \rightarrow P_2$ is a map such that for each $n \in N_1, l \in L_1$, $h_4(nl) = h_3(n)h_2(l)$.

Definition 4.4. Let $\mathcal{S} = (M_1, N_1, L_1, P_1)$ and $\mathcal{T} = (M_2, N_2, L_2, P_2)$ be two semigroups with bi-infinite product. A homomorphism $h: \mathcal{S} \rightarrow \mathcal{T}$ of semigroups with bi-infinite product recognize a fuzzy set $\mu = (\mu_1, \mu_2, \mu_3, \mu_4)$ if there exists a fuzzy set $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ such that the following fuzzy diagrams are commutative

$$\begin{array}{ccc}
 M_1 & \xrightarrow{h_1} & M_2 \\
 \mu_1 \downarrow & & \downarrow \lambda_1 \\
 [0, 1] & \xlongequal{\quad} & [0, 1] \\
 \\
 L_1 & \xrightarrow{h_3} & L_2 \\
 \mu_3 \downarrow & & \downarrow \lambda_3 \\
 [0, 1] & \xlongequal{\quad} & [0, 1] \\
 \\
 N_1 & \xrightarrow{h_2} & N_2 \\
 \mu_2 \downarrow & & \downarrow \lambda_2 \\
 [0, 1] & \xlongequal{\quad} & [0, 1] \\
 \\
 P_1 & \xrightarrow{h_4} & P_2 \\
 \mu_4 \downarrow & & \downarrow \lambda_4 \\
 [0, 1] & \xlongequal{\quad} & [0, 1]
 \end{array}$$

Theorem 4.5. Let $L(\mathfrak{F})$ be a fuzzy bi- ω language recognized by fuzzy bi- ω automata \mathfrak{F} . Then there exists a homomorphism of semigroups with bi-infinite product which recognizes the same fuzzy bi- ω -language.

Proof: Let $\mathfrak{F} = (S, A, f, \mathcal{I}, \mathfrak{B})$ be a fuzzy bi- ω automaton recognizing a fuzzy bi- ω language $L(\mathfrak{F})$ of ${}^\omega A^\omega$. Without loss of generality, let us assume that $L(\mathfrak{F}) = XY$ with X is a fuzzy set of $A^{\bar{\omega}}$ and Y is a fuzzy set of A^ω .

Y is the fuzzy set of right infinite words. So, there exists a homomorphism of semigroup with right infinite product such that the following diagrams are commutative

$$\begin{array}{ccc}
 A^+ & \xrightarrow{h_1} & M \\
 \mu_1 \downarrow & & \downarrow \lambda_1 \\
 [0, 1] & \xlongequal{\quad} & [0, 1] \\
 \\
 A^\omega & \xrightarrow{h_2} & N \\
 \mu_2 \downarrow & & \downarrow \lambda_2 \\
 [0, 1] & \xlongequal{\quad} & [0, 1]
 \end{array}$$

In a similar way, since X is the fuzzy set of left infinite words obtained from $A^{\bar{\omega}}$, by theorem 3.3.1, there exists a homomorphism of semigroup with right infinite product such

that the following diagrams are commutative

$$\begin{array}{ccc}
 A^+ & \xrightarrow{h_3} & M \\
 \mu_3 \downarrow & & \downarrow \lambda_3 \\
 [0, 1] & \equiv & [0, 1]
 \end{array}
 \qquad
 \begin{array}{ccc}
 A^{\bar{\omega}} & \xrightarrow{h_4} & N \\
 \mu_4 \downarrow & & \downarrow \lambda_4 \\
 [0, 1] & \equiv & [0, 1]
 \end{array}$$

Theorem 4.6. Let μ be a fuzzy bi- ω language recognized by a homomorphism $h : (A^+, A^\omega, {}^\omega A, {}^\omega A^\omega) \rightarrow \mathcal{S}$ of semigroups with bi-infinite product. Then there exists a fuzzy bi- ω automata \mathfrak{F} which recognizes the same fuzzy bi- ω language.

Proof: Since h recognize a fuzzy bi- ω language, there exists the following the following commutative fuzzy diagrams

$$\begin{array}{ccc}
 A^+ & \xrightarrow{h_3} & \mathcal{S}_1 \\
 \mu_3 \downarrow & & \downarrow \lambda_3 \\
 [0, 1] & \equiv & [0, 1]
 \end{array}
 \qquad
 \begin{array}{ccc}
 A^\omega & \xrightarrow{h_4} & \mathcal{S}_2 \\
 \mu_4 \downarrow & & \downarrow \lambda_4 \\
 [0, 1] & \equiv & [0, 1]
 \end{array}$$

$$\begin{array}{ccc}
 A^+ & \xrightarrow{h_3} & \mathcal{S}_3 \\
 \mu_3 \downarrow & & \downarrow \lambda_3 \\
 [0, 1] & \equiv & [0, 1]
 \end{array}
 \qquad
 \begin{array}{ccc}
 {}^\omega A & \xrightarrow{h_4} & \mathcal{S}_4 \\
 \mu_4 \downarrow & & \downarrow \lambda_4 \\
 [0, 1] & \equiv & [0, 1]
 \end{array}$$

We construct a fuzzy automaton for each fuzzy diagram. Taking union of these fuzzy automata, we obtain the required fuzzy bi- ω automata \mathfrak{F} to recognize the same fuzzy bi- ω language.

5. Conclusion

This paper defines the notions of fuzzy bi- ω automata and fuzzy bi- ω languages. We have analyzed the relationships among the languages recognized by fuzzy bi- ω automata and fuzzy Buchi recognizable languages. Further, we have proved that the fuzzy bi- ω language is recognizable by fuzzy bi- ω automaton if and only if it is recognizable by homomorphism of semigroups with infinite product.

REFERENCES

1. R.Cokilavany, R.Venkatesan and J.Kavikumar, Fuzzy ω -Automata and its relation -ships, *International Journal of Pure and Applied Mathematics*, 99(1) (2015) 23 - 36.
2. J.Devolder and I.Litovsky, Finitely generated bi- ω Languages, *Theoretical Computer Science*, 85(1) (1991) 33 - 52.
3. K.Krithivasan and K.Sharada, Fuzzy ω -automaton, *Inf. Sci.*, 138 (2001) 257 - 281.
4. W. Kuich and G.Rahonis, Fuzzy regular languages over finite and infinite words, *Fuzzy Sets and Systems*, 157 (2006) 1532 - 1549.
5. M.Nivat and D.Perrin, Ensembles reconnaissables de mots bi-infinis, *Canadian Journal of Mathematics*, 38 (1986) 513 - 537.
6. G.Rahonis, Infinite fuzzy computations, *Fuzzy Sets and Systems*, 153 (2005) 275 - 288.