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A Study on Fuzzy K-Domination Using Strong Arc

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Abstract. Given two fuzzy graphs $G_1 = (\sigma_1, \mu_1)$ and $G_2 = (\sigma_2, \mu_2)$ we consider some parameters of domination such as k-dominating sets, connected-k dominating sets, total – k-dominating sets and composition of G_1 and G_2.

Keywords: *k*-dominating set, Cartesian product, composition

AMS Mathematics Subject Classification (2010): 03E72, 05C72

1. Introduction

The first definition of fuzzy graphs was proposed by Kafmann [3] from the fuzzy relations introduced by Zadeh [14]. Rosenfeld [11] introduced another elaborated definition including fuzzy vertex and fuzzy edges. The concept of domination in fuzzy graphs was introduced by Somasundaram and Somasundaram [12] and Somasundaram [6] developed the concepts of independent domination, total domination, connected domination and domination in Cartesian product and composition of fuzzy graphs. Nagoorgani and Chandrasekaran [8] discussed domination in fuzzy graph using strong arcs. Also Nagoorgani and Vadivel [9] discussed fuzzy independent dominating sets. In this paper we extend domination in Cartesian and composition of fuzzy graphs to another domination parameter such as *k*-domination.

2. Preliminaries

Definition 2.1. [8] Let $G = (\sigma, \mu)$ be a fuzzy graph on V. Let $u, v \in V$. We say that *u* dominates *v* in *G* if (u, v) is a strong arc. A subset D of V is called a dominating set of G if every $v \in V - D$ there exists $u \in D$ such that *u* dominates *v*.

Definition 2.2. [8] The minimum fuzzy cardinality of a dominating set in *G* is called the domination number of *G* and denoted by $\gamma(G)$.

Definition 2.3. [9] Let $G=(\sigma,\mu)$ be a fuzzy graph. Two nodes in a fuzzy graph G are said to be fuzzy independent if there is no strong arc between them. The minimum fuzzy cardinality of an Independent dominating set of G is called the independent domination number of G denoted by $\gamma_i(G)$

Definition 2.4. [9] A fuzzy independent set *S* of *G* is said to be maximal fuzzy independent set if there is no fuzzy independent set whose cardinality is greater than the cardinality of *S*.

Definition 2.5. [9] The maximum cardinality among all maximal fuzzy independent set is called fuzzy independence number of G and is denoted by $\beta(G)$.

Definition 2.6. [9] Let *G* be a fuzzy graph without isolated vertices. A subset *S* of *V* is called a total dominating set of *G* is every vertex in *V* is dominated by a vertex in *S* (or) equivalently *S* is a dominating set of *G* and the induced subgraph *G[S]* has no isolated vertices. The minimum fuzzy cardinality of a total dominating set of *G* is called the total domination number of *G* and denoted by $\gamma_t(G)$.

Definition 2.7. [9] Let *G* be a connected fuzzy graph of *V*. A subset *S* of *V* is called a connected dominating set of *G* if *S* is a dominating set and *G[S]* is connected subgraph of *G*. The minimum fuzzy cardinality of a connected dominating set is called the connected domination number of *G* and is denoted by $\gamma_C(G)$

Definition 2.8. [6] Let $G_1=(\sigma_1,\mu_1)$ and $G_2=(\sigma_2,\mu_2)$ be two fuzzy graphs on V_1 and V_2 respectively. Then the composition of G_1 and G_2, denoted by $G_1\circ G_2$ is the fuzzy graph on $V_1\times V_2$ defined as follows: $G_1\circ G_2=(\sigma_1\circ\sigma_2,\mu_1\circ\mu_2)$

Where $\sigma_1\circ\sigma_2(u_1,u_2)=\sigma_1(u_1)\wedge\sigma_2(u_2)$ and

$$\mu_1\circ\mu_2\left[(u_1,u_2),(v_1,v_2)\right]=\begin{cases}\sigma_1(u_1)\wedge\mu_2(u_2v_2) & if\ u_1=v_1,u_2\neq v_2\\ \sigma_2(u_2)\wedge\sigma_2(v_2)\wedge\mu_1(u_1v_1) & \text{otherwise}\end{cases}$$

Definition 2.9. [6] Let $G_1=(\sigma_1,\mu_1)$ and $G_2=(\sigma_2,\mu_2)$ be two fuzzy graphs on V_1 and V_2 respectively. Then the Cartesian product of G_1 and G_2 denoted by $G_1\square G_2$ is the fuzzy graph on $V_1\times V_2$ defined as follows: $G_1\square G_2=(\sigma_1\square\sigma_2,\mu_1\square\mu_2)$

Where $\sigma_1\square\sigma_2(u_1,u_2)=\sigma_1(u_1)\wedge\sigma_2(u_2)$ and

$$\mu_1\square\mu_2\left[(u_1,u_2),(v_1,v_2)\right]=\begin{cases}\sigma_1(u_1)\wedge\mu_2(u_2v_2) & if\ u_1=v_1\\ \sigma_2(u_2)\wedge\mu_1(u_1v_1) & if\ u_2=v_2\\ 0 & \text{otherwise}\end{cases}$$

Definition 2.10. [6] Let $K\geq 1$ be an integer. Let $G=(V,E)$ be a graph. A set *D* of vertices of *G* is defined to be a total –*k* dominating set of *G* if every vertex in *V* is within distance-*k* from some vertex of D other than itself.

3. Main results

Theorem 3.1. Let G_1, G_2 be connected and D_{K_1}, D_{K_2} be k-dominating sets of G_1 and G_2 respectively. Then $G_1 \square G_2$ is connected and we have

i) If D_{k_1} is connected then $D_{k_1} \times V_2$ is a connected *k*-dominating set of $G_1 \square G_2$.

ii) If D_{k_2} is connected then $V_1 \times D_2$ is a connected *k*-dominating set of $G_1 \square G_2$.

Proof: First we prove that $G_1 \square G_2$ is connected. We consider two arbitrary distinct vertices of $V_1 \times V_2$ such as (x_i, y_j) and (x_l, y_k)

To show: There exists a path between these two vertices in the following 3 cases.

Case (i) $x_i = x_l$

There exists a path $p: y_j, y_{i_1}, y_{i_2}, ..., y_k$ Connectivity of $G_2, \mu_2(uv) > 0$ for each two vertices u, v of path p.

➢ $\mu[(x_i, u), (x_i, v)] = \sigma_1(x_i) \wedge \mu_2(uv) > 0$

Therefore $p': (x_i, y_j)(x_i, y_{i1})(x_i, y_{i2})...(x_i, y_{ik})$ is the path between (x_i, y_j) and (x_l, y_k) in $G_1 \square G_2$

Case (ii) $y_j = y_k$

Now consider a path $q: x_i, x_{ji}, x_{j_2}...x_l$

➢ $\mu[(y_j, u), (y_j, v)] = \sigma_1(y_j) \wedge \mu(uv) > 0$

$q': (x, y_j)(x_{j_1}, y_j)(x_{j_2}, y_j)...(x_l, y_j)$ is the path between (x_i, y_j) and (y_k, y_j)

Case (iii) $x_i \neq x_l, y_j \neq y_k$

$\therefore$ There exists a path $(x_i, y_j)(x_i, y_k)$ by (1) and (x_i, y_k) to (x_l, y_k) by case (4)

The union of this two disjoint path is a path between (x_i, y_j) and (x_1, y_k)

Also if D_1 and D_2 are minimum dominating sets of

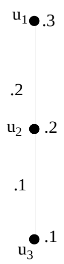
$G_1 = (\sigma_1, \mu_1)$ and $G_2 = (\sigma_2, \mu_2)$ then $\gamma(G_1 \square G_2) \leq \min\{|D_1 \times V_2|_s |V_1 \times D_2|_s\}$

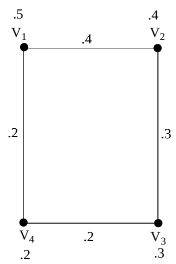
$\therefore D_1 \times V_2$ and $V_1 \times D_2$ are k-dominating sets and the connectivity of them is similarly proved.

Definition 3.1. [5] A set S of vertices of a graph *G* is defined to be k-independent in *G* if every vertex of *S* is at distance atleast *k+1* from every other vertex of *S* in *G* denoted by $i_k(G)$.

Theorem 3.2. Let D_{k_1} and D_{k_2} be k-dominating sets of the fuzzy graphs $G_1 = (\sigma_1, \mu_1)$ and $G_2 = (\sigma_2, \mu_2)$ respectively. Then $D_{k_1} \times D_{k_2}$ is a *k*-dominating set of $G_1 \circ G_2$.

Example 3.1. ***G₁*** ***G₂***

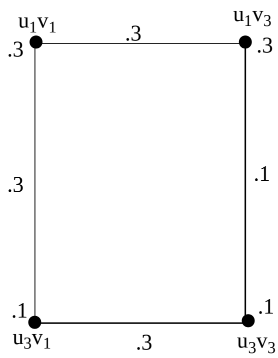




$$D_1 = \{u_2\}$$
$$D_{K_1} = \{v_1, u_3\}$$
$$\sigma_1(u_1) = .3$$
$$\sigma_1(u_3) = .1$$

$$D_2 = \{V_2, V_4\}$$
$$D_{K_2} = \{V_1, V_3\}$$
$$\sigma_2(v_1) = .5$$
$$\sigma_2(v_3) = .3$$

$D_{K_1} \times D_{K_2}$:



$$(u_1, v_3)(u_3, v_3) = \min\left[\sigma_2(v_3) \wedge \mu(u_1, u_3)\right] = \min\{.3 \wedge .1\} = 0.1.$$
$$(u_3, v_3)(u_3, v_1) = \min\left[\sigma_1(u_3) \wedge \mu(v_3, v_1)\right] = \min\{.1 \wedge .3\} = 0.1.$$
$$(u_3 v_1)(u_1, v_1) = \min\left[\sigma_2(v_1) \wedge \mu(u_3 u_1)\right] = \min\{.5 \wedge .1\} = 0.1.$$

Theorem 3.3. Suppose that G_1 has no isolated vertex and D_{k_1} is a total k-dominating set of *G₁*. Then $G_1 \square G_2$ has no isolated vertex and $D_{k_1} \times V_2$ is a total –*k* dominating set of $G_1 \square G_2$.

Proof: For proving $D_{k_1} \times V_2$ is a total –k dominating set of $G_1 \square G_2$.

First let us prove that any vertex from $V_1 \times V_2$ is not an isolated vertex.

Let (u,v) be an arbitrary vertex in $V_1 \times V_2$.

Then there exists a vertex x in D_{k_1} such that $u \in N(x)$

Now $\mu\left[(u,v)(x,v)\right] = \mu_1(ux) \wedge \sigma_2(v)$

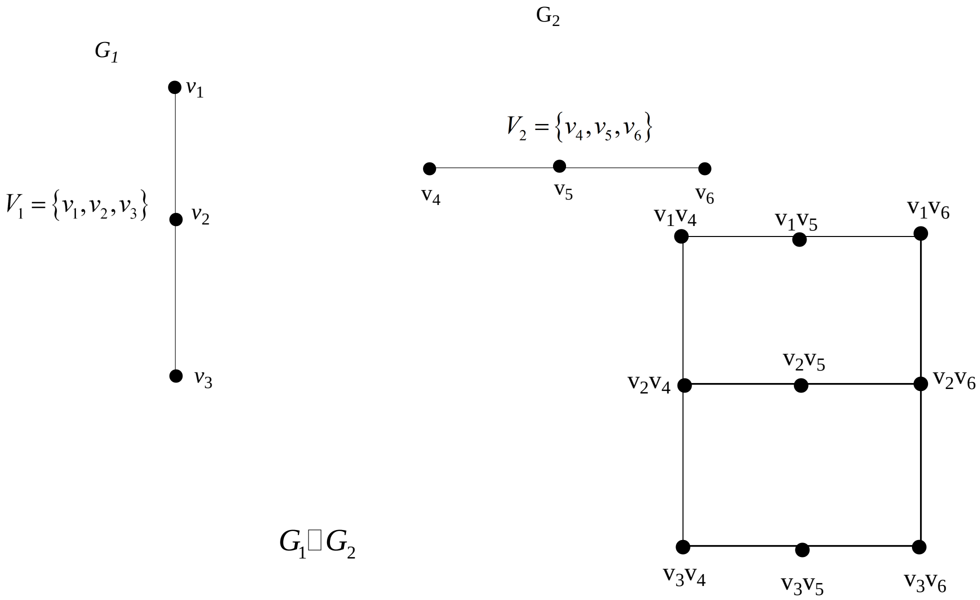
$$= \sigma_1(u) \wedge \sigma_1(x) \wedge \sigma_2(v)$$

$$= \sigma(u,v) \wedge \sigma(x,v)$$

$\Rightarrow (u,v) \in N\left[(x,v)\right]$

$\therefore (u,v)$ is not an isolated vertex and since $(x,v) \in D_{K_1} \times V_2, D_{K_1} \times V_2$ is a total –k dominating set of $G_1 \square G_2$. Hence the proof.

Remark 3.1. Under the similar conditions $V_1 \times D_{K_2}$ is a total –k dominating set of $G_1 \square G_2$.



$D_{k_1} = \{v_2\}$ $\qquad V_2 = \{v_4, v_5, v_6\}$ total –k domination set of $G_1 \square G_2$.

$\therefore D_{k_1} \times v_2 = \{v_2v_4, v_2v_5, v_2v_6\}$ is a

Theorem 3.4. Let D_{k_1} and D_{k_2} be k-dominating sets of G_1 and G_2 respectively.

a. $D_{k_1} \times V_2$ is an independent k-dominating set of $G_1 \square G_2$ if and only if D_{K_1} is k-independent and

$$\mu_1(uv) < \sigma_2(w) \quad u,v \in D_{K_1} \; w \in V_2 \tag{1}$$

$$\left.\begin{array}{l} \text{i) } \mu_2(wz) < \sigma_1(u) u \in D_{K_1}, \quad w,z \in V_2 \\ \text{ii) } \mu_2(wz) < \sigma_2(w) \wedge \sigma_2(z) \quad w,z \in V_2 \end{array}\right\} \tag{2}$$

b. $V_1 \times D_{K_2}$ is an independent k-dominating set of $G_1 \square G_2$ if and only if D_{K_2} is k-independent and

i) $\mu_1(uv) < \sigma_2(w) \qquad u, v \in V_1,\ w \in D_{K_2}$ (3)

ii) $\mu_1(uv) < \sigma_1(u) \wedge \sigma_1(v), \quad u, v \in V_1,$ (4)

$\mu_2(wz) < \sigma_1(u) \qquad u \in V_1, w, z \in D_{K_2}$

Proof: Sufficiency

We show that every two distinct vertices of $D_{K_1} \times V_2$ such as $(x_1, y_1)(x_2, y_2)$ are not adjacent. If $x_1 = x_2$ then by 2(i) and 2(ii)

$$\begin{aligned}\mu\left[(x_1, y_1)(x_2, y_2)\right] &= \sigma_1(x_1) \wedge \mu_2(y_1, y_2)\\ &= \mu_2(y_1, y_2)\\ &< \sigma_2(y_1) \wedge \sigma_2(y_2) \wedge \sigma_1(x_1)\\ &= \sigma(x_1, y_1) \wedge \sigma(x_1, y_2)\end{aligned}$$

If $y_1 = y_2$ the result will be obtained by independence (x_1, y_1) of D_1 and inequality (1).

If $x_1 \neq x_2,\ y_1 \neq y_2$ then by definition we have $\mu(x_1, y_1)(x_2, y_2) = 0$

Hence $(x_1, y_1)(x_2, y_2)$ cannot be adjacent.

Similarly the necessary part

Suppose (2) (ii) is false, i.e., there exists $w, z \in V_2$ such that $\mu_2(wz) = \sigma_2(w) \wedge \sigma_2(z)$

If *u* is any vertex of D_1,

$$\begin{aligned}\mu\left[(u, w)(u, z)\right] &= \sigma_1(u) \wedge \mu_2(wz)\\ &= \sigma(u) \wedge \sigma_2(w) \wedge \sigma_2(z)\\ &= \sigma(u, w) \wedge \sigma(u, z)\end{aligned}$$

- $D_{K_1} \times V_2$ is not independent.

$\therefore$ Our assumption is wrong hence 2 (ii) is true

i.e., $\mu_2(wz) < \sigma_2(w) \wedge \sigma_2(z), w, z \in V_2$. Similarly we can prove (b).

Theorem 3.5. Let G_1 be a connected fuzzy graph, D_1 be a connected *k*-dominating set of G_1 and D_2 be a *k*-dominating set of G_2. Then $G_1 \circ G_2$ is connected and $D_1 \times D_2$ is a connected *k*-dominating set of it.

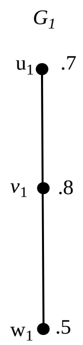
Proof: Suppose that $(u, v), (w, z) \in V_1 \times V_2$

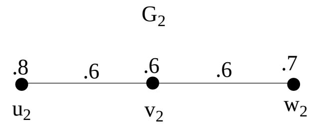
Since G_1 is connected, there exists a path $p : u, x_1, x_2, \ldots, x_{n-1}, w$ in G,

- $p' : (u, v), (x_1, v), (x_2, v) \ldots (x_{n-1}, v)(w, z)$ is a $(u, v) - (w, z)$ path in $G_1 \circ G_2$.

Since $G_1 \circ G_2$ is connected $D_1 \times D_2$ is also connected.

Example 3.2.

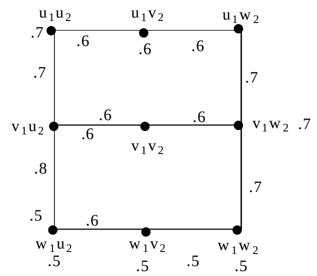




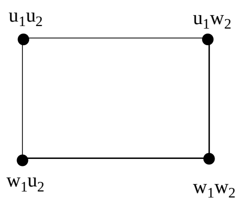
$D_{k_2} = (u_2, w_2)$

$D_{k_1} = \{u_1, w_1\}$

$G_1 \square G_2$



$$D_{K_1} \times D_{K_2} = \{u_1u_2, u_1w_2, w_1u_2, w_1w_2\}$$

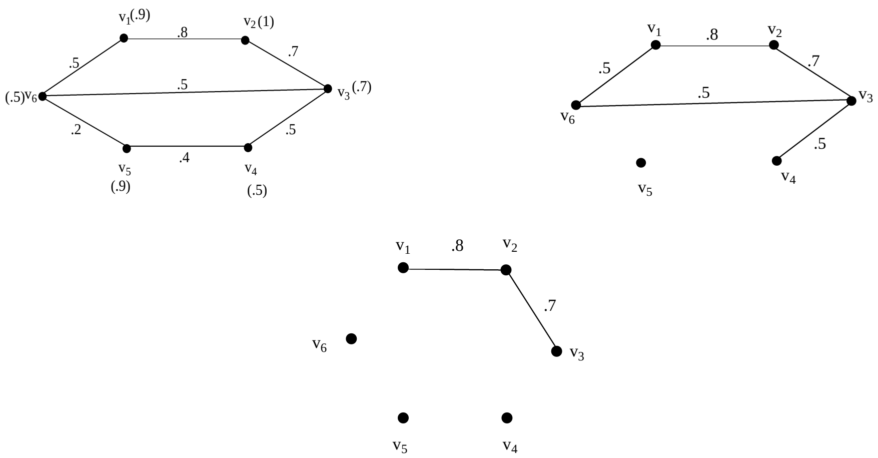


Algorithm 3.1. Algorithm to find a k-dominating set

1. Find $\mu^{\infty}(u,v)$ for all edges (u,v)
2. Delete all the weak edges (G')
3. Select the vertex *u* with maximum δ -edges in G'
4. Group the vertices k-dominated by *u* as V_1
5. Find $G' - V = D_k$
6. Repeat the steps from 3 to 5 until we get isolated vertices.
7. Now the vertices which are selected from step 3 and isolated will form a k-dominating set.

Example 3.3.

G G'



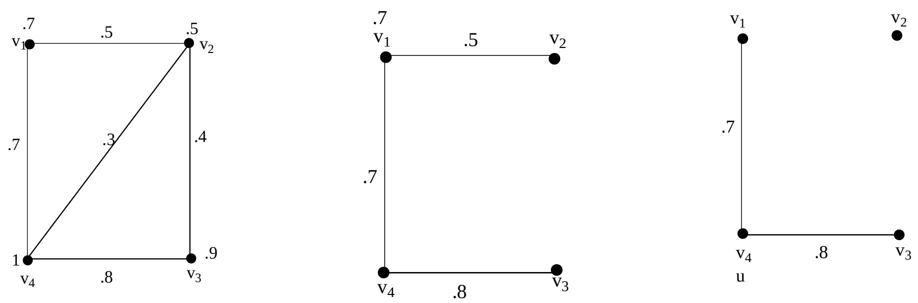
$V_1=\{v_1,v_3\}$

$G'-V_1=\{v_2,v_4,v_5,v_6\}=D_K, V-D_K=(v_1,v_3)$ is within distance-k of

At least one vertex in D_K.

Example 3.4.

G G'



$V_1=\{v_1,v_3\}$ $G'-V_1=\{v_2,v_4\}=D_k$

$V-D_k=\{v_1,v_3\}$ is within distance-k of at least one vertex in D_k.

Result 3.1. For each $v\in D$, there is no vertex in *D* dominates *v*.

Proposition 3.1. For each $v\in D_K$ there is a vertex in D_K such that *k*-dominates *v*.

Proof: Suppose there exists no vertex in D_K, k-dominates v.

This implies there is only one edge between the two vertices and that must be a strong arc.

Therefore, each $v \in D_K$ there is a vertex in D_K dominates v, which is a contradiction to the result.
Hence for each $v \in D_K$ there is a vertex in D_K, k dominates *v*.

Proposition 3.2. Let D be a k-dominating set of a fuzzy graph G then no bridge exist between any two vertices of $V - D_K$.

Proof: Suppose there exists a bridge between any two vertices of $V - D_K$.
i.e., $u, v \in V - D_K$ this implies, (u, v) must be a strong arc

- Either *u* dominates *v* (or) *v* dominates *u.*

But by result (1) there is no vertex in D dominates v.
Hence bridge cannot exist between any two vertices of $V - D_k$.

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