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Edge Chromatic 5-Critical Graphs

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Abstract. In this paper, we study the structure of 5-critical graphs in terms of their size. In particular, we have obtained bounds for the number of major vertices in several classes of 5-critical graphs, that are stronger than the existing bounds.

Keywords: Class one, class two, edge colouring, edge chromatic critical graphs

AMS Mathematics Subject Classification (2010): 94C15

1. Introduction

An edge-colouring of a graph is an assignment of colours to its edges so that no two adjacent edges are assigned the same colour. The chromatic index of a graph G, denoted by χ'(G), is the minimum number of colours used among all colourings of G. If a vertex v of a graph G is of maximum degree Δ(G), then v is called a major vertex; otherwise a minor vertex. If χ'(G) = Δ(G), G is said to be of class one; otherwise G is said to be of class two. A graph G is said to be edge chromatic critical if G is connected, of class 2 and χ'(G–e)<χ'(G) for every edge e of G. If G is critical and Δ(G)=Δ, then G is said to be Δ - critical. In this paper, all graphs are finite, simple, undirected and have n vertices and m edges.

2. Notations

Let G = (V, E) be a graph with n vertices, m edges, maximum degree Δ(G) and minimum degree δ(G). The degree of a vertex v in G is denoted by d(v). The number of vertices of degree j is denoted by n_j. N(x) denotes the set of all vertices adjacent to x in G and N(x, y) =N(x) $\cup$ N(y). If S and T are disjoint subsets of V(G), then [S , T] denotes the set of all edges with one end in S and the other in T. [S] denotes the subgraph of G induced by S and |[S]| denotes |E[S]|. $\pi(G) = 1^{n_1} 2^{n_2} \ldots \Delta^{n_\Delta}$ denotes the degree sequence of G, where if n_j = 0, then the factor j^{n_j} is customarily omitted in $\pi(G)$.

3. Known results

R1 :Vizing's Adjacency Lemma[10] (VAL): In a Δ- critical graph G, if vw is an edge and d(v) = k, then w is adjacent with at least Δ– k + 1 other vertices of degree Δ. In particular, G has at least $\Delta - \delta + 2$ vertices of degree Δ.

R2: Fiorini'sInequality[4]: Let $\Delta \geq 3$ be an integer. If G is a Δ-critical graph, then $n_\Delta \geq 2\sum_{j=2}^{n-1} \frac{n_j}{j-1}$.

R3: Vizing'sConjecture[11]: If G is a Δ- critical simple graph with n vertices, m edgesand $\Delta \geq 3$, then m $\geq \frac{1}{2}[(\Delta-1)n+3]$.

This conjecture has been verified for the graphs with $\Delta \leq 6$ [5, 6, 7, 8].

R4: (Fiorini[4]) If G is a Δ- critical graph, then m $\geq \frac{1}{4}|V|(\Delta+1)$.

The best known bounds are still far from the conjectured bound.

R5:(Zhang [12]) Let G be a Δ- critical graph, xy$\in$ E(G) and d(x) + d(y) = Δ+ 2. Thenthe following hold:

(i) every vertex of N(x , y) \ {x , y} is a major vertex;
(ii) every vertex of N(N(x , y)) \ {x , y} is at least of degree Δ - 1;
(iii) If d(x) , d(y) < Δ, then every vertex of N(N(x , y)) \ {x , y} is a major vertex.

R6: There are no critical graphs of order 4,6,8. [1, 2]

R7: A 2-connected graph of order 7 is critical if and only if its degree sequence is one among the following: 2^7, 23^6, 24^6, $3^2 4^5$, 25^6, 345^5, $4^3 5^4$, $45^2 6^4$, $5^4 6^3$. [2, 3]

3.1. Main theorems

Throughout this section, we use the following notations:

M denotes the set of all major vertices of G and N_2, N_3, N_4 are the sets of all vertices of degree two, three and four respectively. M_3 denotes the set of all major vertices which are adjacent with at least one 3-vertex, but not with any 4-vertex, $M_4$is the set of all major vertices which are adjacent with at least one 4-vertex, but not with any 3-vertex and M_{34} is the set of all major vertices which are adjacent with vertices of degree 3 and 4. Then, M_3 , M_4 andM_{34} are pairwise disjoint. N_3'denotes the set of all 3-vertices which are adjacent with 4-vertices, $N_4'$is the set of all 4-vertices which are adjacent with 3-vertices and $N_4''$is the set of all 4-vertices which are adjacent with at least one 4-vertex. By VAL, each 3-vertex can be adjacent with at most one 4-vertex and vice versa.Hence$|N_3'| = |N_4'|$. Also, let $|M_3| = l'$, $|M_{34}| = m'$, $|M_4| = n'$, $|N_3'| = |N_4'|$ = s, $|N_4''|$ = t and $|[N_4]|$ = r.

Lemma 1. If G is a 5-critical graph, then the following hold:
(i)$|[N_3, N_4]|$=s; (ii) $|[M_{34}, N_3]| = |[M_{34}, N_4]| = m'$;(iii) $|[N_4']|$ = 0 and $N_4' \cap N_4'' = \phi$.

Proof: (i) Since $|N_3'|$ = s, N_3has s 3-vertices which are adjacent with 4-vertices. By VAL, each 3-vertex can be adjacent with at most one 4-vertex. Hence $|[N_3, N_4]|$ = s.

(ii) Each vertex in M_{34} is adjacent with a 3-vertex and a 4-vertex. Hence, by VAL, each vertex in M_{34} has exactly three major neighbours. Hence each vertex in M_{34} is adjacent with exactly one 3-vertex and one 4-vertex.

Thus $|[M_{34}, N_3]| = |[M_{34}, N_4]| = |M_{34}| = m'$.

(iii) Since each vertex in N_4' is adjacent with a 3-vertex, by VAL, each vertex in N_4' has exactly three major neighbours. Hence $|[N_4']| = 0$. Since each vertex in N_4' is adjacent with exactly one 3-vertex and three major vertices, each vertex in N_4' is not adjacent with any 4-vertex. Hence $N_4' \cap N_4'' = \phi$.

In the following theorems, we have obtained bounds for the number of major vertices in several classes of 5-critical graphs, that are stronger than the existing bound given in R2. Also, we have obtained new bounds on size m in terms of order n and minor vertices.

Theorem 3.1.1. Let G be a 5-critical graph. Suppose that every major vertex adjacent with a 4-vertex is adjacent with a 3-vertex.

Then (i) $n_5 \geq 2n_2 + \frac{3}{2}n_3 + n_4$ and (ii) $m \geq 2n + \frac{n_3}{4} + \frac{n_4}{2}$.

Proof: By the Lemma, $|[M_{34}, N_3]| = |[M_{34}, N_4]| = |M_{34}| = m'$ and $|[N_3, N_4]|$=s.

Then, by VAL, $|[N_3, M]| = (n_3 - s)(3) + 2s = 3n_3 - s$.

Now $|[N_3, M_3]| = |[N_3, M]| - |[N_3, M_{34}]| = 3n_3 - s - |[N_4, M_{34}]| = 3n_3 - s - m'$.

Also by VAL, $|[M_3, N_3]| \leq 2l'$. Thus $l' \geq \frac{3n_3 - s - m'}{2}$. By hypothesis, $M_4 = \phi$.

Thus $m' = |[N_4, M_{34}]| = |[N_4, M]| = 4n_4 - |[N_4, N_3]| - 2|[N_4]| = 4n_4$ -s -2r.

Hence $l' + m' \geq \frac{3}{2}n_3 - \frac{s}{2} + 2n_4 - \frac{s}{2} - r$. Thus $n_5 \geq 2n_2 + l' + m' \geq 2n_2 + \frac{3}{2}n_3 + 2n_4 - s - r$. (1)

By VAL, every 4-vertex can be adjacent with at most two 4-vertices.

Hence $2r = \sum_{v \in N_4} d_{N_4}(v) \leq$ 2t and so r $\leq$ t. Then s + r $\leq$ s + t $\leq n_4$. Hence using (1),

$n_5 \geq 2n_2 + \frac{3}{2}n_3 + n_4$. Now $2m = 2n_2 + 3n_3 + 4n_4 + 5n_5 \geq 4n - 2n_2 - n_3 + 2n_2 + \frac{3}{2}n_3 + n_4$.

Thus $m \geq 2n + \frac{n_3}{4} + \frac{n_4}{2}$.

Theorem 3.1.2. If G is a 5-critical graph in which every 3-vertex is adjacent with a 4-vertex, then (i) $n_5 \geq 2n_2 + \frac{14}{5}n_3 + \frac{2}{3}n_4$ and (ii) $m \geq 2n + \frac{9}{10}n_3 + \frac{n_4}{3}$.

Proof: Let u $\in M_3$ be adjacent with y $\in N_3$. By hypothesis, y is adjacent with a 4-vertex,say x. Then d(x) + d(y) = 7 = Δ + 2 and so by R5, all the other neighbours of u are major vertices. Thus, every vertex u $\in M_3$ is incident with an unique edge in

$[M_3, N_3]$ and so $[M_3, N_3] = |M_3| = l'$. (1)

Moreover, every u$\in M_3$ has four major neighbours. By hypothesis and VAL, $|N_4'| = n_3$. By the Lemma, $\|[N_4']\|=0$. Let $\|[N_4', M_4]\| = t'$.Again by using R5, these edges are incident with t' vertices in M_4, each of which has four major neighbours. Thus, by VAL,

$$\sum_{v\in M} d_M(v) \geq 4(2n_2)+4l'+3m'+4t'+(n'-t')(2) = 8n_2 + 4l' + 3m' + 2n' + 2t' \tag{2}$$

By the Lemma and hypothesis, $[N_3, N_4] = n_3$.

Now $|[N_3, M]| = 3n_3 - |[N_3, N_4]| = 2n_3$.

Then $\sum_{v\in M} d_M(v) = 5n_5 - |[N_2 \cup N_3 \cup N_4, M]| = 5n_5 - [2n_2 + 2n_3 + 4n_4 - n_3 - 2r]$

$$= 5n_5 - 2n_2 - n_3 - 4n_4 + 2r \tag{3}$$

By (2) and (3), $5n_5 - 2n_2 - n_3 - 4n_4 + 2r \geq 8n_2 + 4l' + 3m' + 2n' + 2t'$. (4)

Now, by the Lemma, $|[M_{34}, N_4]| = |[M_{34}, N_3]| = |M_{34}| = m'$ and so $|[N_3, M_3]| = |[N_3, M]| - |[N_3, M_{34}]| = 2n_3 - m'$. Thus, by (1), $l' = 2n_3 - m'$. (5)

Also $|[N_4, M]| = 4n_4 - |[N_4, N_3]| - 2|[N_4]| = 4n_4 - n_3 - 2r$.

Hence $|[N_4, M_4]| = |[N_4, M]| - |[N_4, M_{34}]| = 4n_4 - n_3 - 2r - m'$. (6)

Also, by VAL, $\|[M_4, N_4]\| \leq 3|M_4|$ and so, using (6), $n' \geq \dfrac{4n_4 - n_3 - 2r - m'}{3}$. (7)

Using (4), (5) and (7),

$$5n_5 - 2n_2 - n_3 - 4n_4 + 2r \geq 8n_2 + 4(2n_3 - m') + 3m' + 2\left(\frac{4n_4 - n_3 - 2r - m'}{3}\right) + 2t'.$$

Thus $$5n_5 \geq 10n_2 + \frac{25}{3}n_3 + \frac{20}{3}n_4 - \frac{5}{3}m' + 2t' - \frac{10}{3}r. \tag{8}$$

By the Lemma, $N_4'' \cap N_4' = \phi$. Now $|N_4| \geq |N_4'| + |N_4''|$ and so $n_4 \geq n_3 + t$. Also, by VAL, every 4-vertex can be adjacent with at most two 4-vertices.

Hence $2r = \sum_{v\in N_4} d_{N_4}(v) \leq 2t$ and so r $\leq$ t.Thus r$\leq n_4 - n_3$. Also, by VAL, $m' \leq n_3$.

Thus, using (8), we have $$n_5 \geq 2n_2 + 2n_3 + \frac{2}{3}n_4 + \frac{2}{5}t'. \tag{9}$$

Suppose that vw is an edge, where $v \in M_{34}$ and $w \in N_3$. Then, by hypothesis, w is adjacent with a 4-vertex, say z $\in N_4'$. Then, d(w) + d(z) = Δ +2. By R5, any vertex of G at distance two from w is a major vertex in G. But, by hypothesis, v is adjacent with a 4-vertex. Hence z is the 4-vertex adjacent with v .Hence $\|[M_{34}, N_4']\| = |M_{34}| = m' \leq n_3$.

Now $t' = |[N_4', M_4]| = 4|N_4'| - |[N_4', N_3]| - |[N_4', M_{34}]| \geq 4n_3 - n_3 - n_3 = 2n_3$.

Hence, using (9), $n_5 \geq 2n_2 + \frac{14}{5}n_3 + \frac{2}{3}n_4$. Then,

$2m = 2n_2 + 3n_3 + 4n_4 + 5n_5 = 4n - 2n_2 - n_3 + n_5 \geq 4n + \frac{9}{5}n_3 + \frac{2}{3}n_4$. Thus $m \geq 2n + \frac{9}{10}n_3 + \frac{n_4}{3}$.

Corollary 3.1.3. Let G be a 5-critical graph in which each 3-vertex is adjacent with a 4-vertex and no two 4-vertices are adjacent. Then

$$\text{(i) } n_5 \geq 2n_2 + \frac{32}{15}n_3 + \frac{4}{3}n_4 \quad \text{and} \quad \text{(ii) } m \geq 2n + \frac{17}{30}n_3 + \frac{2}{3}n_4.$$

Proof: Using the same notations and proceeding as in Theorem 3.1.2, we have r = 0, $m' \leq n_3$ and $t' \geq 2n_3$. Then equation(8) in Theorem 3.1.2 becomes

$5n_5 \geq 10n_2 + \frac{25}{3}n_3 + \frac{20}{3}n_4 - \frac{5}{3}n_3 + 4n_3$. Hence $n_5 \geq 2n_2 + \frac{32}{15}n_3 + \frac{4}{3}n_4$.

Then $2m = 2n_2 + 3n_3 + 4n_4 + 5n_5 = 4n - 2n_2 - n_3 + n_5 \geq 4n + \frac{17}{15}n_3 + \frac{4}{3}n_4$

and so $m \geq 2n + \frac{17}{30}n_3 + \frac{2}{3}n_4$.

Theorem 3.1.4. If G is a 5-critical graph in which every 3-vertex is adjacent with a 4-vertex and each major vertex is adjacent with at most two 4-vertices, then

$$\text{(i) } n_5 \geq 2n_2 + \frac{14}{5}n_3 + \frac{4}{5}n_4 \quad \text{and} \quad \text{(ii) } m \geq 2n + \frac{9}{10}n_3 + \frac{2}{5}n_4.$$

Proof: Using the same notations and proceeding as in Theorem 3.1.2, we have

$5n_5 - 2n_2 - n_3 - 4n_4 + 2r \geq 8n_2 + 4l' + 3m' + 2n' + 2t'$ (1)

$m' \leq n_3$, $t' \geq 2n_3$, $r \leq n_4 - n_3$, $l' = 2n_3 - m'$ (2)

and $|[N_4, M_4]| = 4n_4 - n_3 - 2r - m'$. (3)

Since each major vertex is adjacent with atmost two 4-vertices, $|[M_4, N_4]| \leq 2|M_4|$.

Hence, by (3), $n' \geq \frac{4n_4 - n_3 - 2r - m'}{2}$. (4)

Using (1), (2) and (4), we have $n_5 \geq 2n_2 + \frac{14}{5}n_3 + \frac{4}{5}n_4$.

Then $2m = 2n_2 + 3n_3 + 4n_4 + 5n_5 \geq 4n - 2n_2 - n_3 + 2n_2 + \frac{14}{5}n_3 + \frac{4}{5}n_4$.

Thus $m \geq 2n + \frac{9}{10}n_3 + \frac{2}{5}n_4$.

Theorem 3.1.5. Let G be a 5-critical graph. Suppose that every major vertex adjacent with a 3-vertex is adjacent with a 4-vertex. Then

$$\text{(i) } n_5 \geq 2n_2 + 2n_3 + \frac{2}{3}n_4 + \frac{s}{3} \text{ and (ii) } m \geq 2n + \frac{1}{2}n_3 + \frac{1}{3}n_4 + \frac{s}{6}.$$

Proof: By the Lemma, $|[M_{34}, N_3]| = |[M_{34}, N_4]| = m'$ and $[N_3, N_4]$=s. (1)

By hypothesis, $M_3 = \phi$.

Then $m' = |[N_3, M_{34}]| = |[N_3, M]| = 3n_3 - |[N_3, N_4]| = 3n_3 - s$. (2)

Suppose that yu is an edge, where y $\in N_3'$ and u$\in M_{34}$. Then, y is adjacent with a 4-vertex, say x in N_4'. Then d(x) + d(y) = 7 = Δ + 2. By R5, any vertex of G at distance two from y is a major vertex in G. But, u is adjacent with a 4-vertex. Hence x is the 4-vertex adjacent with u. Thus, corresponding to each edge from N_3' to M_{34}, there is an edge from M_{34} to N_4' and vice versa. Thus $|[M_{34}, N_4']| = |[N_3', M_{34}]| = 2s$. By the Lemma, $|[N_4']| = 0$.Also, $|[N_4', N_3]| = |[N_4, N_3]| = s$.

Thus $|[N_4', M_4]| = 4|N_4'| - |[N_4', N_3]| - |[N_4', M_{34}]| = 4s - s - 2s = s$.

By R5, these s edges are incident with s vertices in M_4, each of which has four major neighbours. Let M_4' be the set of these s major vertices in M_4. Thus $M_4' \subseteq M_4$ and

$|[N_4, M_4']| = s$. (3)Also $|M_4 - M_4'| = n' - s$.

Now $|[N_4, M_4 - M_4']| = 4n_4 - |N_4, N_3]| - 2|[N_4]| - |[N_4, M_{34}]| - |[N_4, M_4']|$

and by using (1) and (3), $|[N_4, M_4 - M_4']| = 4n_4 - s - 2r - m' - s$. By VAL, every 4-vertex can be adjacent with at most two 4-vertices. Hence $2r = \sum_{v\in N_4} d_{N_4}(v) \le 2t$ and so $r \le t$.

Hence, $s + r \le s + t \le n_4$. Hence $|[N_4, M_4 - M_4']| \ge 2n_4 - m'$.

By VAL, $|[M_4 - M_4', N_4]| \le 3(n' - s)$. Hence $n' - s \ge \frac{2n_4 - m'}{3}$. (4)

Then, using(2)and(4), $m' + n' \ge 2n_3 + \frac{2}{3}n_4 + \frac{s}{3}$.

Thus $n_5 \ge 2n_2 + m' + n' \ge 2n_2 + 2n_3 + \frac{2}{3}n_4 + \frac{s}{3}$.

Then $2m = 2n_2 + 3n_3 + 4n_4 + 5n_5 = 4n - 2n_2 - n_3 + n_5 \ge 4n + n_3 + \frac{2}{3}n_4 + \frac{s}{3}$.

Thus $m \ge 2n + \frac{1}{2}n_3 + \frac{1}{3}n_4 + \frac{s}{6}$.

Theorem 3.1.6. Let G be a 5-critical graph. Suppose that every major vertex adjacent with a 3-vertex is adjacent with a 4-vertex and each major vertex is adjacent with at most two 4-vertices. Then (i) $n_5 \ge 2n_2 + \frac{3}{2}n_3 + n_4 + \frac{s}{2}$ and (ii) $m \ge 2n + \frac{n_3}{4} + \frac{n_4}{2} + \frac{s}{4}$.

Proof: Using the same notations and proceeding in theorem 3.1.5, we have $m' = 3n_3 - s$, $r + s \le n_4$, $|M_4 - M_4'| = n' - s$ and $|[N_4, M_4 - M_4']| \ge 2n_4 - m'$. Since each major vertex is adjacent with at most two 4-vertices, $|[M_4 - M_4', N_4]| \le 2(n' - s)$.

Hence $n' - s \ge \frac{2n_4 - m'}{2}$.Thus $m' + n' \ge \frac{3}{2}n_3 + n_4 + \frac{s}{2}$.

Hence $n_5 \ge 2n_2 + m' + n' \ge 2n_2 + \frac{3}{2}n_3 + n_4 + \frac{s}{2}$.

Then, $2m = 2n_2 + 3n_3 + 4n_4 + 5n_5 = 4n - 2n_2 - n_3 + n_5 \geq 4n + \frac{n_3}{2} + n_4 + \frac{s}{2}$.

Hence $m \geq 2n + \frac{n_3}{4} + \frac{n_4}{2} + \frac{s}{4}$.

Theorem 3.1.7. If G is a 5-critical graph in which every 4-vertex is adjacent with a 3-vertex, then (i) $n_5 \geq 2n_2 + \frac{3}{2}n_3 + \frac{11}{5}n_4$ and (ii) $m \geq 2n + \frac{1}{4}n_3 + \frac{11}{10}n_4$.

Proof: Since each 4-vertex is adjacent with a 3-vertex, by VAL, each 4-vertex has 3 major neighbours and $[N_3, N_4] = n_4$ .Hence $|[N_4]| = 0$.

By the Lemma, $|[M_{34}, N_3]| = |[M_{34}, N_4]| = m'$. (1)

Now, $|[N_3, M_3]| = 3n_3 - |[N_3, N_4]| - |[N_3, M_{34}]| = 3n_3 - n_4 - m'$.

By VAL, $|[M_3, N_3]| \leq 2l'$ and so $l' \geq \frac{|[M_3, N_3]|}{2} = \frac{3n_3 - n_4 - m'}{2}$. (2)

Suppose that yu is an edge, where y $\in N_4$ and u$\in M_4$. By hypothesis, y is adjacent with a 3-vertex, say x. Then d(x) + d(y) = 7 = Δ + 2. By R5, any vertex of G at distance two from y is a major vertex in G. Hence, all the other neighbours of u are major vertices. Thus, every vertex u $\in M_4$ is incident with an unique edge in $[N_4, M_4]$, and so $|[N_4, M_4]| = |M_4| = n'$. Moreover, each vertex of M_4 has 4 major neighbours.

Hence $n' = |[N_4, M_4]| = 4n_4 - |[N_4, N_3]| - |[N_4, M_{34}]|$.

Then, by (1), $n' = 3n_4 - m'$. (3)

Also, $|N_3'| = |N_4'| = n_4$ and $n_3 \geq n_4$. Let $|[N_3', M_3]| = s'$. Again by R5, these s' edges are incident with s' vertices of M_3, each of which has four major neighbours.

Hence $\sum_{v \in M} d_M(v) \geq 4(2n_2) + 4s' + (l' - s')(3) + 3m' + 4n' = 8n_2 + 3l' + 3m' + 4n' + s'$. (4)

Also $\sum_{v \in M} d_M(v) = 5n_5 - |[N_2 \cup N_3 \cup N_4, M]| = 5n_5 - [2n_2 + 3n_3 - n_4 + 4n_4 - n_4]$

$= 5n_5 - 2n_2 - 3n_3 - 2n_4$. (5)

Using (4) and (5), $5n_5 - 2n_2 - 3n_3 - 2n_4 \geq 8n_2 + 3l' + 3m' + 4n' + s'$

$= 8n_2 + \frac{9}{2}n_3 + \frac{21}{2}n_4 - \frac{5}{2}m' + s'$, using (2) and (3).

Hence $n_5 \geq 2n_2 + \frac{3}{2}n_3 + \frac{5}{2}n_4 - \frac{m'}{2} + \frac{s'}{5}$. Each vertex in M_{34} is adjacent with a 4-vertex and a 3-vertex. Hence, by VAL, each vertex in M_{34} is adjacent with exactly one 4-vertex. Thus $m' \leq n_4$ and so $n_5 \geq 2n_2 + \frac{3}{2}n_3 + 2n_4 + \frac{s'}{5}$. (6)Also, $|[N_3', N_4]| = |[N_3, N_4]| = n_4$.

By the hypothesis, R5 and the Lemma, $|[N_3', M_{34}]| = |[N_3, M_{34}]| = m' \leq n_4$.

Now, $s' = |[N_3', M_3]| = 3n_4 - |[N_3', N_4]| - |[N_3', M_{34}]| \geq 3n_4 - n_4 - n_4$.

(i.e.) $s' \geq n_4$. Thus, by (6), $n_5 \geq 2n_2 + \frac{3}{2}n_3 + \frac{11}{5}n_4$.

Then $2m = 2n_2 + 3n_3 + 4n_4 + 5n_5 = 4n - 2n_2 - n_3 + n_5 \geq 4n + \frac{1}{2}n_3 + \frac{11}{5}n_4$.

Thus $m \geq 2n + \frac{1}{4}n_3 + \frac{11}{10}n_4$.

Theorem 3.1.8. Let G be a 5-critical graph. Suppose that no 3-vertex is adjacent with a 4-vertex.Then (i) $n_5 \geq 2n_2 + \frac{3}{2}n_3 + \frac{2}{3}n_4 + \frac{m'}{6}$ and (ii) $m \geq 2n + \frac{n_3}{4} + \frac{1}{3}n_4 + \frac{m'}{12}$.

Proof: By the Lemma, $|[M_{34}, N_3]| = |[M_{34}, N_4]| = |M_{34}| = m'$. Since no 3-vertex is adjacent with a 4-vertex, $|[N_3, N_4]|=0$.

Thus, $|[N_3, M_3]| = 3n_3 - |[N_3, M_{34}]| = 3n_3 - m'$.

By VAL, $|[M_3, N_3]| \leq 2l'$ and so $l' \geq \frac{3}{2}n_3 - \frac{m'}{2}$. Also, by VAL, $2r = \sum_{v \in N_4} d_{N_4}(v) \leq 2n_4$ and so $r \leq n_4$.Then $|[N_4, M_4]| = 4n_4 - 2|[N_4]| - |[N_4, M_{34}]| \geq 2n_4 - m'$.

By VAL, $|[M_4, N_4]| \leq 3n'$ and so $n' \geq \frac{2}{3}n_4 - \frac{m'}{3}$.

Now $n_5 \geq 2n_2 + l' + m' + n' \geq 2n_2 + \frac{3}{2}n_3 + \frac{2}{3}n_4 + \frac{m'}{6}$.

Then $2m = 2n_2 + 3n_3 + 4n_4 + 5n_5 = 4n - 2n_2 - n_3 + n_5 \geq 4n + \frac{n_3}{2} + \frac{2}{3}n_4 + \frac{m'}{6}$.

Thus $m \geq 2n + \frac{n_3}{4} + \frac{1}{3}n_4 + \frac{m'}{12}$.

Theorem 3.1.9. Let G be a 5-critical graph. Suppose that no 3-vertex is adjacent with a 4-vertex and each major vertex is adjacent with at most two 4-vertices. Then

(i) $n_5 \geq 2n_2 + \frac{3}{2}n_3 + n_4$ and (ii) $m \geq 2n + \frac{n_3}{4} + \frac{n_4}{2}$.

Proof: Using the same notations and proceeding as in Theorem 3.1.8, we have $l' \geq \frac{3}{2}n_3 - \frac{m'}{2}$ and $|[N_4, M_4]| \geq 2n_4 - m'$. By hypothesis, $|[M_4, N_4]| \leq 2n'$ and so $n' \geq n_4 - \frac{m'}{2}$. Then $n_5 \geq 2n_2 + l' + m' + n' \geq 2n_2 + \frac{3}{2}n_3 + n_4$.

Hence $2m = 2n_2 + 3n_3 + 4n_4 + 5n_5 = 4n - 2n_2 - n_3 + n_5 \geq 4n + \frac{n_3}{2} + n_4$.

Thus $m \geq 2n + \frac{n_3}{4} + \frac{n_4}{2}$.

Remarks

- If G is a 5-critical graph with $n_4 = 0$ and if each major vertex is adjacent with at most one 3-vertex, then $n_5 \geq 2n_2 + 3n_3$.

- If G is a 5-critical graph with $n_3 = 0$ and if each major vertex is adjacent with at most two 4-vertices, then $n_5 \geq 2n_2 + n_4$.

Theorem 3.1.10. If G is a 5-critical graph, then 2n + 2 ≤ m ≤ 3n – 5.

Proof: By R3, if G is a 5-critical simple graph, then $m \geq 2n + 2$.

If m ≥ 3n – 3, then $4n_2 + 3n_3 + 2n_4 + n_5 \leq 6$ and so n ≤ 6, a contradiction to R6.

If m = 3n – 4, then $n + 3n_2 + 2n_3 + n_4 = 8$ and by R6, n =7 and $\pi(G) = 4\,5^6$, a contradiction to R7. Hence 2n + 2 ≤ m ≤ 3n – 5.

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