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Numerical Solution of Fuzzy Differential Equation by Sixth Order Runge-Kutta Method

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Abstract. In this paper, in order to increase the order of the accuracy of the solution, we propose a method of computing approximate solution of the fuzzy differential equation with initial conditions. The sixth order Runge-Kutta method is discussed in detail followed by a complete error analysis.

Keywords: Fuzzy differential equations, numerical solution, sixth order Runge-kutta method

AMS Mathematics Subject Classification (2010): 34A07

1. Introduction

The topic of fuzzy differential equation has been rapidly growing in recent years. The concept of fuzzy derivatives was first introduced by Chang and Zadeh [7]; it was followed up by Dubois and Prade [8] who used the extension principle in their approach. Puri and Ralesec [16] and Goetschel and Voxman [10] contributed towards the differential of fuzzy functions. The fuzzy differential equation and initial value problems were extensively studied by Kaleva [11, 12] and by Seikkala [17]. Numerical solution of fuzzy differential equations has been introduced by Ma, Friedman, Kandel [14] through Euler method and by Abbasbandy and Allahviranloo [2] by Taylor method. Runge-Kutta methods have also been studied by authors [3, 15]. This paper is organized as follows: In Section 2, we give some basic definitions and results on fuzzy numbers and fuzzy derivatives. Section 3 contains the definition of fuzzy Cauchy problem with initial conditions. In Section 4 we propose the sixth order Runge-Kutta method to solve the fuzzy differential equation with initial condition. The proposed method is illustrated and solved the numerical example in section 5. Also the result of the approximation solution by Runge-Kutta sixth order method is compared with Euler's method and Runge-Kutta fourth order method.

2. Preliminaries

Consider the initial value problem $y'(t) = f(t, y(t))$, $t_0 \le t \le b$, $y(t_0)=y_0$ (2.1)

We assume that

1. $f(t, y(t))$ is defined and continuous in the strip $t_0 \le t \le b$, $-\infty<y<\infty$ with t_0 and b are finite.
2. There exists a constant L such that for any t in $[t_0, b]$ and any two numbers y and y^*

$\left| f(t,y) - f(t,y^*) \right| \le L \left| y - y^* \right|$

These conditions are sufficient to prove that ∃on $[t_0, b]$ a unique continuous differentiable function y(t) satisfying (2.1)

The basis of all Runge-Kutta methods is to express the difference between the value of y at t_{n+1} and t_n as

$$y_{n+1} - y_n = \sum_{i=0}^{m} w_i k_i \text{ ; where } w_i \text{ are constants and } k_i = hf(t_n+a_i h,\ y_n + \sum_{j=1}^{i-1} c_{ij} k_j)$$

Most efforts to increase the order of accuracy of the Runge-Kutta methods have been accomplished by increasing the number of Taylor's series terms used and thus the number of functional evaluations required [6]. The method proposed by Goeken and Johnson. O [9] introduces new terms involving higher order derivatives of 'f' in the Runge-Kutta k_i terms (i > 1) to obtain a higher order of accuracy without a corresponding increase in evaluations of f, but with the addition of evaluations of f^1.

Consider $y(t_{n+1}) = y(t_n) + w_1k_1 + w_2k_2 + w_3k_3 + w_4k_4 + w_5k_5 + w_6k_6 + w_7k_7$

where

$k_1 = hf(t_n, y(t_n))$

$k_2 = hf(t_n+c_2h, y(t_n)+a_{21}k_1)$

$k_3 = hf(t_n+c_3h, y(t_n)+a_{31}k_1+a_{32}k_2)$

$k_4 = hf(t_n+c_4h, y(t_n)+a_{41}k_1+a_{42}k_2+a_{43}k_3)$

$k_5 = hf(t_n+c_5h, y(t_n)+a_{51}k_1+a_{52}k_2+a_{53}k_3+a_{54}k_4)$

$k_6 = hf(t_n+c_6h, y(t_n)+a_{61}k_1+a_{62}k_2+a_{63}k_3+a_{64}k_4+a_{65}k_5+a_{66}k_6)$

$k_7 = h\, f(t_n+c_7h, y(t_n)+a_{71}k_1+a_{72}k_2+a_{73}k_3+a_{74}k_4+ a_{75}k_5+a_{76}k_6 + a_{77}k_7)$ (2.2)

Utilizing the Taylor's series expansion techniques, Runge-Kutta method of order sixth is given by $y_{n+1} = y_n + \dfrac{9k_1 + 64k_3 + 49k_5 + 49k_6 + 9k_7}{180}$

where

$k_1 = hf(t_n, y(t_n))$

$k_2 = hf(t_n+ vh, y(t_n)+ vk_1)$

$k_3 = hf(t_n+\frac{h}{2}, y(t_n)+ ((4v - 1)k_1 + k_2)/(8v))$

$k_4 = hf(t_n+\frac{2h}{3}, y(t_n)+((10v - 2)k_1 + 2k_2 + 8vk_3)/(27v))$

$k_5 = hf(t_n+(7+4.582576)\frac{h}{14}, y(t_n)+(-((77v-56)+(17v-8)4.582576)k_1-8(7+4.582576)k_2$
$+ 48(7+4.582576)vk_3 - 3(21+4.582576)vk_4)/(392v)$

$k_6 = hf(t_n+(7 - 4.582576)\frac{h}{14}, y(t_n)+(-5((287v-56) - (59v-8)4.582576)k_1-40(7-4.582576)k_2$
$+ 320(4.582576)vk_3+ 3(21-121(4.582576))vk_4 + 392(6-4.582576)vk_5) /(1960v)$

$k_7 = hf(t_n+ h, y(t_n)+(15((30v - 8) - 7v(4.582576))k_1+120k_2- 40(5 + 7(4.582576))vk_3$
$+ 63(2 + 3(4.582576))vk_4 - 14(49 - 9(4.582576))vk_5 + 70(7 + 4.582576)vk_6)/(180v)$ (2.3)

Definition 2.1. A fuzzy number u is a fuzzy subset of $\Re$ i.e., u: $\Re \rightarrow [0, 1]$ satisfying the following conditions.

i). u is normal, i.e. $\exists x_0 \in \Re \ni u(x_0) = 1$
ii). u is a convex fuzzy set
i.e.) $u(tx + (1-t)y) \geq \min\{u(x), u(y)\}$, $\forall$ t $\in$ [0, 1] and x, y $\in \Re$
iii). u is upper semi continuous on $\Re$
iv). $\overline{\{x \in R, u(x) > 0\}}$ is compact

The set E is the family of fuzzy numbers and arbitrary fuzzy number is represented by an ordered pair of functions $\left(\underline{u}(r), \overline{u}(r)\right)$, $0 \leq r \leq 1$ that satisfies the following requirements.

1. $\underline{u}(r)$ is a bounded left continuous non-decreasing function over [0, 1] with respect to any 'r'.
2. $\overline{u}(r)$ is a bounded right continuous non-increasing function over [0, 1] with respect to any 'r'.
3. $\underline{u}(r) \leq \overline{u}(r)$, $0 \leq r \leq 1$, r-level cut is $[u]_r = \{x/u(x) \geq r\}$, $0 \leq r \leq 1$ is closed &bounded interval denoted by $[u]_r = \left[\underline{u}(r), \overline{u}(r)\right]$ and clearly $[u]_0 = \overline{\{x/u(x) > 0\}}$ is compact.

Definition 2.2. A triangular fuzzy number u is a fuzzy set in E that is characterized by an ordered triple $(u_l, u_c, u_r) \in R^3$ with $u_l < u_c < u_r$ such that $[u]_0 = [u_l : u_r]$ and $[u]_1 = [u_c]$. The membership function of the triangular fuzzy number u is given by

$$u(x) = \begin{cases} \dfrac{x - u_l}{u_c - u_l}, & u_l \leq x \leq u_c \\ 1 & x = u_c \\ \dfrac{u_r - x}{u_r - u_c}, & u_c \leq x \leq u_r \end{cases} \quad \ldots (2.4)$$

and we will have

i). $u > 0$ if $u_l > 0$
ii). $u \geq 0$ if $u_l \geq 0$
iii). $u < 0$ if $u_c < 0$
iv). $u \leq 0$ if $u_c \leq 0$

Let I be a real interval. A mapping y: I $\rightarrow$ E is called a fuzzy process and its α - level set is denoted by $[y(t)]_\alpha = \left[\underline{y}(t, y), \overline{y}(t, y)\right]$, t $\in$ I, $0 < \alpha \leq$ I.

The Seikkala derivative y(t) of a fuzzy process is defined by $\left[y^1(t)\right]_\alpha = \left[\underline{y}^1(t, y), \overline{y}^1(t, y)\right]$, t $\in$ I, $0 < \alpha \leq$ I provided the equation defines fuzzy number as in [12].For u, v $\in$ E and $\lambda \in \Re$, the u + v and the product λu can be defined by $[u + v]_\alpha = [u]_\alpha + [v]_\alpha$ and $[\lambda u]_\alpha = \lambda[u]_\alpha$, where $\alpha \in [0, 1]$ and $[u]_\alpha + [v]_\alpha$ means the addition of two intervals of $\Re$ and $\lambda[u]_\alpha$ means the product between a scalar λ and a subset of $\Re$.

Arithmetic operations of arbitrary fuzzy numbers u = $\left(\underline{u}(r), \overline{u}(r)\right)$ and v = $\left(\underline{v}(r), \overline{v}(r)\right)$, $\lambda \in \Re$ can be defined as

i). u = v if $\underline{u}(r) = \underline{v}(r)$ and $\overline{u}(r) = \overline{v}(r)$
ii). u + v = $\left(\underline{u}(r) + \underline{v}(r), \overline{u}(r) + \overline{v}(r)\right)$

iii). u - v = $\left(\underline{u}(r)-\overline{v}(r), \overline{u}(r)-\underline{v}(r)\right)$
iv). λu = $\left(\lambda\underline{u}(r), \lambda\overline{u}(r)\right)$ if λ≥ 0
= $\left(\lambda\overline{u}(r), \lambda\underline{u}(r)\right)$ if λ< 0

3. Fuzzy Cauchy problem

Consider the fuzzy initial value problem

$y'(t) = f(t, y(t))$, 0≤ t ≤T, y(0) = y_0 (3.1)

where f is a continuous mapping from $\Re_+ \times \Re \to \Re$ and $y_0 \in$ E with r-level sets $[y_0]_r$ = $\left[\underline{y}(0:r), \overline{y}(0:r)\right]$, r ∈ [0, 1]. The extension principle of Zadeh leads to the following definition of f(t, y),where y = y(t) is a fuzzy number.

f(t, y)(s)=sup{y(τ)\s=f(t, τ)}, s ∈ℜ

$\Rightarrow [f(t,y)]_r = \left[\underline{f}(t,y:r), \overline{f}(t,y:r)\right]$, r∈[0, 1]

It follows that

$\underline{f}(t,y:r) = \min\left\{f(t,u) \setminus u \in \left[\underline{y}(r), \overline{y}(r)\right]\right\}$ and $\overline{f}(t,y:r) = \max\left\{f(t,u) \setminus u \in \left[\underline{y}(r), \overline{y}(r)\right]\right\}$. . . (3.2)

Theorem 3.1. Let f satisfy $\left| f(t,v) - f(t,\overline{v}) \right| \le g\left(t, \left|v - \overline{v}\right|\right)$, t ≥ 0 and v, $\overline{v} \in \Re$, (3.3)

where g : $\Re_+ \times \Re_+ \to \Re_+$ is a continuous mapping such that r → g(t, r) is non-decreasing and the initial value problem $u^1(t)$ = g(t, u(t)), u(0) = u_0 (3.4)
has a solution on $\Re_+$for $u_0 > 0$ and that u(t)≡ 0 is the only solution of (3.4) for $u_0 = 0$. Then the fuzzy initial value problem (3.1) has a unique solution.

Proof: See [17].

3.1. Sixth order Runge–Kutta method

Let the exact solution of the given equation $[y(t)]_r$ = $\left[\underline{y}(t:r), \overline{y}(t:r)\right]$ is approximated by some $[y(t)]_r$ = $\left[\underline{y}(t:r), \underline{y}(t:r)\right]$ and we define

$\underline{y}(t_{n+1}:r) - \underline{y}(t_n:r) = \sum_{i=1}^{7} w_i \underline{k_i}$ and $\overline{y}(t_{n+1}:r) - \overline{y}(t_n:r) = \sum_{i=1}^{7} w_i \overline{k_i}$

where w_i's are constants,

$[k_i(t, y(t,r))]_r = \left[\underline{k_i}(t, y(t,r)), \overline{k_i}(t, y(t,r))\right]$ where i = 1, 2, 3, 4, 5, 6, 7

$\underline{k_1}(t, y(t:r)) = hf\left(t_n, \underline{y}(t_n:r)\right)$

$\overline{k_1}(t, y(t:r)) = hf\left(t_n, \overline{y}(t_n:r)\right)$

$\underline{k_2}(t, y(t:r)) = hf\left(t_n + \frac{h}{2}, \underline{y}(t_n:r) + v\underline{k_1}\right)$

$\overline{k_2}(t, y(t:r)) = hf\left(t_n + \frac{h}{2}, \overline{y}(t_n:r) + v\overline{k_1}\right)$

$\underline{k_3}(t, y(t:r)) = hf\left(t_n + \frac{h}{2}, \underline{y}(t_n:r) + \left((4v-1)\underline{k_1} + \underline{k_2}\right)/8v\right)$

$\overline{k_3}(t, y(t:r)) = hf\left(t_n + \frac{h}{2}, \overline{y}(t_n:r) + \left((4v-1)\overline{k_1} + \overline{k_2}\right)/8v\right)$

$$\underline{k}_4(t, y(t:r)) = hf\left(t_n + \frac{2h}{3}, \underline{y}(t_n:r) + \left((10v-2)\underline{k}_1 + 2\underline{k}_2 + 8v\underline{k}_3\right)/27v\right)$$

$$\overline{k}_4(t, y(t:r)) = hf\left(t_n + \frac{2h}{3}, \overline{y}(t_n:r) + \left((10v-2)\overline{k}_1 + 2\overline{k}_2 + 8v\overline{k}_3\right)/27v\right)$$

$$\underline{k}_5(t, y(t:r)) = hf\left(t_n + (7+4.582576)\frac{h}{14}, \underline{y}(t_n:r) + \begin{pmatrix} -((77v-56)+(17v-8)4.582576)\underline{k}_1 \\ -8(7+4.582576)\underline{k}_2 + 48(7+4.582576)v\underline{k}_3 \\ -3(21+4.582576)v\underline{k}_4 \end{pmatrix}/392v\right)$$

$$\overline{k}_5(t, y(t:r)) = hf\left(t_n + (7+4.582576)\frac{h}{14}, \overline{y}(t_n:r) + \begin{pmatrix} -((77v-56)+(17v-8)4.582576)\overline{k}_1 \\ -8(7+4.582576)\overline{k}_2 + 48(7+4.582576)v\overline{k}_3 \\ -3(21+4.582576)v\overline{k}_4 \end{pmatrix}/392v\right)$$

$$\underline{k}_6(t, y(t:r)) = hf\left(t_n + (7-4.582576)\frac{h}{14}, \underline{y}(t_n:r) + \begin{pmatrix} -5((287v-56)-(59v-8)4.582576)\underline{k}_1 \\ -40(7-4.582576)\underline{k}_2 + 320(4.582576)v\underline{k}_3 \\ +3(21-121(4.582576))v\underline{k}_4 + 392(6-(4.582576))v\underline{k}_5 \end{pmatrix}/1960v\right)$$

$$\overline{k}_6(t, y(t:r)) = hf\left(t_n + (7-4.582576)\frac{h}{14}, \overline{y}(t_n:r) + \begin{pmatrix} -5((287v-56)-(59v-8)4.582576)\overline{k}_1 \\ -40(7-4.582576)\overline{k}_2 + 320(4.582576)v\overline{k}_3 \\ +3(21-121(4.582576))v\overline{k}_4 + 392(6-(4.582576))v\overline{k}_5 \end{pmatrix}/1960v\right)$$

$$\underline{k}_7(t, y(t:r)) = hf\left(t_n + h, \underline{y}(t_n:r) + \begin{pmatrix} 15((30v-8)7v(4.582576))\underline{k}_1 + 120\underline{k}_2 - 40(5+7(4.582576))v\underline{k}_3 \\ +63(2+3(4.582576))v\underline{k}_4 - 14(49-9(4.582576))v\underline{k}_5 \\ +70(7+(4.582576))v\underline{k}_6 \end{pmatrix}/180v\right)$$

$$\overline{k}_7(t, y(t:r)) = hf\left(t_n + h, \overline{y}(t_n:r) + \begin{pmatrix} 15((30v-8)-7v(4.582576))\overline{k}_1 + 120\overline{k}_2 - 40(5+7(4.582576))v\overline{k}_3 \\ +63(2+3(4.582576))v\overline{k}_4 - 14(49-9(4.582576))v\overline{k}_5 \\ +70(7+(4.582576))v\overline{k}_6 \end{pmatrix}/180v\right) \tag{4.1}$$

$$F(t, y(t:r)) = 9\underline{k}_1(t, y(t:r)) + 64\underline{k}_3(t, y(t:r)) + 49\underline{k}_5(t, y(t:r)) + 49\underline{k}_6(t, y(t:r)) + 9\underline{k}_7(t, y(t:r))$$

$$G(t, y(t:r)) = 9\overline{k}_1(t, y(t:r)) + 64\overline{k}_3(t, y(t:r)) + 49\overline{k}_5(t, y(t:r)) + 49\overline{k}_6(t, y(t:r)) + 9\overline{k}_7(t, y(t:r))$$

The exact and approximate solution at t_n, $0 \le n \le N$ are denoted by

$[Y(t_n)]_r = \left[\underline{Y}(t_n:r), \overline{Y}(t_n:r)\right]$ and $[y(t_n)]_r = \left[\underline{y}(t_n:r), \overline{y}(t_n:r)\right]$ respectively.

The solution calculated by grid points at $a = t_0 \le t_1 \le t_2 \le \ldots \le t_N = b$ and

$h = \frac{b-a}{N} = t_{n+1} - t_n$. Therefore we have

$$\underline{Y}(t_{n+1}:r) = \underline{Y}(t_n:r) + \frac{1}{180}F[t_n, \underline{Y}(t_n:r)]$$

$$\overline{Y}(t_{n+1}:r) = \overline{Y}(t_n:r) + \frac{1}{180}G[t_n, \overline{Y}(t_n:r)]$$

$$\underline{y}(t_{n+1}:r) = \underline{y}(t_n:r) + \frac{1}{180}F[t_n, \underline{y}(t_n:r)]$$

$$\overline{y}(t_{n+1}:r)=\overline{y}(t_n:r)+\frac{1}{180}G\left[t_n,\overline{y}(t_n:r)\right] \tag{4.2}$$

Here, we show the convergence of these approximations as

$$\lim_{h\to 0}\underline{y}(t:r)=\underline{Y}(t:r)\text{ and }\lim_{h\to 0}\overline{y}(t:r)=\overline{Y}(t:r)$$

Lemma 3.1.1. Let a sequence of numbers $\{W\}_{n=0}^{N}$ satisfy $|W_{n+1}|\leq A|W_n| + B$, $0 \leq n \leq N-1$ for some given positive constants A and B then $|W_n|\leq A^n|W_0|+B\frac{A^n-1}{A-1}$, $0 \leq n \leq N-1$

Proof: See [14]

Lemma 3.1.2. Let a sequence of numbers $\{W\}_{n=0}^{N}$ and $\{V\}_{n=0}^{N}$ satisfies the conditions $|W_{n+1}|\leq|W_n| + A\max\{|W_n|,|V_n|\}+B$, $|V_{n+1}|\leq|V_n| + A\max\{|W_n|,|V_n|\}+B$ for some given positive constants A &B and denote $U_n = |W_n| + |V_n|$, $0 \leq n \leq N$, then

$$U_n\leq (1 + 2A)^n U_0 + 2B\frac{(1+2A)^n-1}{(1+2A)-1},\quad 0 \leq n \leq N$$

Proof: See [14].

Theorem 3.1.1. Let F(t, u, v) and G(t, u, v) belong to $C^4(K)$ and let the partial derivatives of F and G be bounded over K. Then, for arbitrary fixed r, $0 \leq r \leq 1$, the approximately solutions (4.2) converge to the exact solutions of $\underline{Y}(t_n:r)$ and $\overline{Y}(t_n:r)$ uniformly in it.

Proof: See [14].

3.2. Numerical example

Consider the fuzzy initial value problem,

$$y^1(t) = y(t),\ t \in [0, 1] \text{ with } y(0)= (0.75 + 0.25r, 1.2-0.2r) \text{ where } 0 \leq r \leq 1$$

Solution: The exact solution is given by $\underline{y}(t:r)=\underline{y}(t:r)e^t$ and $\overline{y}(t:r)=\overline{y}(t:r)e^t$ then at $t = 1$, $y(1 : r) = [(0.75+0.25r)e,(1.2-0.2r)e]$, $0 \leq r \leq 1$. The exact and the approximate solutions are obtained by Euler's method, fourth order Runge-Kutta method and sixth order Runge-Kutta method with h = 0.1 is given in the table 3.2.1

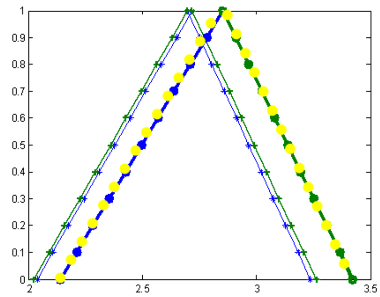
r	Exact Solution		6^{th} order Runge-Kutta Method		4^{th} order Runge-Kutta Method		Euler's Method	
0.0	2.0387113713	3.2619381942	2.0387113094	3.2619383335	2.0225877762	3.2361412048	2.1398372650	3.4237406254
0.1	2.1066684171	3.2075725576	2.1066684723	3.2075719833	2.0900075436	3.1822049618	2.2111654282	3.3666775227
0.2	2.1746254628	3.1532069210	2.1746253967	3.1532073021	2.1574270725	3.1282691956	2.2824935913	3.3096153736
0.3	2.2425825085	3.0988412844	2.2425825596	3.0988414288	2.2248466015	3.0743336678	2.3538210392	3.2525527477
0.4	2.3105395542	3.0444756479	2.3105394840	3.0444755554	2.2922666073	3.0203979015	2.4251492023	3.1954908371
0.5	2.3784965999	2.9901100113	2.3784964085	2.9901101589	2.3596863747	2.9664621353	2.4964776039	3.1384286880
0.6	2.4464536456	2.9357443747	2.4464540482	2.9357445240	2.4271051884	2.9125268459	2.5678048134	3.0813660622
0.7	2.5144106913	2.8813787382	2.5144107342	2.8813788891	2.4945249557	2.8585908413	2.6391329765	3.0243034363

0.8	2.5823677370	2.8270131016	2.5823678970	2.8270127773	2.5619447231	2.8046555519	2.7104613781	2.9672412872
0.9	2.6503247827	2.7726474650	2.6503250599	2.7726476192	2.6293642521	2.7507193089	2.7817888260	2.9101791382
1.0	2.7182818285	2.7182818285	2.7182817459	2.7182817459	2.6967840195	2.6967840195	2.8531169891	2.8531169891

Table 3.2.1: Comparison of 6th order Runge-Kutta method with other methods

r	6th order Runge-Kutta method		4th order Runge-Kutta method		Euler's method	
0.0	0.0000000619	0.0000001393	0.0161235951	0.0257969894	0.1011258937	0.1618024312
0.1	0.0000000552	0.0000005743	0.0166608735	0.0253675958	0.1044970111	0.1591049651
0.2	0.0000000661	0.0000003811	0.0171983903	0.0249377254	0.1078681285	0.1564084526
0.3	0.0000000511	0.0000001444	0.0177359070	0.0245076166	0.1112385307	0.1537114633
0.4	0.0000000702	0.0000000925	0.0182729469	0.0240777464	0.1146096481	0.1510151892
0.5	0.0000001914	0.0000001476	0.0188102252	0.0236478760	0.1179810040	0.1483186767
0.6	0.0000004026	0.0000001493	0.0193484572	0.0232175288	0.1213511678	0.1456216875
0.7	0.0000000429	0.0000001509	0.0198857356	0.0227878969	0.1247222852	0.1429246981
0.8	0.0000001600	0.0000003243	0.0204230139	0.0223575497	0.1280936411	0.1402281856
0.9	0.0000002772	0.0000001542	0.0209605306	0.0219281561	0.1314640433	0.1375316732
1.0	0.0000000826	0.0000000826	0.0214978090	0.0214978090	0.1348351606	0.1348351606

Table 3.1.2: Error analysis



- Runge-Kutta 6th order method
- Runge-Kutta 4th order method
- Euler's method
- Exact Solution

Figure 3.1.1: Numerical solutions of fuzzy differential equation

4. Conclusion

In this paper, we have suggested that the sixth order Runge-Kutta method gives a better numerical solution for fuzzy differential equation. With higher order of convergence, the order of convergence of Euler's method is O(h), the order of convergence of fourth order Runge-Kutta method is $O(h^2)$ [15] whereas order of convergence of sixth order Runge-

Kutta method proposed by us is $O(h^3)$. We have proved that the sixth order Runge-Kutta method proposed by us gives a better solution than Euler's method and fourth order Runge-Kutta method.

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