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Intuitionistic Fuzzy Stability of n-Dimensional Cubic Functional Equation: Direct and Fixed Point Methods

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Abstract. In this paper, the authors investigate generalized Ulam-Hyers stability of a $n-$ dimensional cubic functional equation

$$\sum_{i=1}^{n} h\left(\sum_{j=1}^{i} x_j\right) - \sum_{i=1}^{n}\left(\frac{i^2-5i+6}{16}\right)\sum_{j=1}^{i} h\left(2x_j\right) = \sum_{i=1}^{n}\sum_{1\le j<k<l\le i} h\left(x_j+x_k+x_l\right) - \sum_{i=1}^{n}(i-3)\sum_{1\le j<k\le i} h\left(x_j+x_k\right)$$

with $n\geq 3$, in Intuitioniistic fuzzy normed spaces using direct and fixed point methods.

Keywords: Cubic functional equations, generalized Ulam - Hyers stability, intuitionistic fuzzy normed space, fixed point

AMS Mathematics Subject Classification (2010): 39B52, 32B72, 32B82

1. Introduction

The research of stability problems for functional equations is linked to the renowned Ulam problem [34] (in 1940), concerning the stability of group homomorphisms, which was first elucidated by Hyers [10], in 1941. This stability problem was more widespread by quite a lot of creators [2,9,25,27,28]. Other pertinent research works related to various functional equations using direct and fixed point methods were discussed in (see[1, 3, 5, 6, 11, 14, 15, 21, 22, 27]).

Recently, Murthy et al., [23] introduced and investigate the general solution and generalized Ulam-Hyers stability of a new form of $n-$dimensional cubic functional equation

$$\sum_{i=1}^{n} h\left(\sum_{j=1}^{i} x_j\right) - \sum_{i=1}^{n}\left(\frac{i^2-5i+6}{16}\right)\sum_{j=1}^{i} h\left(2x_j\right) = \sum_{i=1}^{n}\sum_{1\le j<k<l\le i} h\left(x_j+x_k+x_l\right) - \sum_{i=1}^{n}(i-3)\sum_{1\le j<k\le i} h\left(x_j+x_k\right) \tag{1.1}$$

with $n\geq 3$, in Felbins type spaces using direct and fixed point methods.

In this paper, the authors investigate the generalized Ulam-Hyers stability of the above $n-$dimensional cubic functional equation (1.1) in Intuitionistic fuzzy normed spaces using direct and fixed point methods.

In Section 2, we present the solution of the functional equation (1.1). The generalized Ulam-Hyers stability using Banach space in given is Section 3. In Section 4, the basic notations and preliminaries about Intuitionistic Fuzzy Normed Spaces is present. Also the stability of (1.1) in Intuitionistic Fuzzy Normed Spaces using direct and fixed point methods are discussed in Section 5 and 6 respectively.

2. General solution of the functional equation (1.1)

In this section, the authors discuss the general solution of the functional equation (1.1) by considering X and Y are real vector spaces.

Theorem 2.1. [23] If $f: X \to Y$ satisfies the functional equation (1.1) for all $x_1, x_2, x_3, \ldots, x_n \in X$ and $n \geq 3$ then there exists a function $B: X^3 \to Y$ such that $f(x) = B(x,x,x)$ for all $x \in X$ where B is symmetric for each fixed one variable and additive for each fixed two variables.

Hereafter, throughout this paper, we define a mapping $H: X \to Y$ by

$$H(x_1, x_2, \cdots, x_n) = \sum_{i=1}^{n} h\left(\sum_{j=1}^{i} x_j\right) - \sum_{i=1}^{n}\left(\frac{i^2 - 5i + 6}{16}\right)\sum_{j=1}^{i} h(2x_j) - \sum_{i=1}^{n}\sum_{1\leq j<k<l\leq i} h(x_j + x_k + x_l) + \sum_{i=1}^{n}(i-3)\sum_{1\leq j<k\leq i} h(x_j + x_k)$$

for all $x_1, x_2, x_3, \ldots, x_n \in X$.

3. Stability results in Banach space

In this section, the generalized Ulam - Hyers stability of a n-dimensional cubic functional equation (1.1) is provided. Throughout this section, assume X and Y to be a normed space and a Banach space, respectively.

The proof of the following theorem and corollary is similar lines to that of Theorem 4.1 and Corollary 4.2 of [23]. Hence the details of the proofs are omitted.

Theorem 3.1. Let $j = \pm 1$. Let $h: X \to Y$ be a mapping for which there exist a function $\xi: X^n \to [0, \infty)$ with the conditions

$$\lim_{k\to\infty} \frac{1}{2^{3kj}} \xi\left(2^{kj} x_1, \cdots, 2^{kj} x_n\right) \textit{ converges and } \lim_{k\to\infty} \frac{1}{2^{3kj}} \xi\left(2^{kj} x_1, \cdots, 2^{kj} x_n\right) = 0 \tag{3.1}$$

such that the functional inequality

$$\left\| H(x_1, x_2, x_3, \cdots, x_n) \right\| \leq \xi(x_1, x_2, x_3, \cdots, x_n) \tag{3.2}$$

for all $x_1, x_2, x_3, \cdots, x_n \in X$. Then there exists a unique cubic mapping $C: X \to Y$ satisfying the functional equation (1.1) and

$$\|h(x) - C(x)\| \leq \frac{1}{2^3 \ell} \sum_{k=\frac{1-j}{2}}^{\infty} \frac{\xi\left(2^{kj} x, \underbrace{0, \cdots, 0}_{n-1 \text{ times}}\right)}{2^{3kj}}, \quad \ell = \sum_{i=1}^{n} \frac{(i-2)(i-3)}{2} \tag{3.3}$$

for all $x \in X$. The mapping $C(x)$ is defined by $C(x) = \lim_{k\to\infty} \frac{f(2^{kj} x)}{2^{3kj}}$ (3.4) for all $x \in X$.

Corollary 3.1. Let $h: X \to Y$ be a function and there exits real numbers λ and s such that

$$\left\|H\left(x_1,x_2,x_3,\cdots,x_n\right)\right\| \le \begin{cases} \lambda, & \\ \lambda\sum_{k=1}^{n}\|x_k\|^s, & s<3 \; or \; s>3; \\ \lambda\left(\sum_{k=1}^{n}\|x_k\|^{ns}+\prod_{k=1}^{n}\|x_k\|^s\right), & s<\frac{3}{n} \; or \; s>\frac{3}{n}; \end{cases} \tag{3.5}$$

for all $x_1,x_2,x_3,\cdots,x_n \in X$. Then there exists a unique cubic function $C: X \to Y$ such that

$$\|f(x)-C(x)\| \le \begin{cases} \frac{\lambda}{7\ell}, \\ \frac{\lambda\|x\|^s}{\ell|8-2^s|}, \\ \frac{\lambda\|x\|^{ns}}{\ell|8-2^{ns}|}, \end{cases} \text{ for all } x\in X. \tag{3.6}$$

3.1. Stability result: intuitionistic fuzzy normed space

In this section, we give some basic definition and notations about intuitionistic fuzzy metric spaces introduced by J.H. Park [24] and R. Saadati and J.H. Park [30,31].

Definition 3.1.1. Let μ and ν be membership and nonmembership degree of an intuitionistic fuzzy set from $X\times(0,+\infty)$ to [0,1] such that $\mu_x(t)+\nu_x(t)\le 1$ for all $x\in X$ and all $t>0$. The triple $(X,P_{\mu,\nu},M)$ is said to be an intuitionistic fuzzy normed space (briefly IFN-space) if X is a vector space, M is a continuous t−representable and $P_{\mu,\nu}$ is a mapping $X\times(0,+\infty)\to L^*$ satisfying the following conditions: for all $x,y\in X$ and $t,s>0$,

$(IFN1)$ $P_{\mu,\nu}(x,0) = 0_{L^*}$; $(IFN2)$ $P_{\mu,\nu}(x,t) = 1_{L^*}$ if and only if $x=0$;

$(IFN3)$ $P_{\mu,\nu}(\alpha x,t) = P_{\mu,\nu}\left(x,\frac{t}{|\alpha|}\right)$ for all $\alpha\neq 0$; $(IFN4)$ $P_{\mu,\nu}(x+y,t+s) \ge_{L^*} \mathrm{M}\left(P_{\mu,\nu}(x,t),P_{\mu,\nu}(y,s)\right)$.

In this case, $P_{\mu,\nu}$ is called an intuitionistic fuzzy norm. Here $P_{\mu,\nu}(x,t)=\left(\mu_x(t),\nu_x(t)\right)$.

Example 3.1.1. Let $(X,\|.\|)$ be a normed space. Let $T(a,b)=(a,b \text{ min } (a_2+b_2,1))$ for all $a=(a_1,a_2),b=(b_1,b_2)\in L^*$ and μ,ν be membership and non-membership degree of an intuitionistic fuzzy set defined by

$P_{\mu,\nu}(x,t)=\left(\mu_x(t),\nu_x(t)\right)=\left(\frac{t}{t+\|x\|},\frac{\|x\|}{t+\|x\|}\right),\forall t\in R^+$. Then $(X,P_{\mu,\nu},T)$ is an IFN-space.

Definition 3.1.2. A sequence $\{x_n\}$ in an IFN-space $(X,P_{\mu,\nu},T)$ is called a Cauchy sequence if, for any $\varepsilon>0$ and $t>0$, there exists $n_0\in N$ such that $P_{\mu,\nu}(x_n-x_m,t)>_{L^*}\left(N_s(\varepsilon),\varepsilon\right)$, $\forall n,m\ge n_0$, where N_s is the standard negator.

Definition 3.1.3. The sequence $\{x_n\}$ is said to be convergent to a point $x \in X$ (denoted by $x_n \xrightarrow{P_{\mu,\nu}} x$) if $P_{\mu,\nu}(x_n - x, t) \to 1_{L^*}$ as $n \to \infty$ for every $t > 0$.

Definition 3.1.4. An IFN-space $(X, P_{\mu,\nu}, T)$ is said to be complete if every Cauchy sequence in X is convergent to a point $x \in X$ .For further details about IFN space one can see ([4, 7, 8, 12, 13, 16–20, 24, 29–33, 35–37]). Throughout this section, let us consider $X, (Z, P_{\mu,\nu}, M)$ and $(Y, P'_{\mu,\nu}, M)$ are linear space, Intuitionistic fuzzy normed space and Complete Intuitionistic fuzzy normed space.

Theorem 3.1.1. Let $\kappa \in \{-1,1\}$ be fixed and let $\xi : X^n \to Z$ be a mapping such that for some b with $0 < (b/2)^\kappa < 1$, $P'_{\mu,\nu}\left(\xi\left(2^\kappa x, 0, \cdots, 0\right), r\right) \geq_{L^*} P'_{\mu,\nu}\left(b^\kappa \xi\left(x, 0, \cdots, 0\right), r\right)$, (3.1.1)

for all $x \in X$ and all $r > 0, b > 0$, and $\lim_{k\to\infty} P'_{\mu,\nu}\left(\xi\left(2^{\kappa k} x_1, \cdots, 2^{\kappa k} x_n\right), 2^{\kappa k} r\right) = 1_{L^*}$ (3.1.2)

for all $x_1, x_2, x_3, \cdots, x_n \in X$ and all $r > 0$. Suppose that a function $h : X \to Y$ satisfies the inequality $P_{\mu,\nu}\left(H\left(x_1, x_2, x_3, \cdots, x_n\right), r\right) \geq_{L^*} P'_{\mu,\nu}\left(\xi\left(x_1, x_2, x_3, \cdots, x_n\right), r\right)$ (3.1.3)

for all $x_1, x_2, x_3, \cdots, x_n \in X$ and all $r > 0$. Then the limit

$$P_{\mu,\nu}\left(C(x) - \frac{f(2^{\kappa k} x)}{2^{3\kappa k}}, r\right) \to 1_{L^*} \quad as \quad k \to \infty, r > 0 \tag{3.1.4}$$

exists for all $x \in X$ and the mapping $C : X \to Y$ is a unique cubic mapping satisfying (1.1) and

$$P_{\mu,\nu}\left(h(x) - C(x), r\right) \geq_{L^*} P'_{\mu,\nu}\left(\xi\left(x, 0, \cdots, 0\right), \frac{r\ell\,|8-b|}{8}\right) \tag{3.1.5}$$

where $\ell = \sum_{i=1}^{n}\left(\frac{i^2 - 5i + 6}{2}\right)$ for all $x \in X$ and all $r > 0$.

Proof: First assume $\kappa = 1$. Replacing $(x_1, x_2, x_3, \cdots, x_n)$ by $(x, 0, 0, \cdots, 0)$ in (4.3), we arrive

$$P_{\mu,\nu}\left(\left(\sum_{i=1}^{n}\left(\frac{i^2 - 5i + 6}{2}\right)h(x) - \sum_{i=1}^{n}\left(\frac{i^2 - 5i + 6}{16}\right)h(2x)\right), r\right) \geq_{L^*} P'_{\mu,\nu}\left(\xi\left(x, 0, \cdots, 0\right), r\right)$$

for all $x \in X$ and all $r > 0$. Using $(IFN3)$ in the above equation, we get

$$P_{\mu,\nu}\left(\frac{h(2x)}{2^3} - h(x), \frac{r}{\ell}\right) \geq_{L^*} P'_{\mu,\nu}\left(\xi\left(x, 0, \cdots, 0\right), r\right) \tag{3.1.6}$$

where $\ell = \sum_{i=1}^{n}\left(\frac{i^2 - 5i + 6}{2}\right)$ for all $x \in X$ and all $r > 0$. Replacing x by $2^k x$ in (3.1.6), we obtain

$$P_{\mu,\nu}\left(\frac{h(2^{k+1} x)}{2^3} - h(2^k x), \frac{r}{\ell}\right) \geq_{L^*} P'_{\mu,\nu}\left(\xi\left(2^k x, 0, \cdots, 0\right), r\right) \tag{3.1.7}$$

for all $x \in X$ and all $r > 0$. Using (3.1.1), $(IFN3)$ in (3.1.7), we arrive

$$P_{\mu,\nu}\left(\frac{h(2^{k+1} x)}{2^3} - h(2^k x), \frac{r}{\ell}\right) \geq_{L^*} P'_{\mu,\nu}\left(\xi\left(x, 0, \cdots, 0\right), \frac{r}{b^k}\right) \tag{3.1.8}$$

for all $x\in X$ and all $r>0$. It is easy to verify from (3.1.8), that

$$P_{\mu,\nu}\left(\frac{h(2^{k+1}x)}{2^{3(k+1)}}-\frac{h(2^k x)}{2^{3k}},\frac{r}{2^{3k}\cdot\ell}\right)\geq_{L^*} P'_{\mu,\nu}\left(\xi(x,0,\cdots,0),\frac{r}{b^k}\right) \quad (3.1.9)$$

holds for all $x\in X$ and all $r>0$. Replacing r by $b^n r$ in (3.1.9), we get

$$P_{\mu,\nu}\left(\frac{h(2^{k+1}x)}{2^{3(k+1)}}-\frac{h(2^k x)}{2^{3k}},\frac{b^k r}{2^{3k}\cdot\ell}\right)\geq_{L^*} P'_{\mu,\nu}\left(\xi(x,0,\cdots,0),r\right) \quad (3.1.10)$$

for all $x\in X$ and all $r>0$. It is easy to see that

$$\frac{h(2^k x)}{2^{3k}}-h(x)=\sum_{i=0}^{k-1}\frac{h(2^{i+1}x)}{2^{3(i+1)}}-\frac{h(2^i x)}{2^{3i}} \quad (3.1.11)$$

for all $x\in X$. From equations (3.1.10) and (3.1.11), we have

$$P_{\mu,\nu}\left(\frac{h(2^k x)}{2^{3k}}-h(x),\sum_{i=0}^{k-1}\frac{b^i r}{2^{3i}\cdot\ell}\right)\geq_{L^*} M_{i=0}^{n-1}\left\{P_{\mu,\nu}\left(\sum_{i=0}^{k-1}\frac{h(2^{i+1}x)}{2^{3(i+1)}}-\frac{h(2^i x)}{2^{3i}},\frac{b^i r}{2^{3i}\cdot\ell}\right)\right\}$$
$$\geq_{L^*} M_{i=0}^{n-1}\left\{P'_{\mu,\nu}\left(\xi(x,0,\cdots,0),r\right)\right\}\geq_{L^*} P'_{\mu,\nu}\left(\xi(x,0,\cdots,0),r\right)$$

(3.1.12)

for all $x\in X$ and all $r>0$. Replacing x by $2^m x$ in (3.1.12) and using (3.1.1), $(IFN3)$, we obtain

$$P_{\mu,\nu}\left(\frac{h(2^{k+m}x)}{2^{3(k+m)}}-\frac{h(2^m x)}{2^{3m}},\sum_{i=0}^{k-1}\frac{b^i r}{2^{3(i+m)}\cdot\ell}\right)\geq_{L^*} P'_{\mu,\nu}\left(\xi(x,0,\cdots,0),\frac{r}{b^m}\right) \quad (3.1.13)$$

for all $x\in X$ and all $r>0$ and all $m,n\geq 0$. Replacing r by $b^m r$ in (3.1.13), we get

$$P_{\mu,\nu}\left(\frac{h(2^{k+m}x)}{2^{3(k+m)}}-\frac{h(2^m x)}{2^{3m}},\sum_{i=m}^{m+k-1}\frac{b^i r}{2^{3i}\cdot\ell}\right)\geq_{L^*} P'_{\mu,\nu}\left(\xi(x,0,\cdots,0),r\right) \quad (3.1.14)$$

for all $x\in X$ and all $r>0$ and all $m,n\geq 0$. It follows from (3.1.14) that

$$P_{\mu,\nu}\left(\frac{h(2^{k+m}x)}{2^{3(k+m)}}-\frac{h(2^m x)}{2^{3m}},r\right)\geq_{L^*} P'_{\mu,\nu}\left(\xi(x,0,\cdots,0),r\Big/\sum_{i=m}^{m+k-1}\frac{b^i}{2^{3i}\cdot\ell}\right) \quad (3.1.15)$$

for all $x\in X$ and all $r>0$ and all $m,n\geq 0$. Since $0<b<2$ and $\sum_{i=0}^{n}(b/2)^i<\infty$, this implies $\left\{\frac{h(2^k x)}{2^{3k}}\right\}$ is a Cauchy sequence in $(Y,P'_{\mu,\nu},M)$. Since $(Y,P'_{\mu,\nu},M)$ is a complete IFN space, this sequence converges to some point $C(x)\in Y$. So one can define the mapping $C:X\to Y$ by

$$P_{\mu,\nu}\left(C(x)-\frac{f(2^k x)}{2^{3k}},r\right)\to 1_{L^*} \quad as \quad k\to\infty, r>0 \quad (3.1.15a)$$

for all $x\in X$. Letting $m=0$ in (3.1.15), we get

$$P_{\mu,\nu}\left(\frac{h(2^k x)}{2^{3k}}-h(x),r\right)\geq_{L^*} P'_{\mu,\nu}\left(\xi(x,0,\cdots,0),\frac{r}{\sum_{i=0}^{k-1}\frac{b^i}{2^{3i}\cdot\ell}}\right) \quad (3.1.16)$$

for all $x\in X$ and all $r>0$. Letting $k\to\infty$ in (3.1.16), we arrive

$P_{\mu,\nu}\left(h(x)-C(x),r\right)\geq_{L^*} P'_{\mu,\nu}\left(\xi\left(x,0,\cdots,0\right),\frac{r\ell(8-b)}{8}\right)$ for all $x\in X$ and all $r>0$. To prove C satisfies the (1.1), replacing $(x_1,\cdots,x_n)$ by $(2^k x_1,\cdots,2^k x_n)$ and dividing by 2^{3k} in (3.1.2), we obtain $P_{\mu,\nu}\left(\frac{1}{2^{3k}}H\left(2^k x_1,\cdots,2^k x_n\right),r\right)\geq_{L^*} P'_{\mu,\nu}\left(\xi\left(2^k x_1,\cdots,2^k x_n\right),2^{3k}r\right)$ (3.1.17)

for all $x_1,\cdots,x_n\in X$ and all $r>0$. Now,

$$P_{\mu,\nu}\left(\sum_{i=1}^{n}C\left(\sum_{j=1}^{i}x_j\right)-\sum_{i=1}^{n}\left(\frac{i^2-5i+6}{16}\right)\sum_{j=1}^{i}C\left(2x_j\right)-\sum_{i=1}^{n}\sum_{1\leq j<k<l\leq i}C\left(x_j+x_k+x_l\right)+\sum_{i=1}^{n}(i-3)\sum_{1\leq j<k\leq i}C\left(x_j+x_k\right),r\right)$$

$$\geq_{L^*} M\left\{P_{\mu,\nu}\left(\sum_{i=1}^{n}C\left(\sum_{j=1}^{i}x_j\right)-\frac{1}{2^{3k}}\sum_{i=1}^{n}h\left(\sum_{j=1}^{i}2^k x_j\right),\frac{r}{5}\right),\right.$$

$$P_{\mu,\nu}\left(-\sum_{i=1}^{n}\left(\frac{i^2-5i+6}{16}\right)\sum_{j=1}^{i}C\left(2x_j\right)+\frac{1}{2^{3k}}\sum_{i=1}^{n}\left(\frac{i^2-5i+6}{16}\right)\sum_{j=1}^{i}h\left(2^k\cdot 2x_j\right),\frac{r}{5}\right),$$

$$P_{\mu,\nu}\left(-\sum_{i=1}^{n}\sum_{1\leq j<k<l\leq i}C\left(x_j+x_k+x_l\right)+\frac{1}{2^{3k}}\sum_{i=1}^{n}\sum_{1\leq j<k<l\leq i}h\left(2^k\left(x_j+x_k+x_l\right)\right),\frac{r}{5}\right),$$

$$P_{\mu,\nu}\left(+\sum_{i=1}^{n}(i-3)\sum_{1\leq j<k\leq i}C\left(x_j+x_k\right)-\frac{1}{2^{3k}}+\sum_{i=1}^{n}(i-3)\sum_{1\leq j<k\leq i}h\left(2^k\left(x_j+x_k\right)\right),\frac{r}{5}\right),$$

$$P_{\mu,\nu}\left(\frac{1}{2^{3k}}\left(\sum_{i=1}^{n}h\left(\sum_{j=1}^{i}2^k x_j\right)-\sum_{i=1}^{n}\left(\frac{i^2-5i+6}{16}\right)\sum_{j=1}^{i}h\left(2^k\cdot 2x_j\right)\right.\right.$$

$$\left.\left.\left.-\sum_{i=1}^{n}\sum_{1\leq j<k<l\leq i}h\left(2^k\left(x_j+x_k+x_l\right)\right)+\sum_{i=1}^{n}(i-3)\sum_{1\leq j<k\leq i}h\left(2^k\left(x_j+x_k\right)\right)\right),\frac{r}{5}\right)\right\}$$

(3.1.18)

for all $x_1,\cdots,x_n\in X$ and all $r>0$. Using (3.1.15a), (3.1.17), (3.1.2) and $(IFN2)$ in (3.1.18), we arrive

$$\sum_{i=1}^{n}C\left(\sum_{j=1}^{i}x_j\right)-\sum_{i=1}^{n}\left(\frac{i^2-5i+6}{16}\right)\sum_{j=1}^{i}C\left(2x_j\right)=\sum_{i=1}^{n}\sum_{1\leq j<k<l\leq i}C\left(x_j+x_k+x_l\right)-\sum_{i=1}^{n}(i-3)\sum_{1\leq j<k\leq i}C\left(x_j+x_k\right)$$

for all $x_1,\cdots,x_n\in X$. Hence C satisfies the cubic functional equation (1.1). In order to prove $C(x)$ is unique, let $C'(x)$ be another cubic functional mapping satisfying (3.1.4) and (3.1.5). Hence,

$$P_{\mu,\nu}(C(x)-C'(x),r)\geq_{L^*} M\left\{P_{\mu,\nu}\left(\frac{C(2^k x)}{2^{3k}}-\frac{h(2^k x)}{2^{3k}},\frac{r}{2}\right),P_{\mu,\nu}\left(\frac{C'(2^k x)}{2^{3k}}-\frac{h(2^k x)}{2^{3k}},\frac{r}{2}\right)\right\}$$

$$\geq_{L^*} P'_{\mu,\nu}\left(\xi\left(2^k x,0,\cdots,0\right),\frac{r\,2^{3k}\,\ell(8-b)}{2\cdot 8}\right)\geq_{L^*} P'_{\mu,\nu}\left(\xi\left(x,0,\cdots,0\right),\frac{r\,2^{3k}\,\ell(8-b)}{2\cdot 8b^{3k}}\right)$$

for all $x\in X$ and all $r>0$. Since $\lim_{k\to\infty}\frac{r\,2^{3k}\,\ell(8-b)}{8b^{3k}}=\infty$,

we obtain $\lim_{k\to\infty}P'_{\mu,\nu}\left(\xi\left(x,0,\cdots,0\right),\frac{r\,2^{3k}\,\ell(8-b)}{8b^{3k}}\right)=1_{L^*}$. Thus $P_{\mu,\nu}(C(x)-C'(x),r)=1_{L^*}$ for all $x\in X$ and all $r>0$, hence $C(x)=C'(x)$. Therefore $C(x)$ is unique.

For $\kappa = -1$, we can prove the result by a similar method. This completes the proof of the theorem.
From Theorem 3.1.1, we obtain the following corollary concerning the stability for the functional equation (1.1).

Corollary 3.1.1. Suppose that a function $h: X \to Y$ satisfies the inequality

$$P_{\mu,\nu}\left(H\left(x_1,\cdots x_n\right),r\right) \geq_{L^*} \begin{cases} P'_{\mu,\nu}\left(\lambda,r\right), & \\ P'_{\mu,\nu}\left(\lambda\sum_{k=1}^{n} x_k^s, r\right), & s \neq 3 \\ P'_{\mu,\nu}\left(\lambda\left(\sum_{k=1}^{n} x_k^{ns} + \prod_{k=1}^{n} x_k^s\right), r\right), & s \neq \dfrac{3}{n} \end{cases} \tag{3.1.20}$$

for all all $x_1,\cdots x_n \in X$ and all $r > 0$, where λ, s are constants with $\lambda > 0$. Then there exists a unique cubic mapping $C: X \to Y$ such that

$$P_{\mu,\nu}\left(h(x)-C(x),r\right) \geq_{L^*} \begin{cases} P'_{\mu,\nu}\left(\lambda, r\ell\right) \\ P'_{\mu,\nu}\left(\lambda\, x_k^s, \dfrac{r\ell\,|8-2^{3s}|}{8}\right) \\ P'_{\mu,\nu}\left(\lambda x_k^{ns}, \dfrac{r\ell\,|8-2^{ns}|}{8}\right) \end{cases} \tag{4.21}$$

for all $x \in X$ and all $r > 0$.

3.2. Stability results: fixed point method

In this section, the authors present the generalized Ulam - Hyers stability of the functional equation (1.1) in IFN - space using fixed point method.
Now we will recall the fundamental result in fixed point theory.

Theorem 3.2.1. [21] (The alternative of fixed point) Suppose that for a complete generalized metric space (X,d) and a strictly contractive mapping $T: X \to X$ with Lipschitz constant L. Then, for each given element $x \in X$, either

$(B1)$ $$d(T^n x, T^{n+1}x) = \infty \quad \forall\ n \geq 0,$$

or

$(B2)$ there exists a natural number n_0 such that:

(i) $d(T^n x, T^{n+1}x) < \infty$ for all $n \geq n_0$;

(ii) The sequence $(T^n x)$ is convergent to a fixed point y^* of T

(iii) y^* is the unique fixed point of T in the set $Y = \{y \in X : d(T^{n_0}x, y) < \infty\}$;

(iv) $d(y^*, y) \leq \dfrac{1}{1-L}\, d(y, Ty)$ for all $y \in Y$.

For to prove the stability result we define the following: a_i is a constant such that

$a_i = \begin{cases} 2 & if \quad i = 0, \\ \dfrac{1}{2} & if \quad i = 1 \end{cases}$ and A is the set such that $\mathrm{A} = \left\{g \mid g: X \to Y, g(0) = 0\right\}$.

Theorem 3.2.2. Let $h: X \to Y$ be a mapping for which there exist a function $\xi: X^n \to Z$ with the condition $\lim_{k\to\infty} P'_{\mu,\nu}\left(\xi\left(a_i^k x_1,\cdots,a_i^k x_n\right),a_i^{3k} r\right) = 1_{L^*}$, $\forall\, x_1,\cdots,x_n \in X$, $r>0$ (3.2.1)

and satisfying the functional inequality

$$P_{\mu,\nu}\left(H\left(x_1,\cdots,x_n\right),r\right) \geq_{L^*} P'_{\mu,\nu}\left(\xi\left(x_1,\cdots,x_n\right),r\right), \quad \forall\, x_1,\cdots,x_n \in X,\ r>0 \tag{3.2.2}$$

If there exists $L = L(i)$ such that the function $x \to \psi(x) = \xi\left(\frac{x}{2},0,\cdots,0\right)$, has the property

$$P'_{\mu,\nu}\left(L\frac{\psi(a_i x)}{a_i^3},r\right) = P'_{\mu,\nu}\left(\psi(x),r\right), \quad \forall\, x\in X, r>0. \tag{3.2.3}$$

Then there exists unique cubic function $C: X \to Y$ satisfying the functional equation (1.1) and $P_{\mu,\nu}\left(h(x)-C(x),r\right) \geq_{L^*} P'_{\mu,\nu}\left(\psi(x),\left(\frac{L^{1-i}}{1-L}\right)\ell r\right)$, $\forall\, x\in X, r>0.$ (3.2.4)

Proof: Let d be a general metric on A, such that

$$d(g,h) = inf\left\{K\in(0,\infty) \mid P_{\mu,\nu}\left(g(x)-h(x),r\right) \geq_{L^*} P'_{\mu,\nu}\left(\psi(x),Kr\right), x\in X,\ r>0\right\}.$$

It is easy to see that (A,d) is complete. Define $\Upsilon: \mathrm{A}\to\mathrm{A}$ by $\Upsilon g(x) = \frac{1}{a_i^3} g(a_i x)$, for all $x\in X$. By [21], we see that Υ is strictly contractive mapping on A with Lipschitz constant L. It follows from (3.1.6) that

$$P_{\mu,\nu}\left(\frac{h(2x)}{2^3}-h(x),\frac{r}{\ell}\right) \geq_{L^*} P'_{\mu,\nu}\left(\xi\left(x,0,\cdots,0\right),r\right), \ \forall \ \ x\in X, r>0. \tag{3.2.5}$$

Replacing r by $r\ell$ in (3.2.5), we arrive

$$P_{\mu,\nu}\left(\frac{h(2x)}{2^3}-h(x),r\right) \geq_{L^*} P'_{\mu,\nu}\left(\xi\left(x,0,\cdots,0\right),r\ell\right), \ \forall \ \ x\in X, r>0. \tag{3.2.6}$$

With the help of (3.2.3), when $i=1$, it follows from (3.2.6), that

$$P_{\mu,\nu}\left(\frac{h(2x)}{2^3}-h(x),r\right) \geq_{L^*} P'_{\mu,\nu}\left(\psi\left(x\right),r\ell\right), \ \forall \ \ x\in X, r>0.$$

$$\Rightarrow \qquad d\left(\Upsilon h,h\right) \leq 1 = L^0 = L^{1-i}. \tag{3.2.7}$$

Replacing x by $\frac{x}{2}$ in (3.2.5), we obtain

$$P_{\mu,\nu}\left(h(x)-2^3 h\left(\frac{x}{2}\right),r\right) \geq_{L^*} P'_{\mu,\nu}\left(\xi\left(\frac{x}{2},0,\cdots,0\right),\frac{\ell r}{2^3}\right), \ \forall \ \ x\in X, r>0. \tag{3.2.8}$$

With the help of (3.2.3), when $i=0$, it follows from (3.2.8), that

$$P_{\mu,\nu}\left(h(x)-2^3 h\left(\frac{x}{2}\right),r\right) \geq_{L^*} P'_{\mu,\nu}\left(\psi(x),L\ell r\right), \quad \forall \ \ x\in X, r>0,$$

$$\Rightarrow \qquad d(h,\Upsilon h) \leq L = L^1 = L^{1-i}. \tag{3.2.9}$$

Then from (3.2.7) and (3.2.9), we can conclude $d\left(h,\Upsilon h\right) \leq L^{1-i} < \infty$.

Now from the fixed point alternative in both cases, it follows that there exists a fixed point C of Υ in A such that

$$C(x)\xrightarrow{P_{\mu,\nu}}\frac{h\left(a_i^k x\right)}{a_i^{3k}},\ k\to\infty,\ \ \forall\ x\in X. \tag{3.2.10}$$

Replacing $\left(x_1,\cdots,x_n\right)$ by $\left(a_i^k x_1,\cdots,a_i^k x_n\right)$ in (3.2.2), we arrive

$$P_{\mu,\nu}\left(\frac{1}{a_i^{3k}}H\left(a_i^k x_1,\cdots,a_i^k x_n\right),r\right)\geq_{L^*} P'_{\mu,\nu}\left(\xi\left(a_i^k x_1,\cdots,a_i^k x_n\right),a_i^{3k}r\right),\quad \forall\ \ x_1,\cdots,x_n\in X, r>0.$$

By proceeding the same procedure in the Theorem 3.1.1, we can prove the function, $C:X\to Y$ is cubic and it satisfies the functional equation (1.1). Since C is unique fixed point of Υ in the set $\mathrm{B}=\{h\in \mathrm{A}\mid d(h,C)<\infty\}$, such that

$$P_{\mu,\nu}\left(h(x)-C(x),r\right)\geq_{L^*} P'_{\mu,\nu}\left(\psi(x),Kr\right),\ \ \forall\ \ x\in X, r>0. \tag{3.2.11}$$

Again using the fixed point alternative, we obtain

$$d\left(h,C\right)\leq\frac{1}{1-L}d\left(h,\Upsilon h\right)\Rightarrow d\left(h,C\right)\leq\frac{L^{1-i}}{1-L}.$$

Hence, we have $P_{\mu,\nu}\left(h(x)-C(x),r\right)\geq_{L^*} P'_{\mu,\nu}\left(\psi(x),\left(\frac{L^{1-i}}{1-L}\right)\ell r\right),\ \forall\ x\in X, r>0.$ (3.2.12)

This completes the proof of the theorem. From Theorem 3.2.2, we obtain the following corollary concerning the stability for the functional equation (1.1).

Corollary 3.2.1. Suppose that a function $h:X\to Y$ satisfies the inequality

$$P_{\mu,\nu}\left(H\left(x_1,\cdots x_n\right),r\right)\geq_{L^*}\begin{cases} P'_{\mu,\nu}\left(\varepsilon,r\right), & \\ P'_{\mu,\nu}\left(\varepsilon\sum_{i=1}^{n}x_i^s,r\right), & s\neq 3;\\ P'_{\mu,\nu}\left(\varepsilon\left(\prod_{i=1}^{n}x_i^s+\sum_{i=1}^{n}x_i^{ns}\right),r\right), & s\neq\frac{3}{n};\end{cases} \tag{3.2.13}$$

for all $x_1,\cdots x_n\in X$ and all $r>0$, where ε, s are constants with $\varepsilon>0$. Then there exists a unique cubic mapping $C:X\to Y$ such that

$$P_{\mu,\nu}\left(h(x)-C(x),r\right)\geq_{L^*}\begin{cases} P'_{\mu,\nu}\left(\varepsilon,\left(\frac{8}{|7|}\right)r\ell\right),\\ P'_{\mu,\nu}\left(\varepsilon x^s,r\ell\left(\frac{2^s}{|8-2^s|}\right)\right)\\ P'_{\mu,\nu}\left(\varepsilon x_i^{ns},r\ell\left(\frac{2^{ns}}{|8-2^{ns}|}\right)\right)\end{cases} \tag{3.2.14}$$

for all $x\in X$ and all $r>0$.

Proof: The proof follows by replacing

$L=2^3\quad for\ i\ =\ 0\ \ and\ \ L\ =\ 2^{-3}\quad for\ \ i\ =\ 1,$

$L=2^{3-s}\quad for\ \ s>3, i\ =\ 0\ \ and\ \ L\ =\ 2^{3-s}\quad for\ \ s<3, i\ =\ 1,$

$L=2^{3-ns}\quad for\ \ s>\frac{3}{n}, i\ =\ 0\ \ and\ \ L\ =\ 2^{3-ns}\quad for\ \ s<\frac{3}{n}, i\ =\ 1,$

in Theorem 3.2.2, we desired our results.

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