

Stability Criteria for BAM Neural Networks With Time–Varying Delays and Leakage Delays

N.Gopalakrishnan¹ and S.Muralisankar²

¹School of Mathematics, Madurai Kamaraj University, Madurai – 625 021, Tamilnadu
India. e-mail: gopukannan.016@gmail.com
Corresponding Author

²School of Mathematics, Madurai Kamaraj University, Madurai – 625 021, Tamilnadu
India. e-mail: muralimku@gmail.com

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Abstract. This paper is concerned with the stability analysis problem of bidirectional associative memory neural networks with time-varying delays and leakage delays. A new sufficient stability criterion is derived for the addressed system in terms of linear matrix inequalities by constructing a Lyapunov-Krasovskii functional and employing delay decomposition approach. Finally, the effectiveness of derived theoretical result is illustrated by a numerical example.

Keywords: BAM neural networks, asymptotic stability, Lyapunov-Krasovskii functional, time-varying delays, linear matrix inequality

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1. Introduction

It is well known that, the Bidirectional Associative Memory (BAM) is a type of recurrent neural networks which was introduced by Kosko [1] in 1988, who generalized the single auto- associative Hebbian correlator to a two-layer pattern-matched heteroassociative circuit. Recently, BAM neural networks have received remarkable consideration because of their potential applications in different fields such as automatic control engineering, image processing, parallel computation, signal processing, optimization and associative memories, and pattern recognition. Since these applications rely on the dynamical behaviors of the equilibrium point of the BAM neural networks, it is very important to investigate the stability of BAM neural networks and a large number of results have been reported, see [2, 3, 5]. Due to the finite switching speed of neuron amplifiers and the finite speed of signal propagation, time delays are unavoidable in very large-scale integration implementation of neural systems. The existence of time delay may lead to some more complicated dynamic behaviors such as oscillation, divergence, chaos, instability or other poor performance of the neural networks. Therefore, the equilibrium and stability analysis of neural networks with time delays have received much interest in recent years; see [2-7, 10]. In many practical problems, the leakage delay exists in the negative feedback term of the system, such term is called leakage term. In fact, the leakage term has also a great impact on the dynamical behavior of neural networks. The authors in [6]

pointed out the time delay in stabilizing negative feedback term has a tendency to destabilize the system. In recent years, the stability analysis with time delay in the leakage term has been studied in [2, 3, 4, 7]. Based on the above discussions, the stability problem for BAM neural networks with time-varying delays and leakage delays have been investigated in this paper. By constructing a Lyapunov-Krasovskii functional and employing a delay decomposition approach, a sufficient stability criterion is derived for the addressed system in terms of linear matrix inequalities (LMIs), which can be easily calculated by MATLAB LMI solver. Finally, a numerical example is provided to show the effectiveness of the proposed method.

Notations: Throughout this paper, \mathbb{R}^n and $\mathbb{R}^{n \times n}$ denotes the n -dimensional Euclidean space and the set of all $n \times n$ real matrices respectively. The superscript T denotes the transpose of the matrix X . The notation $X \geq Y$ (respectively, $X > Y$), where X and Y are symmetric matrices, means that $X - Y$ is positive semi-definite (respectively, positive definite). The notation $*$ always denotes the symmetric block in one symmetric matrix. I_n is the $n \times n$ identity matrix. $\|\cdot\|$ is the Euclidean norm in \mathbb{R}^n .

2. Problem description and preliminaries

Consider the following delayed BAM NNs with time-varying delays and leakage delays described as

$$\begin{aligned} \dot{u}(t) &= -Au(t - \rho) + B_0\tilde{f}(v(t)) + B_1\tilde{f}(v(t - \tau(t))) + I, \\ \dot{v}(t) &= -Cv(t - \sigma) + D_0\tilde{g}(u(t)) + D_1\tilde{g}(u(t - d(t))) + J, \end{aligned} \quad (1)$$

where $u(t) = [u_1(t), \dots, u_n(t)]^T \in \mathbb{R}^n$ and $v(t) = [v_1(t), \dots, v_m(t)]^T \in \mathbb{R}^m$ are neuron state vectors, $A = \text{diag}\{a_1, a_2, \dots, a_n\} > 0$, $C = \text{diag}\{c_1, c_2, \dots, c_m\} > 0$ are diagonal matrices with positive entries $a_i > 0$ and $c_i > 0$, B_0 and D_0 are the connection weight matrices, B_1 and D_1 are the delayed connection weight matrices, $\tilde{f}(v(t)) = [\tilde{f}_1(v_1(t)), \dots, \tilde{f}_m(v_m(t))]^T$, $\tilde{g}(u(t)) = [\tilde{g}_1(u_1(t)), \dots, \tilde{g}_n(u_n(t))]^T$ denote neuron activation functions, $I = [I_1, I_2, \dots, I_n]^T$ and $J = [J_1, J_2, \dots, J_m]^T$ are external inputs, the leakage delays $\rho \geq 0, \sigma \geq 0$ are constants, the time-varying delays $\tau(t)$ and $d(t)$ satisfy $0 \leq \tau(t) \leq \tau$ and $0 \leq d(t) \leq d$, where τ and d are positive constants. The initial conditions of the system (1) are assumed to be $u(s) = \phi(s), s \in [-\tau, 0]$, $v(s) = \varphi(s), s \in [-d, 0]$.

Assumptions:

The neuron activation functions $\tilde{f}_j(\cdot)$ and $\tilde{g}_i(\cdot)$ satisfy $l_j^- \leq \frac{\tilde{f}_j(u) - \tilde{f}_j(v)}{u - v} \leq l_j^+, k_i^- \leq \frac{\tilde{g}_i(u) - \tilde{g}_i(v)}{u - v} \leq k_i^+$, for any $u, v \in \mathbb{R}, u \neq v$, where l_j^-, l_j^+, k_i^- and k_i^+ are positive real constants.

Assume that the neural network (1) has only one equilibrium point $u^* = [u_1^*, u_2^*, \dots, u_n^*], v^* = [v_1^*, v_2^*, \dots, v_m^*]$. Then, we will shift the equilibrium points u^* and v^* to the origin. By using the transformation $y(t) = u(t) - u^*$ and $z(t) = v(t) - v^*$, the system (1) into the following form:

$$\begin{aligned} \dot{y}(t) &= -Ay(t - \rho) + B_0f(z(t)) + B_1f(z(t - \tau(t))), \\ \dot{z}(t) &= -Cz(t - \sigma) + D_0g(y(t)) + D_1g(y(t - d(t))), \end{aligned} \quad (2)$$

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Where $f(z(t)) = [f_1(z_1(t)), \dots, f_m(z_m(t))]^T$, $g(y(t)) = [g_1(y_1(t)), \dots, g_n(y_n(t))]^T$,
 $f_j(z_j(t)) = \tilde{f}_j(z_j(t) + v^*) - \tilde{f}_j(v^*)$, $g_i(y_i(t)) = \tilde{g}_i(y_i(t) + u^*) - \tilde{g}_i(u^*)$. According
to the Assumption 2.1, one can obtain that $l_j^- \leq \frac{f_j(u)}{u} \leq l_j^+$, $k_i^- \leq \frac{g_i(u)}{u} \leq k_i^+$, (3)
 $f_j(0) = g_i(0) = 0$, $i = 1, 2, \dots, n$, $j = 1, 2, \dots, m$. In order to obtain the main result, we
need the following lemmas.

Lemma 2.1. [8] For any constant matrix $M \in \mathbb{R}^{n \times n}$, $M = M^T > 0$, scalar $\eta > 0$, vector
function $\omega: [0, \eta] \rightarrow \mathbb{R}^n$ such that the integrations are well defined, the following
inequality holds: $(\int_0^\eta \omega(s) ds)^T M (\int_0^\eta \omega(s) ds) \leq \eta \int_0^\eta \omega^T(s) M \omega(s) ds$.

Lemma 2.2. [9] Let M, P, Q be the given matrices such that $Q > 0$, then

$$\begin{bmatrix} P & M^T \\ M & -Q \end{bmatrix} < 0 \Leftrightarrow P + M^T Q^{-1} M < 0.$$

Lemma 2.4. [10] Let D and N be real constant matrices of appropriate dimensions, matrix
 $F(t)$ satisfies $F^T(t)F(t) \leq I$.

Then (i) for any scalar $\varepsilon > 0$, $DF(t)N + N^T F^T(t)D^T \leq \varepsilon^{-1}DD^T + \varepsilon NN^T$, (ii) For any
 $P > 0$, $2a^T b \leq a^T P^{-1} a + b^T P b$.

3. Main results

Using a simple transformation, the model (2) has an equivalent form as follows

$$\begin{aligned} \frac{d}{dt} \left[y(t) - A \int_{t-\rho}^t y(s) ds \right] &= -Ay(t) + B_0 f(z(t)) + B_1 f(z(t - \tau(t))), \\ \frac{d}{dt} \left[z(t) - C \int_{t-\sigma}^t z(s) ds \right] &= -Cz(t) + D_0 g(y(t)) + D_1 g(y(t - d(t))). \end{aligned} \quad (4)$$

For representation convenience, we use the following notations:

$$\begin{aligned} L_1 &= \text{diag}\{l_1^- l_1^+, \dots, l_n^- l_n^+\}, & L_2 &= \text{diag}\left\{\frac{l_1^- + l_1^+}{2}, \dots, \frac{l_n^- + l_n^+}{2}\right\}, \\ K_1 &= \text{diag}\{k_1^- k_1^+, \dots, k_n^- k_n^+\}, & K_2 &= \text{diag}\left\{\frac{k_1^- + k_1^+}{2}, \dots, \frac{k_n^- + k_n^+}{2}\right\}. \end{aligned}$$

Now, a delay-dependent stability analysis of delayed BAM NNs (4) is given in the
following theorem.

Theorem 3.1. For given scalars $\tau > 0, d > 0, \rho > 0, \sigma > 0, 0 < \delta < 1$ and $0 < \gamma < 1$,
the delayed BAM neural networks (4) are globally asymptotically stable, if there exist
symmetric positive definite matrices $P_1 > 0, P_2 > 0, Q_a > 0$ ($a = 1, 2, \dots, 8$), $R_b >$
 0 ($b = 1, 2, \dots, 6$), $M_k > 0, N_k > 0, E_k > 0$, ($k = 1, 2$), positive diagonal matrices
 W_l ($l = 1, 2, 3, 4$) and real matrices U_1, U_2 of appropriate dimensions such that the
following LMIs hold:

$$\Theta_1 = \begin{bmatrix} \Theta_{i,j} & \Gamma_1 & \Gamma_2 & \Gamma_3 \\ * & -M_1 & 0 & 0 \\ * & * & -N_2 & 0 \\ * & * & * & -E_1 \end{bmatrix} < 0, \quad \Theta_2 = \begin{bmatrix} \Theta_{i,j} & \Gamma_1 & \Gamma_2 & \Gamma_3 \\ * & -M_1 & 0 & 0 \\ * & * & -N_2 & 0 \\ * & * & * & -E_1 \end{bmatrix} < 0,$$

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$$\Psi_1 = \begin{bmatrix} \Psi_{i,j} & \Sigma_1 & \Sigma_2 & \Sigma_3 \\ * & -M_2 & 0 & 0 \\ * & * & -N_1 & 0 \\ * & * & * & -E_2 \end{bmatrix} < 0, \quad \Psi_2 = \begin{bmatrix} \Psi_{i,j} & \Sigma_1 & \Sigma_2 & \Sigma_3 \\ * & -M_2 & 0 & 0 \\ * & * & -N_1 & 0 \\ * & * & * & -E_2 \end{bmatrix} < 0,$$

where $\Theta_{9 \times 9}$ with entries :

$$\begin{aligned} \Theta_{1,1} &= -2P_1A + Q_1 + Q_3 + Q_5 + \rho R_5 - 2L_1W_1, \quad \Theta_{1,7} = 2L_2W_1, \\ \Theta_{1,9} &= A^T P_1 A, \quad \Theta_{2,2} = -2L_1W_2, \quad \Theta_{2,8} = 2L_2W_2, \quad \Theta_{3,3} = -Q_1 - Q_6, \quad \Theta_{4,4} = -Q_3, \\ \Theta_{4,6} &= -A^T U_1^T, \quad \Theta_{5,5} = -Q_5 + Q_6, \quad \Theta_{6,6} = \delta d R_1 + (d - \delta d) R_2 - 2U_1, \\ \Theta_{7,7} &= N_1 - 2W_1 + E_2, \quad \Theta_{8,8} = N_1 - 2W_2 + E_2, \quad \Theta_{9,9} = A^T M_2 A - \frac{1}{\rho} R_5, \\ \Gamma_1 &= [P_1(B_0 + B_1) \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0], \\ \Gamma_2 &= [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad D_0^T P_2 \quad D_1^T P_2 \quad 0], \\ \Gamma_3 &= [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad U_1(B_0 + B_1) \quad 0 \quad 0 \quad 0]. \end{aligned}$$

LMI 1 entries:

$$\begin{aligned} \Theta_{1,1} &= \frac{-1}{\delta d} R_1, \quad \Theta_{1,2} = \frac{1}{\delta d} R_1, \quad \Theta_{2,2} = -\frac{2}{\delta d} R_1, \quad \Theta_{2,5} = \frac{1}{\delta d} R_1, \quad \Theta_{3,3} = -\frac{1}{d-\delta d} R_2, \\ \Theta_{3,5} &= \frac{1}{d-\delta d} R_2, \quad \Theta_{5,5} = -\frac{1}{\delta d} R_1 - \frac{1}{d-\delta d} R_2. \end{aligned}$$

LMI 2 entries:

$$\begin{aligned} \Theta_{1,1} &= \frac{-1}{\delta d} R_1, \quad \Theta_{1,5} = \frac{1}{\delta d} R_1, \quad \Theta_{2,2} = -\frac{2}{d-\delta d} R_2, \quad \Theta_{2,3} = \frac{1}{d-\delta d} R_2, \\ \Theta_{2,5} &= \frac{1}{d-\delta d} R_2, \quad \Theta_{3,3} = -\frac{1}{d-\delta d} R_2, \quad \Theta_{5,5} = -\frac{1}{\delta d} R_1 - \frac{1}{d-\delta d} R_2. \end{aligned}$$

$\Psi_{9 \times 9}$ with entries :

$$\begin{aligned} \Psi_{1,1} &= -2P_2C + Q_2 + Q_4 + Q_7 + \sigma R_6 - 2K_1W_3, \quad \Psi_{1,7} = 2K_1W_3, \\ \Psi_{1,9} &= C^T P_2 C, \quad \Psi_{2,2} = -2K_1W_4, \quad \Psi_{2,8} = 2K_2W_4, \quad \Psi_{3,3} = -Q_2 - Q_8, \quad \Psi_{4,4} = -Q_4, \\ \Psi_{4,6} &= -C^T U_2^T, \quad \Psi_{5,5} = -Q_7 + Q_8, \quad \Psi_{6,6} = \gamma \tau R_3 + (\tau - \gamma \tau) R_4 - 2U_2, \\ \Psi_{7,7} &= M_1 - 2W_3 + E_1, \quad \Psi_{8,8} = M_1 - 2W_4 + E_1, \quad \Psi_{9,9} = C^T N_2 C - \frac{1}{\sigma} R_6, \\ \Sigma_1 &= [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad B_0^T P_1 \quad B_1^T P_1 \quad 0], \\ \Sigma_2 &= [P_2(D_0 + D_1) \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0], \\ \Sigma_3 &= [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad U_2(D_0 + D_1) \quad 0 \quad 0 \quad 0]. \end{aligned}$$

LMI 3 entries:

$$\begin{aligned} \Psi_{1,1} &= \frac{-1}{\gamma \tau} R_3, \quad \Psi_{1,2} = \frac{1}{\gamma \tau} R_3, \quad \Psi_{2,2} = -\frac{2}{\gamma \tau} R_3, \quad \Psi_{2,5} = \frac{1}{\gamma \tau} R_3, \quad \Psi_{3,3} = -\frac{1}{\tau - \gamma \tau} R_4, \\ \Psi_{3,5} &= \frac{1}{\tau - \gamma \tau} R_4, \quad \Psi_{5,5} = -\frac{1}{\gamma \tau} R_3 - \frac{1}{\tau - \gamma \tau} R_4. \end{aligned}$$

LMI 4 entries:

$$\begin{aligned} \Psi_{1,1} &= \frac{-1}{\gamma \tau} R_3, \quad \Psi_{1,5} = \frac{1}{\gamma \tau} R_3, \quad \Psi_{2,2} = -\frac{2}{\tau - \gamma \tau} R_4, \quad \Psi_{2,3} = \frac{1}{\tau - \gamma \tau} R_4, \quad \Psi_{2,5} = \frac{1}{\tau - \gamma \tau} R_4, \\ \Psi_{3,3} &= -\frac{1}{\tau - \gamma \tau} R_4, \quad \Psi_{5,5} = -\frac{1}{\gamma \tau} R_3 - \frac{1}{\tau - \gamma \tau} R_4. \end{aligned}$$

The remaining entries are zero.

Proof. Choose the Lyapunov-Krasovskii functional as follows

$$V(t) = V_1(t) + V_2(t) + V_3(t) + V_4(t), \quad (5)$$

$$\text{where } V_1(t) = \left[y(t) - A \int_{t-\rho}^t y(s) ds \right]^T P_1 \left[y(t) - A \int_{t-\rho}^t y(s) ds \right]$$

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$$\begin{aligned}
& + \left[z(t) - C \int_{t-\sigma}^t z(s) ds \right]^T P_2 \left[z(t) - C \int_{t-\sigma}^t z(s) ds \right], \\
V_2(t) &= \int_{t-d}^t y^T(s) Q_1 y(s) ds + \int_{t-\tau}^t z^T(s) Q_2 z(s) ds + \int_{t-\rho}^t y^T(s) Q_3 y(s) ds \\
& + \int_{t-\sigma}^t z^T(s) Q_4 z(s) ds, \\
V_3(t) &= \int_{t-\delta d}^t y^T(s) Q_5 y(s) ds + \int_{t-d}^{t-\delta d} y^T(s) Q_6 y(s) ds + \int_{t-\gamma\tau}^t z^T(s) Q_7 z(s) ds \\
& + \int_{t-\tau}^{t-\gamma\tau} z^T(s) Q_8 z(s) ds, \\
V_4(t) &= \int_{-\delta d}^0 \int_{t+\theta}^t \dot{y}^T(s) R_1 \dot{y}(s) ds d\theta + \int_{-d}^{-\delta d} \int_{t+\theta}^t \dot{y}^T(s) R_2 \dot{y}(s) ds d\theta \\
& + \int_{-\gamma\tau}^0 \int_{t+\theta}^t \dot{z}^T(s) R_3 \dot{z}(s) ds d\theta + \int_{-\tau}^{-\gamma\tau} \int_{t+\theta}^t \dot{z}^T(s) R_4 \dot{z}(s) ds d\theta, \\
& \int_{-\rho}^0 \int_{t+\theta}^t y^T(s) R_5 y(s) ds d\theta + \int_{-\sigma}^0 \int_{t+\theta}^t z^T(s) R_6 z(s) ds d\theta.
\end{aligned}$$

Let \mathcal{L} be the infinitesimal operator of $V(t)$ and using Lemma 2.1, we have

$$\mathcal{L}V(t) = \mathcal{L}V_1(t) + \mathcal{L}V_2(t) + \mathcal{L}V_3(t) + \mathcal{L}V_4(t), \quad (6)$$

where

$$\begin{aligned}
\mathcal{L}V_1(t) &= 2 \left[y(t) - A \int_{t-\rho}^t y(s) ds \right]^T P_1 \left[-Ay(t) + B_0 f(z(t)) + B_1 f(z(t-\tau(t))) \right] \\
& \left[z(t) - C \int_{t-\sigma}^t z(s) ds \right]^T P_2 \left[-Cz(t) + D_0 g(y(t)) + D_1 g(y(t-d(t))) \right], \quad (7)
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}V_2(t) &= y^T(t) [Q_1 + Q_3] y(t) - y^T(t-d) Q_1 y^T(t-d) - y^T(t-\rho) Q_3 y^T(t-\rho) \\
& + z^T(t) [Q_2 + Q_4] z(t) - z^T(t-\tau) Q_2 z^T(t-\tau) - z^T(t-\sigma) Q_4 z^T(t-\sigma), \quad (8)
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}V_3(t) &= y^T(t) Q_5 y(t) - y^T(t-\delta d) Q_5 y(t-\delta d) + y^T(t-\delta d) Q_6 y(t-\delta d) \\
& - y^T(t-d) Q_6 y(t-d) + z^T(t) Q_7 z(t) - z^T(t-\gamma\tau) Q_7 z(t-\gamma\tau) \\
& + z^T(t-\gamma\tau) Q_8 z(t-\gamma\tau) - z^T(t-\tau) Q_8 z(t-\tau), \quad (9)
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}V_4(t) &= \delta d \dot{y}^T(t) R_1 \dot{y}(t) - \int_{t-\delta d}^t \dot{y}^T(s) R_1 \dot{y}(s) ds + (d - \delta d) \dot{y}^T(t) R_2 \dot{y}(t) \\
& - \int_{t-d}^{t-\delta d} \dot{y}^T(s) R_2 \dot{y}(s) ds + \gamma\tau \dot{z}^T(t) R_3 \dot{z}(t) - \int_{t-\gamma\tau}^t \dot{z}^T(s) R_3 \dot{z}(s) ds \\
& + (\tau - \gamma\tau) \dot{z}^T(t) R_4 \dot{z}(t) - \int_{t-\tau}^{t-\gamma\tau} \dot{z}^T(s) R_4 \dot{z}(s) ds + \rho y^T(t) R_5 y(t) \\
& - \frac{1}{\rho} \left(\int_{t-\rho}^t y(s) ds \right)^T R_5 \left(\int_{t-\rho}^t y(s) ds \right) + \sigma z^T(t) R_6 z(t) \\
& - \frac{1}{\sigma} \left(\int_{t-\sigma}^t z(s) ds \right) R_6 \left(\int_{t-\sigma}^t z(s) ds \right). \quad (10)
\end{aligned}$$

In addition, for any $n \times n$ diagonal matrices $W_l > 0$ ($l = 1, 2, 3, 4$), the following inequalities hold:

$$\begin{bmatrix} y(t) \\ g(y(t)) \end{bmatrix} \begin{bmatrix} L_1 W_1 & -L_2 W_1 \\ -L_2 W_1 & W_1 \end{bmatrix} \begin{bmatrix} y(t) \\ g(y(t)) \end{bmatrix} \leq 0, \quad (11)$$

$$\begin{bmatrix} y(t-d(t)) \\ g(y(t-d(t))) \end{bmatrix} \begin{bmatrix} L_1 W_2 & -L_2 W_2 \\ -L_2 W_2 & W_2 \end{bmatrix} \begin{bmatrix} y(t-d(t)) \\ g(y(t-d(t))) \end{bmatrix} \leq 0, \quad (12)$$

$$\begin{bmatrix} z(t) \\ f(z(t)) \end{bmatrix} \begin{bmatrix} K_1 W_3 & -K_2 W_3 \\ -K_2 W_3 & W_3 \end{bmatrix} \begin{bmatrix} z(t) \\ f(z(t)) \end{bmatrix} \leq 0, \quad (13)$$

$$\begin{bmatrix} z(t-\tau(t)) \\ f(z(t-\tau(t))) \end{bmatrix} \begin{bmatrix} K_1 W_4 & -K_2 W_4 \\ -K_2 W_4 & W_4 \end{bmatrix} \begin{bmatrix} z(t-\tau(t)) \\ f(z(t-\tau(t))) \end{bmatrix} \leq 0, \quad (14)$$

Furthermore, the following equality holds for any real matrices U_1 and U_2 with compatible dimensions

$$0 = 2\dot{y}^T(t)U_1 \left\{ -Ay(t - \rho) + B_0f(z(t)) + B_1f(z(t - \tau(t))) - \dot{y}(t) \right\}, \quad (15)$$

$$0 = 2\dot{z}^T(t)U_2 \left\{ -Cz(t - \sigma) + D_0g(y(t)) + D_1g(y(t - d(t))) - \dot{z}(t) \right\}. \quad (16)$$

Case 1: If $0 \leq d(t) \leq \delta d$, we have

$$\begin{aligned} & - \int_{t-\delta d}^t \dot{y}^T(s)R_1\dot{y}(s)ds - \int_{t-d}^{t-\delta d} \dot{y}^T(s)R_2\dot{y}(s)ds \\ &= - \int_{t-d(t)}^t \dot{y}^T(s)R_1\dot{y}(s)ds - \int_{t-\delta d}^{t-d(t)} \dot{y}^T(s)R_1\dot{y}(s)ds - \int_{t-d}^{t-\delta d} \dot{y}^T(s)R_2\dot{y}(s)ds. \end{aligned}$$

Note that $R_1 > 0$ and from Lemma 2.1, it follows

$$- \int_{t-d(t)}^t \dot{y}^T(s)R_1\dot{y}(s)ds \leq \frac{-1}{\delta d} [y(t) - y(t - d(t))]R_1[y(t) - y(t - d(t))], \quad (17)$$

$$- \int_{t-\delta d}^{t-d(t)} \dot{y}^T(s)R_1\dot{y}(s)ds \leq \frac{-1}{\delta d} [y(t - d(t)) - y(t - \delta d)]R_1[y(t - d(t)) - y(t - \delta d)], \quad (18)$$

$$- \int_{t-d}^{t-\delta d} \dot{y}^T(s)R_2\dot{y}(s)ds \leq \frac{-1}{d-\delta d} [y(t - \delta d) - y(t - d)]R_2[y(t - \delta d) - y(t - d)]. \quad (19)$$

Substituting (7) – (10), (15) – (19) into (6) and subtracting (11) – (14) from (6), we obtain

$$\begin{aligned} \mathcal{L}V(t) \leq & \zeta_1^T(t) \{ \Theta_{i,j} + P_1(B_0 + B_1)M_1^{-1}(B_0 + B_1)^T P_1 + (D_0^T + D_1^T)P_2N_2^{-1}P_2(D_0 + D_1) \\ & + U_1(B_0 + B_1)E_1^{-1}(B_0 + B_1)^T U_1 \} \zeta_1(t). \end{aligned}$$

Then, by Lemma 2.4, we have

$$\mathcal{L}V(t) \leq \zeta_1^T(t)\Theta_1\zeta_1(t), \quad (20)$$

$$\text{where } \zeta_1^T(t) = [y^T(t) \quad y^T(t - d(t)) \quad y^T(t - d) \quad y^T(t - \rho) \quad y^T(t - \delta d) \quad y^T(t) \quad g^T(y(t)) \quad g^T(y(t - d(t))) \quad \left(\int_{t-\rho}^t y(s)ds \right)^T]^T,$$

$$\text{and } \Theta_1 = \Theta_{i,j} + P_1(B_0 + B_1)M_1^{-1}(B_0 + B_1)^T P_1 + (D_0^T + D_1^T)P_2N_2^{-1}P_2(D_0 + D_1) + U_1(B_0 + B_1)E_1^{-1}(B_0 + B_1)^T U_1.$$

Case 2: If $\delta d \leq d(t) \leq d$, we have

$$\begin{aligned} & - \int_{t-\delta d}^t \dot{y}^T(s)R_1\dot{y}(s)ds - \int_{t-d}^{t-\delta d} \dot{y}^T(s)R_2\dot{y}(s)ds \\ &= - \int_{t-\delta d}^t \dot{y}^T(s)R_1\dot{y}(s)ds - \int_{t-d}^{t-d(t)} \dot{y}^T(s)R_2\dot{y}(s)ds - \int_{t-d(t)}^{t-\delta d} \dot{y}^T(s)R_2\dot{y}(s)ds. \end{aligned}$$

Note that $R_2 > 0$ and from Lemma 2.1, it is similar as Case 1, we have

$$\mathcal{L}V(t) \leq \zeta_1^T(t)\Theta_2\zeta_1(t). \quad (21)$$

Case 3: If $0 \leq \tau(t) \leq \gamma\tau$, we have

$$\begin{aligned} & - \int_{t-\gamma\tau}^t \dot{z}^T(s)R_3\dot{z}(s)ds - \int_{t-\tau}^{t-\gamma\tau} \dot{z}^T(s)R_4\dot{z}(s)ds \\ &= - \int_{t-\tau(t)}^t \dot{z}^T(s)R_3\dot{z}(s)ds - \int_{t-\gamma\tau}^{t-\tau(t)} \dot{z}^T(s)R_3\dot{z}(s)ds - \int_{t-\tau}^{t-\gamma\tau} \dot{z}^T(s)R_4\dot{z}(s)ds. \end{aligned}$$

Note that $R_3 > 0$ and from Lemma 2.1, it follows

$$- \int_{t-\tau(t)}^t \dot{z}^T(s)R_3\dot{z}(s)ds \leq \frac{-1}{\gamma\tau} [z(t) - z(t - \tau(t))]R_3[z(t) - z(t - \tau(t))], \quad (22)$$

$$- \int_{t-\gamma\tau}^{t-\tau(t)} \dot{z}^T(s)R_3\dot{z}(s)ds \leq \frac{-1}{\gamma\tau} [z(t - \tau(t)) - z(t - \gamma\tau)]R_3 [z(t - \tau(t)) - z(t - \gamma\tau)], \quad (23)$$

$$- \int_{t-\tau}^{t-\gamma\tau} \dot{z}^T(s)R_4\dot{z}(s)ds \leq \frac{-1}{\tau-\gamma\tau} [z(t - \gamma\tau) - z(t - \tau)]R_4[z(t - \gamma\tau) - z(t - \tau)]. \quad (24)$$

Substituting (7) – (10), (15), (16), (22) – (24) into (6) and subtracting (11) – (14) from (6), we obtain

$$\begin{aligned} \mathcal{L}V(t) \leq & \zeta_2^T(t) \{ \Psi_{i,j} + (B_0 + B_1)^T P_1 M_2^{-1} P_1 (B_0 + B_1) + P_2 (D_0 + D_1) N_1^{-1} (D_0^T + D_1^T) P_2 \\ & + U_2 (D_0 + D_1) E_2^{-1} (D_0 + D_1)^T U_2 \} \zeta_2(t). \end{aligned}$$

Then, by Lemma 2.4, we have

$$\mathcal{L}V(t) \leq \zeta_2^T(t)\Psi_1\zeta_2(t), \quad (25)$$

$$\text{where } \zeta_2^T(t) = [z^T(t) \quad z^T(t - \tau(t)) \quad z^T(t - \tau) \quad z^T(t - \sigma) \quad z^T(t - \gamma\tau) \quad \dot{z}^T(t) \\ f^T(z(t)) \quad f^T(z(t - \tau(t))) \quad \left(\int_{t-\sigma}^t z(s)ds\right)^T],$$

$$\text{and } \Psi_1 = \Psi_{i,j} + (B_0 + B_1)^T P_1 M_2^{-1} P_1 (B_0 + B_1) + P_2 (D_0 + D_1) N_1^{-1} (D_0^T + D_1^T) P_2 \\ + U_2 (D_0 + D_1) E_2^{-1} (D_0 + D_1)^T U_2.$$

Case 4: If $\gamma\tau \leq \tau(t) \leq \tau$, we have

$$- \int_{t-\gamma\tau}^t \dot{z}^T(s) R_3 \dot{z}(s) ds - \int_{t-\tau}^{t-\gamma\tau} \dot{z}^T(s) R_4 \dot{z}(s) ds \\ = - \int_{t-\gamma\tau}^t \dot{z}^T(s) R_3 \dot{z}(s) ds - \int_{t-\tau(t)}^{t-\gamma\tau} \dot{z}^T(s) R_4 \dot{z}(s) ds - \int_{t-\tau}^{t-\tau(t)} \dot{z}^T(s) R_4 \dot{z}(s) ds.$$

Note that $R_4 > 0$ and from Lemma 2.1, it is similar as Case 3, we have

$$\mathcal{L}V(t) \leq \zeta_2^T(t)\Psi_2\zeta_2(t). \quad (26)$$

Hence, from (20) and (21), we have

$$\mathcal{L}V(t) \leq -\zeta_1^T(t)\{\Theta_i^*\}\zeta_1(t), \quad \forall i = 1, 2, \quad (27)$$

where $\Theta_i^* = -\Theta_i > 0$.

Hence, from (25) and (26), we have

$$\mathcal{L}V(t) \leq -\zeta_2^T(t)\{\Psi_i^*\}\zeta_2(t), \quad \forall i = 1, 2, \quad (28)$$

where $\Psi_i^* = -\Psi_i > 0$.

Taking expectation on both sides of (27) and (28) and integrating from 0 to t , we get

$$\mathbb{E}[V(t)] + \int_0^t \mathbb{E}[\zeta_1^T(s)\Theta_i^*\zeta_1(s) + \zeta_2^T(s)\Psi_i^*\zeta_2(s)] ds \leq \mathbb{E}[V(0)] < \infty, t \geq 0, \forall i = 1, 2.$$

Applying Lemma 2.1, we have

$$\mathbb{E}\left\{\left\|A \int_{t-\rho}^t y(s)ds\right\|^2\right\} = \mathbb{E}\left\{\left[A \int_{t-\rho}^t y(s)ds\right]^T \left[A \int_{t-\rho}^t y(s)ds\right]\right\} \\ \leq \lambda_{\max}(A^2) \mathbb{E}\left\{\left[\int_{t-\rho}^t y(s)ds\right]^T \left[\int_{t-\rho}^t y(s)ds\right]\right\} \\ \leq \frac{\lambda_{\max}(A^2)}{\lambda_{\min}(Q_3)} \mathbb{E}\left\{\left[\int_{t-\rho}^t y(s)ds\right]^T \left[\int_{t-\rho}^t y(s)ds\right]\right\} \\ \leq \rho \frac{\lambda_{\max}(A^2)}{\lambda_{\min}(Q_3)} \mathbb{E}\left\{\left[\int_{t-\rho}^t y^T(s)Q_3y(s)ds\right]\right\} \leq \rho \frac{\lambda_{\max}(A^2)}{\lambda_{\min}(Q_3)} \mathbb{E}[V(t)] \\ \leq \rho \frac{\lambda_{\max}(A^2)}{\lambda_{\min}(Q_3)} \mathbb{E}[V(0)], \quad t \geq 0.$$

Similarly, $\mathbb{E}\left\{\left\|C \int_{t-\sigma}^t z(s)ds\right\|^2\right\} \leq \sigma \frac{\lambda_{\max}(C^2)}{\lambda_{\min}(Q_4)} \mathbb{E}[V(0)]$, $t \geq 0$. Further

$$\mathbb{E}\left\{\left\|y(s) - A \int_{t-\rho}^t y(s)ds\right\|^2\right\} = \mathbb{E}\left\{\left[y(s) - A \int_{t-\rho}^t y(s)ds\right]^T \left[y(s) - A \int_{t-\rho}^t y(s)ds\right]\right\} \\ \leq \frac{\mathbb{E}[V_1(t)]}{\lambda_{\min}(P_1)} \leq \frac{\mathbb{E}[V(t)]}{\lambda_{\min}(P_1)} \leq \frac{\mathbb{E}[V(0)]}{\lambda_{\min}(P_1)}.$$
 Hence, it can be obtained that

$$\mathbb{E}\{\|y(t)\|^2\} = \mathbb{E}\left\{\left\|y(t) - A \int_{t-\rho}^t y(s)ds + A \int_{t-\rho}^t y(s)ds\right\|^2\right\} \leq 2\mathbb{E}\left\{\left\|A \int_{t-\rho}^t y(s)ds\right\|^2\right\}$$

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$$+2\mathbb{E}\left\{\left\|y(t) - A \int_{t-\rho}^t y(s)ds\right\|^2\right\} \leq \rho \frac{\lambda_{\max}(A^2)}{\lambda_{\min}(Q_3)} \mathbb{E}[V(0)] + 2 \frac{\mathbb{E}[V(0)]}{\lambda_{\min}(P_1)} < \infty, \quad t \geq 0.$$

$$\text{Similarly, } \mathbb{E}\{\|z(t)\|^2\} \leq \sigma \frac{\lambda_{\max}(C^2)}{\lambda_{\min}(Q_4)} \mathbb{E}[V(0)] + 2 \frac{\mathbb{E}[V(0)]}{\lambda_{\min}(P_2)} < \infty, \quad t \geq 0.$$

$$\mathbb{E}\{\|y(t)\|^2 + \|z(t)\|^2\} \leq \left(\rho \frac{\lambda_{\max}(A^2)}{\lambda_{\min}(Q_3)} + \sigma \frac{\lambda_{\max}(C^2)}{\lambda_{\min}(Q_4)} + \frac{2}{\min\{\lambda_{\min}(P_1), \lambda_{\min}(P_2)\}}\right) \times [\alpha_1 \mathbb{E}\|\phi\|_{\tau^*}^2 + \alpha_2 \mathbb{E}\|\varphi\|_{d^*}^2] < \infty,$$

where

$$\begin{aligned} \alpha_1 &= \lambda_{\max}(P_1)(1 + \rho^2 \max_{i \in \Lambda} a_i) + d\lambda_{\max}(Q_1) + \rho\lambda_{\max}(Q_3) + \delta d\lambda_{\max}(Q_5) \\ &\quad (d - \delta d)\lambda_{\max}(Q_5) + (\delta d)^2\lambda_{\max}(R_1) + (d - \delta d)^2\lambda_{\max}(R_2), \\ \alpha_2 &= \lambda_{\max}(P_2)(1 + \sigma^2 \max_{i \in \Lambda} c_i) + \tau\lambda_{\max}(Q_2) + \sigma\lambda_{\max}(Q_4) + \gamma\tau\lambda_{\max}(Q_6) \\ &\quad (\tau - \gamma\tau)\lambda_{\max}(Q_7) + (\gamma\tau)^2\lambda_{\max}(R_3) + (\tau - \gamma\tau)^2\lambda_{\max}(R_4). \end{aligned}$$

This implies that the trivial solution of (4) is locally stable. Thus, considering the continuity of activation function $f(\cdot), g(\cdot)$, the solutions $y(t) = y(t, 0, \phi)$ and $z(t) = z(t, 0, \varphi)$ of system (4) is bounded on $[0, \infty)$. Considering (4), we know that $\mathbb{E}\|y(t)\|^2$ and $\mathbb{E}\|z(t)\|^2$ are bounded on $[0, \infty)$, which leads to the uniform continuity of the solution $y(t)$ and $z(t)$ on $[0, \infty)$. By Barbalat's lemma [6], it holds that $\mathbb{E}\|y(t)\|^2 \rightarrow 0$ and $\mathbb{E}\|z(t)\|^2 \rightarrow 0$ as $t \rightarrow \infty$. Hence the system (4) is globally asymptotically stable in the mean square.

4. Numerical example

In this section, a numerical example is provided to illustrate the effectiveness of the proposed method.

Consider the delayed BAM neural networks (4) with the following parameters

$$\begin{aligned} A &= \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}, B_0 = \begin{bmatrix} 0.4 & 0.3 \\ -0.4 & 0.5 \end{bmatrix}, B_1 = \begin{bmatrix} -0.4 & -0.3 \\ -0.4 & -0.2 \end{bmatrix}, C = \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix}, D_0 = \begin{bmatrix} -0.5 & 0.4 \\ -0.6 & -0.5 \end{bmatrix}, \\ D_1 &= \begin{bmatrix} -0.6 & -0.5 \\ 0.4 & -0.6 \end{bmatrix}, L_1 = 0I, \quad L_2 = 0.5I, \quad K_1 = 0I, \quad K_2 = 0.5I. \end{aligned}$$

The activation functions are described by $g(y(t)) = \frac{1}{2}[|y(t) + 1| - |y(t) - 1|]$ and

$f(z(t)) = \frac{1}{2}[|z(t) + 1| - |z(t) - 1|]$. The time-varying delays are taken as

$d(t) = 0.25 + 0.25\sin(t)$ and $\tau(t) = 0.25 + 0.25\sin(t)$. The leakage delay and the time-varying delays satisfy $\rho = 0.1, \sigma = 0.1, d = 0.5, \tau = 0.5, \gamma = 0.1$ and $\delta = 0.1$.

By using the Matlab LMI solver, in order to see that the LMIs given in Theorem 3.1 is feasible. Therefore, it follows from Theorem 3.1 that the delayed BAM neural network (4) is globally asymptotically stable in the mean square.

5. Conclusion

In this paper, the stability problem of BAM neural networks with time-varying delays and leakage delays has been studied. By constructing a suitable Lyapunov-Krasovskii functional and employing a delay decomposition approach, a sufficient stability criterion has been obtained for the given addressed system. These conditions are expressed in terms of LMIs, which can be easily calculated by MATLAB LMI solver. Finally, a numerical example has been provided to show the effectiveness of the proposed method.

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