Intern. J. Fuzzy Mathematical Archive
Vol. 6, No. 2, 2015,111-118
ISSN: 2320 –3242 (P), 2320 –3250 (online)
Published on 22 January 2015
www.researchmathsci.org

International Journal of
Fuzzy Mathematical Archive

Global Stability of Nonlinear Transports Equations

***S.Balamuralitharan*[1] and *D.Seethalakshmi*[2]**

[1]Faculty of Engineering and Technology, Department of Mathematics
SRM University, Kattankulathur - 603 203,Tamil Nadu, India.
E-mail: balamurali.maths@gmail.com
Corresponding Author

[2]Department of Mathematics, CIPET, Bhubaneswar, Odissa, India
E-mail: seetha_subash@rediffmail.com

Received 22 October 2014; accepted 29 November 2014

Abstract. The aim of this paper is to study the global stability and associated delay differential equation of the nonlinear transports equations

$$\frac{\partial v}{\partial t}+g(x)\frac{\partial v}{\partial x}=f\left(t,v(x,t),v(h(x),t-\tau)\right)$$

Applying suitable local stability we present new criteria for asymptotically stable behaviors of nonlinear transport solutions of above equation. The results obtained basically improve and complement previous ones.

Keywords: Global stability, local stability, delay differential equation, nonlinear transport equations

AMS Mathematics Subject Classification (2010): 34D23, 37J25|

1. Introduction

For the recent contribution, we refer the reader to [1—13] and the references cited therein. It is observed that very few papers on global stability behavior of solutions of nonlinear transport equations are available to meet our growing interest of research.

$$\frac{\partial v}{\partial t}+g(x)\frac{\partial v}{\partial x}=f\left(t,v(x,t),v(h(x),t-\tau)\right) \tag{1}$$

Equation (1) can be written in the form

$$\frac{\partial v}{\partial t}+g(x)\frac{\partial v}{\partial x}=f\left(t,v,v_{t-\tau}\right) \tag{2}$$

where $v_{t-\tau}(x,t)=v(h(x),t-\tau),\tau>\mathbf{0}$.

We assume that the functions $g:(\mathbf{0},\mathbf{1})\to R, h:(\mathbf{0},\mathbf{1})\to(\mathbf{0},\mathbf{1})$ and $f:(\mathbf{0},\infty)\times R\times R\to R$ are continuously differentiable and satisfy the following conditions:

(i) $g(\mathbf{0})=\mathbf{0}, g(x)>\mathbf{0}$ *for* $x>\mathbf{0}$,

(ii) $h(\mathbf{0})=\mathbf{0}, h(x)<x$ *for* $x\in(\mathbf{0},\mathbf{1})$,

Equation (1) is considered with the initial condition

$$v(x,t)=\Omega(x,t) \quad \text{for} \quad (x,t)\in(\boldsymbol{0},\boldsymbol{1})\times(-\tau,\boldsymbol{0}). \tag{3}$$

A function $v:(\boldsymbol{0},\boldsymbol{1})\times(-\tau,\infty)\to R$ is called a classical solution of the problem (2), (3) if v is a continuous function in its domain, v satisfies the initial condition (i, ii), the $\frac{\partial v}{\partial t}$ and $\frac{\partial v}{\partial x}$ exist for $(x,t)\in(\boldsymbol{0},\boldsymbol{1})\times(\boldsymbol{0},\infty)$. and v satisfies equation (2) for $(x,t)\in(\boldsymbol{0},\boldsymbol{1})\times(\boldsymbol{0},\infty)$. First we show that if Ω is a continuously differentiable function then there exists exactly one classical solution of (2), (3). Let $\pi_s x$ be the solution of the equation $\pi_s x=\int_{\boldsymbol{0}}^{\infty} g(\pi_s x)ds$ with the initial condition $\pi_{\boldsymbol{0}} x=x$ for $x\in(\boldsymbol{0},\boldsymbol{1})$. The solution is well defined for $\pi_s x\le\boldsymbol{1}$. We can omit the problem of the global existence of the solutions of differential equation by extending the function g on the interval $(\boldsymbol{0},\infty)$ provided that g is bounded and continuously differentiable on $(\boldsymbol{0},\infty)$. Then the function $(x,s)\to\pi_s x$ is well defined on $R\times(\boldsymbol{0},\infty)$ and continuously differentiable with respect to (s,x). If v is a solution of the problem (2) and (3), then the function $\psi(c,s)=v(\pi_s c,s)$ is well defined for $s>\boldsymbol{0}$ and c such that $\boldsymbol{0}\le\pi_s c\le\boldsymbol{1}$.

The function ψ satisfies the equation

$$\frac{\partial\psi}{\partial s}=f(s,\psi,v(h(\pi_s c),s-\tau)), \tag{4}$$

for $s\in(\boldsymbol{0},\tau),\boldsymbol{0}\le\pi_s c\le\boldsymbol{1}$. We can rewrite equation (4) in the form

$$\frac{\partial\psi}{\partial s}=\bar{f}(s,c,\psi) \tag{5}$$

where $\bar{f}(s,c,\psi)=f(s,z,v(h(\pi_s c),s-\tau))$.

The function $\bar{f}$ is continuously differentiable in its domain and grows at most linearly with respect to z. From this it follows that for any $c\in(\boldsymbol{0},\boldsymbol{1})$ there exists a unique solution of (5) and this solution is defined for s such that $\pi_s c\in(\boldsymbol{0},\boldsymbol{1})$. Moreover the function $\psi(c,s)$ is continuously differentiable with respect to (c,s). On the other hand if t is a solution of (5) such that $\psi(c,\boldsymbol{0})=v(c,\boldsymbol{0})=\Omega(c,\boldsymbol{0})$, then the function $v(x,t)=\psi(\pi_{-t}x,t)$ is well defined for $(x,t)\in(\boldsymbol{0},\boldsymbol{1})\times(\boldsymbol{0},\tau)$, $\frac{\partial v}{\partial t}$ and $\frac{\partial v}{\partial x}$ exist and are continuous functions for $(x,t)\in(\boldsymbol{0},\boldsymbol{1})\times(\boldsymbol{0},\tau)$, and v satisfies equation (2) in this set. In this way we obtain the solution of (5) for $t\in(\boldsymbol{0},\tau)$. Using this method we can solve (2) successively for $t\in(\tau,\boldsymbol{2}\tau),t\in(\boldsymbol{2}\tau,\boldsymbol{3}\tau),$........ We will check that it does for $t=\tau$.

Let ψ be a solution of the equation

$$\frac{\partial\psi}{\partial s}=f(s,\psi(c,s),v(h(\pi_{s-\tau}c),s-\tau))$$

with the initial condition $\psi(c,\tau)=v(c,\tau)$. Then

$$\frac{\partial \overset{+}{\psi}}{\partial S}(c,\tau)=f(\tau,\psi(c,\tau),v(h(c),\mathbf{0})).$$

Let $v(x,t)=\psi(\pi_{\tau-t}x,t)$. Then $\frac{\partial \psi^{+}}{\partial t}(c,\tau)=\frac{\partial v^{+}}{\partial t}(c,\tau)+g(c)\frac{\partial v}{\partial x}(c,\tau)$

$$\frac{\partial v^{+}}{\partial t}+g(x)\frac{\partial v}{\partial x}=f(t,v,v_{t-\tau}) \quad for\, t=\tau.$$

This implies that $\frac{\partial v^{+}}{\partial t}=\frac{\partial v^{-}}{\partial t}$ for $t=\tau$ and v satisfies (2) for $t=\tau$.

2. Main results

Now we consider the following delay differential equation associated with (2):

$$z^{'}(t)=f(t,z,z_{\tau}). \tag{6}$$

Theorem 2.1. Let $v(x,t)$ be a solution of the problem (2) and (3). Let $z(t)$ be the solution of (6) satisfying the initial condition $z(t)=v(\mathbf{0},t)\ for\, t\in[-\tau,\mathbf{0}]$. Then for every $t_0\geq\mathbf{0}$ and $\varepsilon>\mathbf{0}$ there exist $t_1>\mathbf{0}$ and another solution $\bar{v}(x,t)$ of (2) such that

$$\min_{(x,t)\in(\mathbf{0},\mathbf{1})\times(-\tau,t_0)}\left|\bar{v}(x,t)-z(t)\right|<\varepsilon,$$

$$\bar{v}(x,t)=v(x,t) \text{ for } (x,t)\in(\mathbf{0},\mathbf{1})\times(t_1,\infty)$$

If $z_0(t)$ is a globally asymptotically stable solution of (6) and $v_0(x,t)=z_0(t)$ is a locally asymptotically stable solution of (2) then $v_0(x,t)$ is globally asymptotically stable solution of (2). The determining the global stability of a solution of (2) can be reduced to the problem of tentative the global stability of the differential delay equation (6) and the local stability of (2). Therefore, in the general case it is necessary to focus on the global stability of the associated differential delay equation (6).

Proof: We first show that for every $\delta\in(\mathbf{0},\mathbf{1})$ there exists $t_1>\mathbf{0}$ such that if $\Omega_1,\Omega_2\in(\mathbf{0},\mathbf{1})\times(-\tau,\mathbf{0})\rightarrow R$ are continuous functions and $\Omega_1(x,t)=\Omega_2(x,t)$ for $(x,t)\in(\mathbf{0},\delta)\times(-\tau,\mathbf{0})$, then the solutions v_1 and v_2 of (2) corresponding to Ω_1,Ω_2 satisfy $v_1(x,t)=v_2(x,t)\ for\,(x,t)\in(\mathbf{0},\mathbf{1})\times(t_1,\infty)$. Let $f:(\mathbf{0},\mathbf{1})\times(\mathbf{0},\mathbf{1})$ be a function given by the formula

$$f(x)=\min\left\{\pi_{-\tau}x,\max_{y\leq x}h(y)\right\}.$$

Then f is a continuous function. Since $\pi_{-\tau}x<x\ for\, x\in(\mathbf{0},\mathbf{1})$ and $h(y)<y\leq x$ for $\mathbf{0}<y\leq x\leq\mathbf{1}$, we have $f(x)<x\ for\, x\in(\mathbf{0},\mathbf{1})$. First, we check that if $\alpha\in(\mathbf{0},\mathbf{1})$ and v_1,v_2 are two solutions of (5) such that $v_1(x,t)=v_2(x,t)\ for\,(x,t)\in(\mathbf{0},f(\alpha))\times(-\tau,t_0)$

then $v_1(x,t)=v_2(x,t)$ *for* $(x,t)\in(0,\alpha)\times(t_0+\tau,\infty)$. Indeed, if $(x,t)\in(0,\alpha)\times(t_0,t_0+\tau)$, then $h(x)\le f(\alpha)$, $t-\tau\in(-\tau,t_0)$. This implies that $v_{1\tau}(y,t)=v_{2\tau}(y,t)$ for $(x,t)\in(0,\alpha)\times(t_0,t_0+\tau)$. From this it follows that v_1 and v_2 are the solutions of the same partial differential equation for $(x,t)\in(0,\alpha)\times(t_0,t_0+\tau)$. Moreover, if $\pi_{t_0-t}\le f(\alpha)$, then $v_1(\pi_{t_0-t}x,t_0)=v_2(\pi_{t_0-t}x,t_0)$. This implies that the solutions v_1 and v_2 are the same along the characteristic $\gamma(t)=(\pi_{t_0-t}x,t)$. In particular, since $\pi_{t_0-t}x\le f(\alpha)$ *for* $x\in(0,\alpha)$, we have $v_1(x,t_0+\tau)=v_2(x,t_0+\tau)$.

For $x\in(0,\alpha)\times(t_0+\tau,t_0+2\tau)$ the functions v_1 and v_2 are the solutions of the same partial differential equation with the same initial condition. This implies that $v_1(x,t)=v_2(x,t)$ for $(x,t)\in(0,\alpha)\times(t_0+\tau,\infty)$. Consider the sequence is $\{f^n(1)\}$. Since f is a continuous function and $f(x)<x$ *for* $x\in(0,1)$, we have $\lim_{n\to\infty} f^n(1)=0$. Let k be an integer such that $f^k(1)\le\delta$. If $\Omega_1(x,t)=\Omega_2(x,t)$ for $(x,t)\in(0,\delta)\times(-\tau,0)$, then $v_1(x,t)=v_2(x,t)$ for $(x,t)\in(0,1)\times((2k-1)\tau,\infty)$.

Now let $\varepsilon>0$ and $t_0>0$ be given constants. Let $v(x,t)$ be a solution of (2) and $z(t)$ be the solution of (6) with the initial condition $z(t)=v(0,t)$ *for* $t\in(-\tau,0)$. The function $v(x,t)=z(t)$ is also a solution of (2). From the continuous dependence of the solutions of (2) on the initial condition it follows that there exists $\varepsilon_1>0$ such that if

$$\left|\bar{\Omega}(x,t)-z(t)\right|<\varepsilon_1 \ \ for\ (x,t)\in(0,1)\times(-\tau,0) \tag{7}$$

Then

$$\left|\bar{v}(x,t)-z(t)\right|<\varepsilon\ for\ (x,t)\in(0,1)\times(-\tau,t_0), \tag{8}$$

where $\bar{v}$ is the solution of (2) which satisfies the initial condition $\bar{v}(x,t)=\bar{\Omega}(x,t)$ for $(x,t)\in(0,1)\times(-\tau,0)$. Since $v(x,t)$ is a continuous function and $z(t)=v(0,t)$ for $t\in(0,1)$. there exists $\delta>0$ such that $|v(x,t)-z(t)|<\varepsilon_1$ for $(x,t)\in(0,\delta)\times(-\tau,0)$.
Now, let

$$\bar{\Omega}(x,t)=\begin{Bmatrix} v(x,t):(x,t)\in(0,\delta)\times(-\tau,0), \\ v(\delta,t):(x,t)\in(\delta,1)\times(-\tau,0) \end{Bmatrix}.$$

Then $\bar{\Omega}$ satisfies (7). If $\bar{v}$ is a solution of (3.22) with the initial condition $\bar{\Omega}$, then $\bar{v}$ satisfies (8). Since $v(x,t)=\bar{v}(x,t)$ *for* $(x,t)\in(0,\delta)\times(-\tau,0)$, from the first step it follows that $v(x,t)=\bar{v}(x,t)$ *for* $(x,t)\in(0,1)\times(t_1,\infty)$.

We use to considerations of the local stability of the full partial differential equation (2). We assume that the function f does not depend on t. Then equation (2) takes the form

$$\frac{\partial v}{\partial t}+g(x)\frac{\partial v}{\partial x}=f(v,v_\tau). \tag{9}$$

Let $\bar{v}(x,t)$ be a given solution of (9)**.** We say that the solution $\bar{v}$ of (9) is exponentially stable on the set A if there exists $\mu > \boldsymbol{0}$ such that for every $\Omega \in A$ the solution of the problem (9), (3) satisfies the condition

$$\min\left\{ |v(x,t)| = \bar{v}(x,t) : x \in (\boldsymbol{0},\boldsymbol{1}) \right\} \le Ce^{-\mu t}, \tag{10}$$

where c is a constant which depends only on Ω.

We say that $\bar{v}$ is locally exponentially stable if there exist an $\varepsilon > \boldsymbol{0}$, μ and c such that condition (10) holds for every solution of the problem (9), (3) with $\Omega \in A_\varepsilon$.

Theorem 2.2. Let u be a constant such that $f(u,u) = \boldsymbol{0}$ and $\frac{\partial f}{\partial v}(u,u) < -\left| \frac{\partial f}{\partial v_\tau}(u,u) \right|$. Then the solution $\bar{v}(x,t) \equiv u$ of (9) is locally exponentially stable.

Proof: Without loss of generality we can assume that $u = \boldsymbol{0}$ Let

$$a = \frac{\partial f}{\partial v}(\boldsymbol{0},\boldsymbol{0}) \ \ and \ \ b = \frac{\partial f}{\partial v_\tau}(\boldsymbol{0},\boldsymbol{0})$$

then $a < -|b|$. In the space $c((\boldsymbol{0},\boldsymbol{1}),(-\tau,\boldsymbol{0}))$, we introduce an auxiliary norm

$$\|\Omega\|_\lambda = \min\left\{ |\Omega(x,t)| e^{\lambda t} : (x,t) \in (\boldsymbol{0},\boldsymbol{1}) \times (-\tau,\boldsymbol{0}) \right\},$$

where $\lambda \in R$.. If $\lambda = \boldsymbol{0}$ then $\|\Omega\| = \|\Omega\|_{\boldsymbol{0}}$ is the standard norm in $c\left[(0,1)\times(-\tau,0)\right]$.

For any $\lambda_1, \lambda_2 \in R$ the norms $\|\bullet\|_{\lambda_1}$ *and* $\|\bullet\|_{\lambda_2}$ are equivalent and

$$\|\Omega\|_{\lambda_2} \le \|\bullet\|_{\lambda_1} \le e^{(\lambda_2 - \lambda_1)\tau} \|\Omega\|_{\lambda_2} \ \ for \ \ \lambda_1 \le \lambda_2.$$

Let v be a solution of (9), (3) and be a transformation given by $(T\Omega) = v(x, t+\tau)$. We check that there exists $\lambda, \varepsilon > \boldsymbol{0}$ and $\gamma \in (\boldsymbol{0},\boldsymbol{1})$ such that $\|T\Omega\|_\lambda \le \gamma \|\Omega\|_\lambda$ *for* $\|\Omega\|_\lambda \le \varepsilon$.

From the Lagrange mean value theorem it follows that for every $(v, v_\tau) \in R^{\boldsymbol{2}}$ there exists $\theta \in (\boldsymbol{0},\boldsymbol{1})$ such that

$$f(v,v_\tau) = \frac{\partial f}{\partial v}(\theta v, \theta v_\tau) v + \frac{\partial f}{\partial v_\tau}(\theta v, \theta v_\tau) v_\tau$$

Let $\delta > \boldsymbol{0}$ be a given constant and $\delta_1 > \boldsymbol{0}$ be a constant such that

$$\left| \frac{\partial f}{\partial v}(v,v_\tau) - a \right| < \delta, \ \left| \frac{\partial f}{\partial v_\tau}(v,v_\tau) - b \right| < \delta \ \ for \ \ |v| < \delta_1, |v_\tau| < \delta_1.$$

From the continuous dependence of the solution of (9) on the initial condition, it follows that there exists $\delta_2 \in (\boldsymbol{0}, \delta_1)$ such that $\|T\Omega\| < \delta_1$ *for* $\|\Omega\| < \delta_2$. This implies that if $\|\Omega\| < \delta_2$, then the solution of the problem (9), (3) satisfies

$$\left| \frac{\partial v}{\partial t} + g(x) \frac{\partial v}{\partial x} \right| \le (|a| + \delta)|v| + (|b| + \delta)\|\Omega\| \tag{11}$$

Let $(x, y)\in(\boldsymbol{0},\boldsymbol{1})\times(\boldsymbol{0},\tau)$ and $\psi(s)= v(\pi_{s-t}x, s)$ *for* $s\in(\boldsymbol{0}, t)$.
Then from (11) it follows that

$$\left|\psi^{'}(s)\right|\leq (|a|+\delta)|\psi(s)|+(|b|+\delta)\|\Omega\| \tag{12}$$

Inequality (12) implies that

$$|\psi(s)|\leq\left(|\psi(\boldsymbol{0})|+\frac{(|b|+\delta)\|\Omega\|}{|a|+\delta}\right)e^{(|a|+\delta)s}. \tag{13}$$

Since $|\psi(\boldsymbol{0})| \leq \|\Omega\|$, from (13) we obtain

$$|\psi(s)| \leq k_{\tau}\|\Omega\| , \tag{14}$$

From (14) and from the formula $v(x,t)=\psi(t)$ it follows that

$$|v(x,t)|\leq k_{\tau}\|\Omega\| \quad for \quad (x,t)\in (\boldsymbol{0},\boldsymbol{1})\times(\boldsymbol{0},\tau).$$

Let $\Omega\in c[(\boldsymbol{0},\boldsymbol{1})\times(-\tau,\boldsymbol{0})]$ be a function such that $\|\Omega\|<\delta_{\boldsymbol{2}}$ and let λ be a positive constant. Then

$$|\psi(x,t)|\leq e^{-\lambda t}\|\Omega\|_{\lambda} \quad for\,(x,t)\in(\boldsymbol{0},\boldsymbol{1})\times(-\tau,\boldsymbol{0})$$

Let $v(x,t)$ be a solution of (9), (3). From (11) it follows that $v(x,t)$ satisfies the inequality

$$\left|\frac{\partial v}{\partial t}+g(x)\frac{\partial v}{\partial x}-av\right| \leq \delta|v(x,t)|+ (|b|+\delta)|v(h(x),t-\tau)| \quad for(x,t)\in (\boldsymbol{0},\boldsymbol{1})\times(\boldsymbol{0},\tau).$$

$$|V(x,t)|\leq k_{\tau}e^{\lambda\tau}\|\Omega\|_{\lambda} \;\; and \;\; |v(h(x),t-\tau)| \leq e^{\lambda(\tau-\tau)}\|\Omega\|_{\lambda}$$

$$\left|\frac{\partial v}{\partial t}+g(x)\frac{\partial v}{\partial x}-av\right| \leq \left(\delta k_{\tau}e^{\lambda\tau}+(|b|+\delta)e^{\lambda(\tau-t)}\right)\|\Omega\|_{\lambda}$$

Let $(x,t)\in (\boldsymbol{0},\boldsymbol{1})\times(\boldsymbol{0},\tau)$ *and* $\psi(s)=e^{-as}v(\pi_{s-t}x,s)$ *for* $s\in[\boldsymbol{0},t]$.

$$\left|\psi^{'}(s)\right| \leq \left(\delta k_{\tau}e^{\lambda\tau-as}\right)+ (|b|+\delta)e^{\lambda\tau-\lambda s-as}\|\Omega\|_{\lambda}.$$

Since $\psi(\boldsymbol{0})=v(\pi_{s-t}x,\boldsymbol{0}) = \Omega(\pi_{s-t}x,\boldsymbol{0})$, we have $|\psi(\boldsymbol{0})|\leq\|\Omega\|_{\lambda}$.

$$|\psi(t)| \leq \left[\boldsymbol{1}+\frac{\delta k_{\tau}}{a}e^{\lambda\tau}\left(\boldsymbol{1}-e^{-at}\right)+\frac{|b|+\delta}{a+\lambda}e^{\lambda\tau}\left(\boldsymbol{1}-e^{-(\lambda+a)t}\right)\right]\|\Omega\|_{\lambda,} \tag{15}$$

Since $v(x,t)= e^{at}\psi(t)$ and $T\Omega(x,t)=v(x,t+\tau)$ we have

$$e^{\lambda t}\,T\Omega(x,t)=e^{at+\lambda t+a\tau}\,\psi(t+\tau) \tag{16}$$

From (15), (16) and inequalities $a<\boldsymbol{0}, t\leq\boldsymbol{0}$ it follows that

$$e^{\lambda t}|T\Omega(x,t)| \leq \left[e^{at+\lambda t+a\tau}+\frac{\delta k_{\tau}}{|a|}e^{\lambda\tau}+\frac{|b|+\delta}{a+\lambda}e^{\lambda\tau}\left(e^{(\lambda+a)(t+\tau)}-\boldsymbol{1}\right)\right]\|\Omega\|_{\lambda}.$$

$$\|T\Omega\|_{\lambda}\leq\gamma(\delta,\lambda)\|\Omega\|_{\lambda},$$

where $\gamma(\delta,\lambda)=\min\left\{e^{-\lambda\tau}+\frac{\delta k_{\tau}}{|a|}e^{\lambda\tau}, e^{a\tau}+\frac{\delta k_{\tau}}{|a|}e^{\lambda\tau}+\frac{|b|+\delta}{a+\lambda}\left(e^{(\lambda+a)\tau}-\boldsymbol{1}\right)\right\}$.

For any $\lambda>\boldsymbol{0}$ we can choose $\delta>\boldsymbol{0}$ sufficiently small so that the first term in the above minimum is less than one. The second term in the minimum equals for $\delta,\lambda=\boldsymbol{0}$. Since

$a\tau < \mathbf{0}$ and $|a| > |b|$, this term is less than one. In this way we obtain inequality for $\|\Omega\| \le \delta_2$. If we take $\varepsilon = e^{-\lambda\tau}\delta_2$, then for any Ω such that $\|\Omega\|_\lambda \le \varepsilon$ we have $\|\Omega\|_\lambda \le \delta_2$ and consequently holds for $\|\Omega\|_\lambda \le \varepsilon$. Now, let v be a solution of the problem (9), (3) such that $\|\Omega\|_\lambda \le \varepsilon$.

Since $V(x,t) = T^n\Omega(x, t - n\tau)$ *for* $t \in [(n-\mathbf{1})\tau, n\tau]$ *and* $\|T^n\Omega\|_\lambda \le \gamma^n\|\Omega\|_\lambda$
we have

$$|V(x,t)| \le e^{\lambda\tau}\gamma^n\|\Omega\|_\lambda \quad for \ (x,t) \in (\mathbf{0},\mathbf{1}) \times ((n-\mathbf{1})\tau, n\tau).$$

If we take $\mu = -\tau^{-\mathbf{1}}\log\gamma$, then

$$|v(x,t)| \le e^{\lambda\tau}\|\Omega\|_\lambda e^{-\mu t} \le e^{\lambda\tau - \mu}\|\Omega\| \quad for\,(x,t) \in (\mathbf{0},\mathbf{1}) \times (-\tau, \infty),$$

This completes the proof.

3. Conclusion

Let z_0 be a solution of (6). Assume that there exists a solution v_0 of (2) such that $v_0(x,t) = z_0(t)$ for $t \in (-\tau, \mathbf{0})$ and assume that it converges to zero. Then z_0 is a stable solution of (2). Indeed, for each $\varepsilon > \mathbf{0}$ we can find another solution $\bar{v}$ of (2) such that $\left|\bar{v}(x,t) - z_0(t)\right| < \varepsilon / \mathbf{3}$ *for* $x \in (\mathbf{0},\mathbf{1}), t \in (-\tau, \mathbf{0})$ and $v_0(x,t) = \bar{v}(x,t)$ for $x \in (\mathbf{0},\mathbf{1})$ and sufficiently large t. After a small modification of theorem 2.1 and theorem 2.2 can check that any solution v of (2) such that $v_0(\mathbf{0},t) = z_0(t)$ for $t \in (-\tau, \mathbf{0})$ is also stable.

REFERENCES

1. R.Crabb, J.Losson and M.C.Mackey, Dependence on initial conditions in nonlocal PDE's and hereditary dynamical systems, *Proc. Inter. Conf. Nonlin. Anal.*, 4 (1996) 3125-3136.
2. O.Diekmann, H.J.A.M.Heijmans and H.R.Thieme, On the stability of the cell size distribution, *J. Math. Biology*, 19 (1984) 227-248.
3. J.Dyson, R.Villella-Bressan and G.F.Webb, A singular transport equation modeling a proliferating maturity structured cell population, *Can. Appl. Math. Quar.*, 4 (1996) 65-95.
4. J.Hale, Theory of Functional Differential Equations. Springer-Verlag, New York, 1977.
5. A.Lasota and M.C.Mackey, Globally asymptotic properties of proliferating cell populations, *J. Math. Biology*, 19 (1984) 43-62.
6. K.Loskot, Bounded solutions of partial differential equations describing interacting cell populations. *Bull. Ac. Pol. Math.*, 42 (1994) 315-343.
7. M.C.Mackey and R.Rudnicki, Global stability in a delayed partial differential equation describing cellular replication, *J. Math. Biol.*, 33 (1994) 89-109.
8. A.Rey and M.C.Mackey, Bifurcations and traveling waves in a delayed partial differential equation, *Chaos*, 2 (1992) 231-244.

9. A.Rey and M.C.Mackey, Multistability and boundary layer development in a transport equation with retarded arguments, *Can. Appl. Math. Quar.*, 1 (1993) 1-21.
10. R.Rudnicki, Invariant measures for the flow of a first order partial differential equation, *Ergod. Th. & Dynam. Sys.*, 5 (1985) 437-443.
11. A.Tiryaki and S.Yaman, Asymptotic behivior of a class of nonlinear functional differential equations of third order, *Appl. Math. Lett.*, 14 (2001) 327–332.
12. X.Liu and Q.Wang, On stability in terms of two measures for impulsive systems of functional differential equations, *Journal of Mathematical Analysis and Applications*, 326 (2007) 252–265.
13. X.Li, New results on global exponential stabilization of impulsive functional differential equations with infinite delays or finite delays, *Nonlinear Analysis Series B: Real World Applications*, 11 (2010) 4194–4201.