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Uniform Boundedness Principle on 2-Fuzzy Normed Linear Spaces

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Abstract. This paper defines the concept of 2-fuzzy continuity and 2-fuzzy bounded linear operator. Relation between weakly 2-fuzzy continuity and weakly 2-fuzzy boundedness is studied. The concept of second weak 2-fuzzy dual for 2-fuzzy normed linear space is developed. Uniform Boundedness and Banach Alaoglu theorems are established.

Keywords: 2-fuzzy norm, weakly 2-fuzzy continuity, weakly 2-fuzzy bounded linear mapping, 2-fuzzy dual space

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1. Introduction

The concept of fuzzy set was introduced by Zadeh [8] in 1965. A satisfactory theory of 2-norm on a linear space has been introduced and developed by Gahler in [3]. The concept of fuzzy norm and α-norm were introduced by Bag and Samanta and the notions of convergence and Cauchy sequence were also discussed in [1]. Jialuzhang [4] has defined fuzzy linear space in a different way. Bag and Samanta [2] defined fuzzy bounded linear operator and fuzzy bounded linear functional on the fuzzy dual space. Somasundaram and Beaula [6] defined the notion of 2-fuzzy 2-normed linear space. They also established the famous closed graph theorem and Riesz theorem in 2-fuzzy 2-normed linear space. Beaula and Gifta [7] defined 2-fuzzy normed linear space and proved some standard results. They also defined 2-fuzzy dual space and proved it is complete 2-fuzzy normed linear space. In section 2, we recall some preliminary concepts and in section 3, we define 2-fuzzy continuity and boundedness and a theorem is established related to these concepts. In section 4, the Uniform Boundedness theorem is established and in section 5, we introduce second fuzzy dual for the 2-fuzzy normed linear space. Banach Alaoglu theorem is developed in this space.

2. Preliminaries

For the sake of completeness, we reproduce the following definitions due to Saadati[5], Bag and Samanta [1], Zhang [4] and Somasundaram [6].

Definition 2.1. [5] Let X be a linear space over K (field of real or complex numbers). A fuzzy subset N of $X \times R$ (R, the set of real numbers) is said to be a fuzzy norm on X if and only if for all $x, u \in X$ and $c \in K$.

(N_1) For all $t \in R$ with $t \le 0$, $N(x, t) = 0$.

(N_2) For all $t \in R$ with $t > 0$, $N(x, t) = 1$ if and only if $x = 0$.

(N_3) For all $t \in R$ with $t > 0$, $N(cx, t) = N\left(x, \frac{t}{|c|}\right)$ if $c \neq 0$.

(N_4) For all $s, t \in R$, $x, u \in X$, $N(x+u, s+t) \geq \min\{N(x, s), N(u, t)\}$.

(N_5) $N(x, \cdot)$ is a non decreasing function of R and $\lim N(x, t) = 1$.

The pair (X, N) will be referred to as a fuzzy normed linear space.

Theorem 2.1. Let (X, N) be a fuzzy normed linear space. Assume further that (N_6)$N(x, t) > 0$ for all $t > 0$ implies $x = 0$. Define $||x||_\alpha = \inf\{ t : N(x, t) \geq \alpha\}, \alpha \in (0, 1)$. Then $\{||\cdot||_\alpha : \alpha \in (0, 1)\}$ is an ascending family of norms on X (or) α-norms on X corresponding to the fuzzy norm on X.

Definition 2.2. [4] Let X be any non-empty set and F(X) be the set of all fuzzy sets on X. For $U, V \in F(X)$ and $k \in K$, the field of real numbers, define

$$U + V = \{(x + y, \lambda \wedge \mu) / (x,\lambda) \in U, (y,\mu) \in V\} \quad \text{and}$$

$$kU = \{(kx,\lambda) / (x,\lambda) \in U\}.$$

Definition 2.3. [6] A fuzzy linear space $\tilde{X} = X \times (0,1]$, over the number field K where the addition and scalar multiplication operation on $\tilde{X}$ defined by

$$(x,\lambda) + (y,\mu) = (x + y, \lambda \wedge \mu),$$

$$k(x,\lambda) = (kx,\lambda)$$

is a fuzzy normed space if to every $(x,\lambda) \in \tilde{X}$ there is associated a nonnegative real numbers $||(x, \lambda)||$ called the fuzzy norm of (x, λ) in such a way that

(1) $||(x, \lambda)|| = 0$ if and only if $x = 0$ the zero element of X and $\lambda \in (0, 1]$.

(2) $||k(x, \lambda)|| = |k|\, ||(x, \lambda)||$ for all $(x,\lambda) \in \tilde{X}$ and $k \in K$.

(3) $||(x, \lambda) + (y, \mu)|| \le ||(x, \lambda \wedge \mu)|| + ||(y, \lambda \wedge \mu)||$ for all $(x, \lambda), (y, \mu) \in \tilde{X}$.

(4) $\left\| x, \vee_t \lambda_t \right\| = \wedge_t \left\| (x, \lambda_t) \right\|$ for $\lambda_t \in (0, t]$.

Definition 2.4. [6] Let X be a nonempty set and F(X) be the set of all fuzzy sets in X. If $f \in F(X)$ then $f = \{(x, \mu) \mid x \in X \text{ and } \mu \in [0, 1]\}$. Clearly f is a bounded function for $|f(x)| \le 1$ for every $x \in X$. Let K be the space of real numbers, then F(X) is a linear space over the field K where the addition and scalar multiplication are defined by

$$\begin{aligned} f + g &= \{(x,\mu) + (y,\eta) / x,y \in X \text{ and } \mu, \eta \in [0,1] \} \\ &= \{(x + y, \mu \wedge \eta) / (x,\mu) \in f \text{ and } (y,\eta) \in g\} \\ kf &= \{k(x,\mu) / x \in X \text{ and } \mu \in [0,1] \\ &= \{(kf,\mu) / (x,\mu) \in f\} \quad \text{where } k \in K. \end{aligned}$$

The linear space F(X) is said to be a normed space if to every $f \in F(X)$, there is associated a nonnegative real number $||f||$ called the norm of f in such a way that

(1) $||f|| = 0$ if and only if $f = 0$

$$\|f\| = 0 <=> \{\|(x,\mu)\| \ / (x,\mu)\in f\} = 0$$
$$<=> x = 0, \mu \in [0,1]$$
$$<=> f = 0$$

(2) $||kf|| = |k|\ ||f||, k \in K$

For,

$$\|kf\| = \{\|k(x,\mu)\| / (x,\mu) \in f \text{ and } k \in K\}$$
$$= \{|k|\ \|(x,\mu)\| / (x,\mu) \in f\}$$
$$= |k|\ \|f\|$$

(3) $||f+g|| \le ||f|| + ||g||$ for every $f, g \in F(X)$

For,

$$\|f+g\| = \{\|(x,\mu)+(y,\eta)\| \ / x,y \in X \text{and } \mu,\eta \in [0,1]\}$$
$$= \{\|(x+y),\mu \wedge \eta\| \ / x,y \in X \text{ and } \mu,\eta \in [0,1]\}$$
$$\le \{\|x,\mu \wedge \eta\| + \|y,\mu \wedge \eta\| \ / (x,\mu) \in f \text{ and } (y,\eta) \in g\}$$
$$= \|f\| + \|g\|$$

And so $(F(X), ||\cdot||)$ is a normed linear space.

Definition 2.5. [6] A 2-fuzzy set on X is a fuzzy set on F(X).

Definition 2.6. Let F(X) be a linear space over the real field K. A fuzzy subset N of $F(X) \times R$ (the set of real numbers) is called a 2-fuzzy norm on F(X) if and only if,

(N_1) for all $t \in R$ with $t \le 0$, $N(f, t) = 0$.

(N_2) for all $t \in R$ with $t > 0$, $N(f, t) = 1$ if and only if $f = \bar{0}$.

(N_3) for all $t \in R$ with $t \ge 0$, $N(cf,t) = N\left(f, \frac{t}{|c|}\right)$, if $c \neq 0$, $c \in K$ (field).

(N_4) for all $s, t \in R$, $N(f_1 + f_2, s+ t) \ge \min\{N(f_1, s), N(f_2, t)\}$.

(N_5) $N(f, \cdot)$ is a non decreasing function of R with $\lim_{t\to\infty} N(f,t) = 1$.

Then the pair (F(X), N) is said to be a 2-fuzzy normed linear space.

3. 2-Fuzzy continuity

Definition 3.1. Let $(F(X), N_1)$ and $(F(Y), N_2)$ be 2-fuzzy normed linear spaces. The mapping T from F(X) to F(Y) is said to be 2-fuzzy continuous at $f_0 \in F(X)$ if for given $\delta > 0$, $\varepsilon \in (0, 1)$, there exists $\gamma = \gamma(\varepsilon, \delta) > 0$, $r = r(\varepsilon, \delta) \in (0, 1)$ such that for every $f \in F(X)$

$$N_1(f - f_0, \gamma) > 1 - r \text{ implies } N_2(T(f) - T(f_0), \delta) > 1 - \varepsilon.$$

If T is a 2-fuzzy continuous at each element of F(X), then T is said to be 2-fuzzy continuous on F(X).

Definition 3.2. Let $(F(X), N_1)$ and $(F(Y), N_2)$ be 2-fuzzy normed linear spaces. The mapping T from F(X) to F(Y) is said to be weakly 2-fuzzy continuous at $f_0 \in F(X)$ if for each $\delta > 0$ and $r \in (0, 1)$ there exist $\gamma > 0$ such that for every $f \in F(X)$

$$N_1(f - f_0, \gamma) \ge 1 - r \text{ implies } N_2(T(f) - T(f_0), \delta) \ge 1 - r.$$

Definition 3.3. Let $(F(X), N_1)$ and $(F(Y), N_2)$ be two 2-fuzzy normed linear spaces. A linear mapping T from F(X) to F(Y) is said to be 2-fuzzy bounded if there exist $M \in R^+$ such that for every $f \in F(X)$ and for each $t > 0$,

$$N_2(T(f), t) \geq N_1(f, t/M).$$

Definition 3.4. A linear mapping T from a 2-fuzzy normed linear spaces $(F(X), N_1)$ to a 2-fuzzy normed linear spaces $(F(X),N_2)$ is said to be weakly 2-fuzzy bounded if for any $r\in(0,1)$ there exist $M\in R^+$ such that for all $f\in F(X)$ and $t > 0$,

$$N_1(f, t/M) \geq 1-r \text{ implies } N_2(T(f), t) \geq 1-r.$$

Theorem 3.1. Let $(F(X), N_1)$ and $(F(Y), N_2)$ be 2-fuzzy normed linear spaces and T is a linear mapping from F(X) to F(Y) then

(i) T is weakly 2-fuzzy continuous on F(X) if T is weakly 2-fuzzy continuous at $f_0\in F(X)$

(ii) T is weakly continuous if and only if T is weakly 2-fuzzy bounded.

Proof:

(i) If T is weakly 2-fuzzy continuous then by definition, for given $\delta > 0$ and $r\in(0, 1)$ there exist $\gamma > 0$ such that for every $f\in F(X)$,

$$N_1(f-f_0, \gamma)\geq 1-r \text{ implies } N_2(T(f)-T(f_0),\delta)\geq 1-r \qquad (1)$$

Let $g \in F(X)$ and replace f by $f + f_0- g$ in (1)

$$N_1(f+f_0-g-f_0, \gamma) \geq 1-r \text{ implies } N_2(T(f+f_0-g)-T(f_0), \delta) \geq 1-r$$

So, $N_1(f-g, \gamma) \geq 1-r$ implies $N_2(T(f-g)-T(f_0), \delta) \geq 1-r$

Hence T is weakly 2-fuzzy continuous on F(X).

(ii) Suppose T is weakly 2-fuzzy bounded then for any $r \in (0, 1)$ there exist $M \in R^+$ such that for every $f \in F(X)$ and $t > 0$, $N_1(f, t/M) \geq 1-r$ implies $N_2(T(f), t) \geq 1-r$

that is, $N_1(f-\overline{0}, t/M) \geq 1-r$ implies $N_2(T(f)-\overline{0}, t) \geq 1-r$

that is, $N_1(f-\overline{0}, t_0) \geq 1-r$ implies $N_2(T(f)-\overline{0}, t) \geq 1-r$ where $t_0 = t/M$ implies T is weakly 2-fuzzy continuous at $f=\overline{0}$ and so T is weakly 2-fuzzy continuous on F(X).

Conversely, assume that T is weakly 2-fuzzy continuous on F(X).

Case (i) Considering T is continuous at $f=\overline{0}$ and taking $\delta = 1$ we have for every $r\in(0,1)$there exist $\gamma>0$ such that for every $f \in F(X)$,

$$N_1(f-\overline{0}, \gamma) \geq 1-r \text{ implies } N_2(T(f)-\overline{0}, 1) \geq 1-r$$

that is, $N_1(f, \gamma) \geq 1-r$ implies $N_2(T(f), 1) \geq 1-r$

Case (ii) Suppose that $f \neq \overline{0}$. Take $f = u/t$, $t > 0$ then

$$N_1(u/t, \gamma) \geq 1-r \text{ implies } N_2(T(u/t), 1) \geq 1-r$$

$$N_1(u, t\gamma) \geq 1-r \text{ implies } N_2(T(u), t) \geq 1-r$$

$$N_1(u, t/M) \geq 1-r \text{ implies } N_2(T(u), t) \geq 1-r \text{ where } \gamma = 1/M$$

implies T is weakly 2-fuzzy bounded.

Case (iii) If $f \neq \overline{0}$ and $t \leq 0$ then $N_1(u, t/M) = N_2(T(u), t) = 0$ for any $M > 0$

Case (iv) If $f = \overline{0}$ and $M > 0$

$N_1(u, t/M) = N_2(T(u), t) = 1$ if $t > 0$

$N_1(u, t/M) = N_2(T(u), t) = 0$ if $t \leq 0$

from all the above cases we get

$N_1(u, t/M) \geq 1-r$ implies $N_2(T(u), t) \geq 1-r$, for all $u \in F(X)$, $r \in (0, 1)$ and $t > 0$ implies T is weakly 2-fuzzy bounded.

3.1. 2-Fuzzy Banach space

Definition 3.1.1. Every complete 2-fuzzy normed linear space is said to be a 2-fuzzy Banach space.

Definition 3.1.2. A mapping T from a 2-fuzzy Banach space A to a 2-fuzzy Banach space B is said to be a 2-fuzzy linear mapping if it satisfies the following conditions

(i) $T(f + g) = T(f) + T(g)$, where $f, g \in A$

(ii) $T(\alpha f) = \alpha T(f)$, where $\alpha \in R^+$, $f \in A$.

Definition 3.1.3. T is said to be bounded with respect to α-norm if there exist a constant $M \in [0, 1]$ such that $T(f) \leq M \|f\|_\alpha$ for every $f \in A$. If T is bounded, define

$\|T\| = \text{glb}\{M : |T(f)| \leq M\|f\|_\alpha, \text{for every } f \in A\}$ (or)

$\|T\| = \sup\{M : |T(f)| > M\|f\|_\alpha, \text{for every } f \in A\}$

Theorem 3.1.1. If $B[f_0, r_0, t] = \{f \in F(X) / N(f-f_0, t) \geq 1-r_0\}$ then

(i) $B[f_0, r_0, t] - f_0$ is a closed ball centered at the origin

(ii) $\frac{1}{r_0}[B[f_0, r_0, t] - f_0]$ is a unit closed ball centered at the origin.

Proof:

(i) Let $g_0 \in B[f_0, r_0, t] - f_0$ then $g_0 = f - f_0$ where $f \in B[f_0, r_0, t]$

$$N(g_0, t) = N(f-f_0, t) \geq 1-r_0$$

implies that $g_0 \in B[0, r_0, t]$, a closed ball centered at the origin.

(ii) Let $g \in \frac{1}{r_0}[B[f_0, r_0, t] - f_0]$ then $g = \frac{1}{r_0}[f - f_0]$ where $f \in B[f_0, r_0, t]$

$$\begin{aligned} N(g, t) &= N\left(\frac{1}{r_0}(f - f_0), t\right) \\ &= N(f - f_0, r_0 t) \\ &\geq 0 \ \text{(by definition)} \end{aligned}$$

which implies $\frac{1}{r_0}[B[f_0, r_0, t] - f_0]$ is a unit closed ball centered at the origin.

Theorem 3.1.2. (Uniform Boundedness Theorem)

Let $(F(X), N_1)$ be a 2-fuzzy Banach space and $(F(Y), N_2)$ a 2-fuzzy normed linear space. If $\{T_i\}$ is a nonempty set of continuous linear function from F(X) into F(Y) and $\{T_i(f)\}$ is a bounded subset of F(Y) for every f in F(Y) with the property that $N_2(T_i(f), t_{f,k}) \geq 1- k$ then $N_1(T_i, t) \geq k$ is a sequence of values in [0, 1], that is $\{T_i\}$ is bounded as a subset of $\mathcal{B}(F(X), F(Y))$, (the set of all bounded functions from F(X) to F(Y)).

Proof: For $t_{f,k} > 0$, define $A_{t_{f,k}} = \{f \in F(X) : N_2(T_i(f), t_{f,k}) \geq 1 - k\}$

which is a subspace of F(X), a 2-fuzzy Banach space.

Let $B[\bar{0},M,t]=\{f\in F(X)\,/\,N(f,t)>1-M\}$ be a closed ball centered at the origin $\bar{0}$ and radius M in (F(X), N).Now, if $f\in A_{t_{f,k}}$ then $N_2(T_i(f), t_{f,k})\geq 1-k$

which implies $T_i(f)\in B[\bar{0},k,t_{f,k}]$.

Thus $A_{t_{f,k}}$ is a closed subspace of F(X), and $F(X)=\bigcup_{t_{f,k}>0} A_{t_{f,k}}$

Since F(X) is a 2-fuzzy Banach space it is complete. By Baires theorem, $\overline{A}_{t_{f_0,k_0}}$ has a nonempty interior for any t_{f_0,k_0} and so it contains a closed ball say $B[f_0, r_0, t]$.

Hence $N_2(T_i(f), t_{f,k})\geq 1-k$ for $f\in\overline{A}_{t_{f_0,k_0}}$ and so $N_2(T_i(B[f_0,r_0,t]), t_{f,k})\geq 1-k$.Thus

$$N_2\left(T_i\left(\frac{B[f_0,r_0,t)-f_0}{r_0}\right),t_{f,k}\right)=N_2(T_i(B[f_0,r_0,t]-f_0),r_0t_{f,k})$$
$$> 1-k \qquad (1)$$

As $\frac{B[f_0,r_0,t]-f_0}{r_0}$ is a unit closed ball centered at origin let it be $B[\bar{0},\bar{1},t]$ and from (1) we get $N_2(T_i(B[\bar{0},\bar{1},t]),t_{f,k})\geq 1-k$ (2)

By definition $\|T_i(f)\|_\alpha=\sup\{t: N_2(T_i(f),t)\geq 1-\alpha\}$, for every $\alpha\in(0, 1)$

and $\|T\|_\alpha=\sup\{t: N(T,t)\geq 1-\alpha\}$, from (2) we get, $\|T_i(f)\|_\alpha\geq t_0$ as $N_2(T_i(f), t_0)\geq 1-k$ further $\|T_i(f)\|=\sup\{\|T_i(f)\|_\alpha:\|f\|_\alpha\leq 1\}$

$$\geq t_0$$

equivalently, $$N_1(T_i,t)=\sup\{k:\|T_i\|_\alpha\geq 1-t\}$$
$$\geq k,\quad k\in[0,1]$$

Hence $\{T_i\}$ is bounded as a subset of $\mathcal{B}$(F(X), F(Y))

3.2. 2-Fuzzy dual space

Definition 3.2.1. Let (F(X), N) be a 2-fuzzy normed linear space. A weak bounded 2-fuzzy linear mapping defined from (F(X), N) to R (the set of real numbers) is said to be weakly 2-fuzzy functional. The set of all weakly 2-fuzzy functionals is known as the first weak 2- fuzzy dual space denoted by $(F(X)^*, N^*)$.

Definition 3.2.2. Let (F(X), N) be a 2-fuzzy normed linear space and $T\in F(X)^*$ and $\|\;\|_\alpha^*$ be the family of all 2-fuzzy α-norms on F(X). Define $\|T\|_\alpha^*=\sup\left\{\frac{|T(f)|}{\|f\|_\alpha}\right\}$, for every $f\in F(X)$ and $\alpha\in(0, 1)$. Then $\{\|\cdot\|_\alpha^*:\alpha\in(0,1)\}$ is an ascending family of α- norms on $F(X)^*$.

Define $$N^*(T,t)=\begin{cases}\sup\{\alpha:\|T\|_\alpha^*\geq t\}, & (T,t)\neq 0\\ 1, & (T,t)=0\end{cases}$$

Then N^* is said to be a 2-fuzzy norm on $F(X)^*$ and the weak 2-fuzzy dual space $(F(X)^*, N^*)$ is a 2-fuzzy normed linear space.

Definition 3.2.3. The set of all weakly 2-fuzzy functional from $(F(X)^*, N^*)$ to R^+ is the dual of $(F(X)^*, N^*)$ known as the second weak 2- fuzzy dual of (F(X), N). It is denoted by $F(X)^{**}$.
Let $\mathcal{S} \in F(X)^{**}$ is a mapping from $F(X)^*$ to R^+. Define $N^{**}(\mathcal{S}, t) = \sup\{\alpha : \|\mathcal{S}\|_\alpha^{**} \geq t\}$
where $\|\mathcal{S}\|_\alpha^{**} = \sup\{t : N^*(\mathcal{S}, t) \leq \alpha\}$ (or) $\|\mathcal{S}\|_\alpha^{**} = \sup\left\{\frac{|\mathcal{S}(T)|}{\|T\|_\alpha^*}\right\}$

Theorem 3.2.1. $(F(X)^{**}, N^{**})$ is a 2-fuzzy normed linear space.

Proof:

(i) For all $t \in R$, $t \leq 0$, $N^*(\mathcal{S}, t) = 0$

$$\|\mathcal{S}\|_\alpha^{**} = \sup\{t : N^*(\mathcal{S}, t) \leq \alpha\} = 0.$$ Therefore $N^{**}(\mathcal{S}, t) = 0$.

(ii) For all $t \in R$, $t > 0$, $N^*(\mathcal{S}, t) = 1$

$$\|\mathcal{S}\|_\alpha^{**} = \sup\{t : N^*(\mathcal{S}, t) \leq \alpha\} = 1.$$ Therefore $N^{**}(\mathcal{S}, t) = 1$.

(iii) For all $t \in R$, $t > 0$ and $c \neq 0$

$N^*(c\mathcal{S}, t) = N^*(\mathcal{S}, t/|c|)$ and $\|c\mathcal{S}\|_\alpha^{**} = |c|\,\|\mathcal{S}\|_\alpha^{**}$

$$\begin{aligned} N^{**}(c\mathcal{S}, t) &= \sup\{\alpha : |c|\,\|\mathcal{S}\|_\alpha^{**} \geq t\} \\ &= \sup\{\alpha : \|\mathcal{S}\|_\alpha^{**} \geq t/|c|\} \\ &= N^{**}(\mathcal{S}, t/|c|) \end{aligned}$$

Therefore $N^{**}(c\mathcal{S}, t) = N^{**}(\mathcal{S}, t/|c|)$

(iv) For all $s, t \in R$

$N^*(\mathcal{S} + \mathcal{T}, s + t) \geq \min\{N^*(\mathcal{S}, s), N^*(\mathcal{T}, t)\}$ and $\|\mathcal{S} + \mathcal{T}\|_\alpha^{**} \leq \|\mathcal{S}\|_\alpha^{**} + \|\mathcal{T}\|_\alpha^{**}$

$$\begin{aligned} N^{**}(\mathcal{S} + \mathcal{T}, s + t) &= \sup\{\alpha : \|\mathcal{S} + \mathcal{T}\|_\alpha^{**} \geq s + t\} \\ &= \sup\{\alpha_1 + \alpha_2 : \|\mathcal{S}\|_\alpha^{**} + \|\mathcal{T}\|_\alpha^{**} \geq s + t\} \\ &\geq \sup\{\alpha_1 : \|\mathcal{S}\|_\alpha^{**} \geq s\} + \sup\{\alpha_2 : \|\mathcal{T}\|_\alpha^{**} \geq t\} \\ &= N^{**}(\mathcal{S}, s) + N^{**}(\mathcal{T}, t) \end{aligned}$$

So, $N^{**}(\mathcal{S} + \mathcal{T}, s + t) \geq \min\{N^{**}(\mathcal{S}, s), N^{**}(\mathcal{T}, t)\}$

(v) $N^{**}(\mathcal{S}, \cdot)$ be the non decreasing function such that $\lim N^{**}(\mathcal{S}, t) = 1$.

By definition, $N^{**}(\mathcal{S}, t) = \sup\{\alpha : \|\mathcal{S}\|_\alpha^{**} \geq t\}$ where $\alpha \in (0,1)$ and $\lim_{t\to\infty} N^{**}(\mathcal{S}, t) = 1$.

Therefore $(F(X)^{**}, N^{**})$ is 2-fuzzy normed linear space.

Theorem 3.2.2. Let $F(X)^{**}$ be the second weak 2-fuzzy dual of (F(X), N). If every element of f in (F(X),N) gives rise to a functional say $\mathcal{S}_f$ in $F(X)^{**}$ defined as $\mathcal{S}_f(T) = T(f)$ where $T \in F(X)^*$ then $\mathcal{S}_f$ belongs to $F(X)^{**}$ is linear for every element f in F(X).

Proof: Consider (i) $\mathcal{S}_{f_1+f_2}(T) = \mathcal{S}_{f_1}(T) + \mathcal{S}_{f_2}(T)$ and (ii) $\mathcal{S}_{kf}(T) = k\,\mathcal{S}_f(t)$ where $k \in R$, the space of real numbers

(i) $\mathcal{S}_{f_1+f_2}(T) = T(f_1 + f_2)$

$= T(f_1) + T(f_2)$

$= \mathcal{S}_{f_1}(T) + \mathcal{S}_{f_2}(T)$

(ii) $\mathcal{S}_{kf}(T) = T(kf)$

$= k\,T(f)$

$= k\,\mathcal{S}_f(T)$. Hence $\mathcal{S}_f$ is linear.

Theorem 3.2.3. $(F(X)^{**}, N^{**})$ is a 2-fuzzy normed linear space where N^{**} is defined as $N^{**}(\mathcal{S}, t) = \sup\{\alpha : \|\mathcal{S}\|_\alpha^{**} \geq t\}$ such that $N^{**}(\mathcal{S}_f, t) = N^*(T, t)$ for $T \in F(X)^*$

Proof: By definition

$$N^{**}(\mathcal{S}_f, t) = \sup\{\alpha : \|\mathcal{S}_f\|_\alpha^{**} \geq t\}$$

$$= \sup\{\alpha : \|\mathcal{S}_f\|_\alpha^{*} \geq t\} \qquad (\text{by } [7])$$

$N^{**}(\mathcal{S}_f, t) = \sup\{\alpha : \|T\|_\alpha^{*} \geq t\}$, since $\mathcal{S}_f(T) = T(f)$

$= N^*(T, t)$

Therefore $N^{**}(\mathcal{S}_f, t) = N^*(T, t)$.

Definition 3.2.4. The weak * topology on $F(X)^*$ is the weakest topology such that all $\mathcal{S} \in F(X)^{**}$ are weakly 2- fuzzy continuous. This topology is generated by the subbasis element

$S(f, T_0, r, t) = \{T \in F(X)^* \mid N^*(T-T_0, t) > 1-r\} = \{T \in F(X)^* \mid N^{**}(\mathcal{S}_f(T) - \mathcal{S}_f(T_0,)\ t) > 1-r\}$

Theorem 3.2.4. (Banach Alaoglu Theorem)

If F(X) is a 2-fuzzy normed linear space then the fuzzy closed set $\mathcal{B}^*$ in $F(X)^*$ is a fuzzy compact hausdorff space in the weak * topology on $F(X)^*$.

Proof: If T and U are distinct functionals in $\mathcal{B}^*$ then there exist $f \in F(X)$ such that $T(f) \neq U(f)$. If $\varepsilon = \frac{|T(f) - U(f)|}{3}$ then $S(f, T, \varepsilon, t)$ and $S(f, U, \varepsilon, t)$ are disjoint neighborhoods of T and U in weak * topology. Hence $\mathcal{B}^*$ is a hausdorff space.

For each $f \in F(X)$ define a closed interval $f_x = \left[\inf_{x \in X} f(x), \sup_{x \in X} f(x)\right]$ then the product of f_x, $\underset{f \in F(X)}{X} f_x$ is a closed subspace and by classical Tychonoff theorem, it is compact.

From the definition of weak * topology on $\mathcal{B}^*$ it is the same as the topology as a subspace of F(X). Since $\underset{f \in F(X)}{X} f_x$ is compact it is enough to show that $\mathcal{B}^*$ is closed as a subspace of $\underset{f \in F(X)}{X} f_x$. It is enough to show that if $T \in \overline{\mathcal{B}^*}$ then $T \in \mathcal{B}^*$. We know that $\overline{\mathcal{B}^*} = \inf\{C^* : C^*$ is a closed subspace containing $\mathcal{B}^*\}$. If $T \in F(X)$ then clearly $T \in \underset{f \in F(X)}{X} f_x$ and so $T \in \mathcal{B}^*$. Finally let us show that T is linear in F(X).For any $\varepsilon > 0$, let $f, g \in F(X)$, since $T \in \overline{\mathcal{B}^*}$ every subbasic open set containing T intersects $\mathcal{B}^*$ say at H such that $|T(f_1) - H(f_1)| < \varepsilon/3$, $|T(f_2) - H(f_2)| < \varepsilon/3$ and $|T(f_1 + f_2) - H(f_1 + f_2)| < \varepsilon/3$. Since H is linear, $H(f_1 + f_2) - H(f_1) - H(f_2) = 0$ and $H(kf) - k\,H(f) = 0$, $k \in K$, therefore

$$\begin{aligned}\left|T(f_1+f_2)-T(f_1)-T(f_2)\right| &= \left|(T(f_1+f_2)-H(f_1+f_2))-((T(f_1)-H(f_1))\right.\\ &\qquad \left.-(T(f_2)-H(f_2))\right|\\ &\le \left|T(f_1+f_2)-H(f_1+f_2)\right|+\left|T(f_1)-H(f_1)\right|\\ &\qquad +\left|T(f_2)-H(f_2)\right|\\ &< \frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon\end{aligned}$$

implies $T(f_1 + f_2) = T(f_1) + T(f_2)$, for $\varepsilon > 0$ and for any $k \in K$, the space of real numbers we have $\left|T(kf)-H(kf)\right| < \frac{\varepsilon}{2}$.Now

$$\begin{aligned}\left|T(kf)-kT(f)\right| &= \left|(T(kf)-H(kf))-(kT(f)-kH(f))\right|\\ &\le \left|T(kf)-H(kf)\right|+\left|k\right|\left|T(f)-H(f)\right|\\ &< \frac{\varepsilon}{2}+\left|k\right|\frac{\varepsilon}{2\left|k\right|}\\ &= \varepsilon\end{aligned}$$

implies $T(kf) = k\,T(f)$. Therefore T is linear.

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