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An M/(G$_1$,G$_2$)/1 Feedback Retrial Queue with Two Phase Service, Variant Vacation Policy Under Delaying Repair for Impatient Customer

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Abstract. In this paper, we discuss about the concept of single server feedback retrial queueing system with two types of service and each type consist of two essential phases under variant vacation policy. Customers are allowed to balk and renege at particular times. By using the supplementary variables method, steady state probability generating function for system size and orbit size are obtained. The system performance measures and important special cases are discussed. Numerical illustrations are analyzed to see the effect of system parameters.

Keywords: Feedback, two type service, variant vacation, balking, reneging

AMS Mathematics Subject Classification (2010): 60K25, 90B22, 68M20

1. Introduction

Retrial queues (or queues with repeated attempts) are characterized by the phenomenon that an arriving customer who finds the server busy upon arrival is obliged to leave the service area and repeat his demand after some time. Between trials, a blocked customer who remains in a retrial group is said to be in orbit. Queues in which customers are allowed to conduct retrials have wide applications in telephone switching systems and computers competing to gain service from a central processing unit. There is an extensive literature on the retrial queues. We refer the works by Artalejo and Gomez-Corral [1] and Artalejo [2] as a few. In a retrial queue, an arriving customer who finds the server busy has to leave the system or may join into the orbit. Later, after entering into orbit the reneging customers may decide to go to service area or leave the system. Such queues model many real world situations like web access, including call centers and computer systems, etc. Some of the authors like, Baruah et al. [3], Wang and Li [12] and Rajadurai et al. [10] discussed about the concept balking and reneging. In a vacation queueing system, the server may not be available for a period of time due to many reasons like, being checked for maintenance, working at other queues, scanning for new work or simply taking break. This period of time, when the server is unavailable for primary customers is referred as a vacation. Chang and Ke [4] examined on a batch retrial model with *J* vacations in which if orbit becomes empty, the server takes at most *J* vacations repeatedly until at least one customer appears in the orbit upon returning from a vacation. Using the supplementary variable technique, system characteristics are derived. Later, Ke and Chang [7] developed a model with the concept of retrial queueing system with balking and feedback.

The service interruptions are unavoidable phenomenon in many real life situations. In most of the studies, it is assumed that the server is available in the service station on a permanent basis and service station never fails. However, these assumptions are practically unrealistic. Ke and Choudhury [8] discussed about the batch arrival retrial queue with two phases of service under the concept of breakdown and delaying repair. While the server is working with any phase of service, it may breakdown at any instant and the service channel will fail for a short interval of time. The repair process does not start immediately after a breakdown and there is a delay time for repair to start. Chen et al. [6] studied a retrial queues with modified vacation and breakdowns. Authors like Choudhury et al. [5], Mokaddis et al. [9], Rajadurai et al. [11] and Wang and Li [12] discussed about the retrial queueing systems with the concept of breakdown and repair. However, no work has been down in the concept of impatient customers with two phases and two types of service under delaying repair. The suggested model has also potential application in the transfer model of an email system. In Simple Mail Transfer Protocol (SMTP) mail system uses to deliver the messages between mail servers for relaying. The results of this paper finds other applications in LAN, client-server communication, telephone network and software designs of various computer communications systems, packet switched networks, production lines and mail systems, etc.

2. Description of the model

In this paper, we consider a single server feedback retrial queueing system for impatient customers under variant vacation policy where subject to breakdown and delaying repair. Then the server provides two types of service and each types consists two essential phases. The detailed description of model is given as follows:

Arrival process: New customers arrive from outside according to a Poisson process with rateλ. We assume that there is no waiting space and therefore if an arriving customers find the server free, the arrival beings his service. If an arriving customer finds the server being busy, vacation or breakdown, the arrivals either leave the service area with probability *b* and join pool of blocked customers called an orbit, or balk the system with probability *1-b*. Measured from the moment when the server becomes idle, the customer at the head of the retrial queue competes with potential primary customers to decide which customer will enter service next. If a primary customer arrives first, the retrial customer may cancel its attempt for service and either returns to its position in the retrial queue with probability *r* or quits the system with probability *1-r*. Inter-retrial times have an arbitrary distribution $R(x)$ with corresponding Laplace-Stieltijes transform (LST) $R^*(t)$.

Service process: There is a single server who provides two types of service and each type contains two phases in succession, the first phase service (FPS) followed by the second phase service (SPS). If an arriving customer finds the server free, then he choose first type of service with probability $p_1$or choose second type of service with probability $p_2(p_1+p_2 = 1)$. After completion of two phase service, if the customer is unsatisfied with service, then he may rejoin the orbit as a feedback customer with probability p or may leave the system with probability $q = 1 - p$. The service times follow a general random variable on both types and both phases $S_i^{(k)}$ with distribution function (d.f) $S_i^{(k)}(x)$ and Laplace- Stieltijes transform (LST) $S_i^{(k)*}(t)$ (for k = 1,2 (phases) and i = 1, 2(types)) .

Vacation rule: Whenever the orbit is empty, the server leaves for a vacation of random length V. If no customer appears in the orbit when the server returns from a vacation, it leaves again for another vacation with the same length. Such pattern continues until it returns from a vacation to find at least one customer found in the orbit or it has already taken J vacations. If the orbit is empty at the end of the J^{th} vacation, the server remains idle for new arrivals in the system. At a vacation completion epoch the orbit is nonempty, the server waits for the customers in the orbit or new customers to arrive. The vacation time V has distribution function $V(x)$ and LST $V^*(t)$.

Breakdown process: While the server is working with any types of service, it may breakdown at any time and the service channel will fail for a short interval of time i.e. server is down for a short interval of time. The breakdowns i.e. server's life times are generated by exogenous Poisson processes with rates for $\alpha_1^{(k)}$ FPS on both types and for $\alpha_2^{(k)}$ SPS on both types, which we may call some sort of disaster during FPS on both types and SPS on both types periods respectively (k = 1,2).

Repair process: As soon as breakdown occurs the server is sent for repair, during that time it stops providing service to the arriving customer and waits for repair to start, which we may refer to as waiting period of the server. We define the waiting time as delay time. The delay time $D_i^{(k)}$ of the server for k^{th} type and i^{th} phase of service follows with d.f. $D_i^{(k)}(y)$ and LST $D_i^{(k)*}(t)$ (for i = 1,2 and k = 1,2). The customer who was just being served before server breakdown waits for the remaining service to complete. The repair time (denoted by $G_1^{(k)}$ for FPS on both types and $G_2^{(k)}$ for SPS on both types) distributions of the server for both the phases of service are assumed to be arbitrarily distributed with d.f. $G_i^{(k)}(y)$ and LST $G_i^{(k)*}(t)$ (for i=1,2 and k=1,2). Various stochastic processes involved in the system are assumed to be independent of each other.

3. System analysis

In this section, we develop the steady state difference-differential equations for the retrial system by treating the elapsed retrial time, the elapsed service times, the elapsed vacation times, the elapsed delay times and the elapsed repair times as supplementary variables. Then we derive the probability generating functions (PGF) for the server states, the PGF for number of customers in the system and orbit. Further, we assume that $R(0)=0,\ R(\infty)=1,$ $S_i^{(k)}(0)=0,\ S_i^{(k)}(\infty)=1,\ V(0)=0,\ V(\infty)=1$ (for i=1,2 and k=1,2) are continuous at x=0 and $D_i^{(k)}(0)=0, D_i^{(k)}(\infty)=1, G_i^{(k)}(0)=0,\ G_i^{(k)}(\infty)=1$ (for i=1,2 and k=1,2) are continuous at y=0. In addition, let $R^0(t),\ S_i^{0(k)}(t),\ V^0(t),\ D_i^{0(k)}(t),$ and $G_i^{0(k)}(t)$ be elapsed retrial times, service times, vacation times, delay times and repair times respectively at time t.The state of system at time t can be described by bivariate Markov process $\{C(t), N(t); t\geq 0\}$ where $C(t)$ denotes the server state (0,1,2,…$J+6$) depending on the server is idle, busy on both types in FPS or SPS, delay time on both types in FPS or SPS, repair on both types in FPS or SPS, 1st vacation,…or

J^{th} vacation. $N(t)$denotes the number of customers in the orbit at time t. So that the function $\theta(x),\ \mu_i^{(k)}(x),\ \gamma(x),\ \eta_i^{(k)}(y)$ and $\xi_i^{(k)}(y)$ are the conditional completion rates for repeated attempts, service, vacation, delay time and repair time respectively (for i =1,2 and k= 1,2)

$$\theta(x)dx=\frac{dR(x)}{1-R(x)},\mu_i^{(k)}(x)dx=\frac{dS_i^{(k)}(x)}{1-S_i^{(k)}(x)},\gamma(x)dx=\frac{dV(x)}{1-V(x)},\ \eta_i^{(k)}(y)dy=\frac{dD_i^{(k)}(y)}{1-D_i^{(k)}(y)},\xi_i^{(k)}(y)dy=\frac{dG_i^{(k)}(y)}{1-G_i^{(k)}(y)}.$$

Let $\{t_n;\ n = 1,2,\ldots\}$ be the sequence of epochs at which either a service period completion occurs or a vacation time ends. The sequence of random vectors $Z_n=\{C(t_n+),\ N(t_n+)\}$ forms a Markov chain which is embedded in the retrial queueing system. The embedded Markov chain $\{Z_n;\ n\in N\}$ is ergodic if and only if $\rho<1$, where $\rho=r(1-R^*(\lambda))+p+\varpi b,$

$$\varpi=\lambda\left\{\sum_{i=1}^{2}p_i\left(E^{(i)}(S_1)\left[1+\alpha_1^{(i)}\left(E^{(i)}(D_1)+E^{(i)}(G_1)\right)\right]+E^{(i)}(S_2)\left[1+\alpha_2^{(i)}\left(E^{(i)}(D_2)+E^{(i)}(G_2)\right)\right]\right)\right\}$$

The following probabilities are used in sequent sections for for $t\ge 0,\ x\ge 0,\ y\ge 0$ and $n\ge 0$ (i=1,2, k=1,2 and j=1,2…J). For the process $\{N(t),\ t\ge 0\}$, we define the probabilities $I_0(t)=P\{C(t)=0,\ N(t)=0\}$ and the probability densities

$$I_n(x,t)dx=P\left\{C(t)=0,N(t)=n,\ x\le R^0(t)<x+dx\right\},\ P_{1,n}^{(k)}(x,t)dx=P\left\{C(t)=1,N(t)=n,\ x\le S_1^{\,0}(t)<x+dx\right\},$$

$$P_{2,n}^{(k)}(x,t)dx=P\left\{C(t)=2,\ N(t)=n,\ x\le S_2^{\,0}(t)<x+dx\right\},\ Q_{1,n}^{(k)}(x,y,t)dy=P\left\{C(t)=3,\ N(t)=n,\ y\le D_1^{\,0}(t)<y+dy/S_1^{\,0}(t)=x\right\}$$

$$Q_{2,n}^{(k)}(x,y,t)dy=P\left\{C(t)=4,\ N(t)=n,\ y\le D_2^{\,0}(t)<y+dy/S_2^{\,0}(t)=x\right\},\ R_{1,n}^{(k)}(x,y,t)dy=P\left\{C(t)=5,\ N(t)=n,\ \ y\le G_1^{\,0}(t)<y+dy/S_1^{\,0}(t)=x\right\},$$

$$R_{2,n}^{(k)}(x,y,t)dy=P\left\{C(t)=6,\ N(t)=n,\ y\le G_2^{\,0}(t)<y+dy/S_2^{\,0}(t)=x\right\},\ \Omega_{j,n}(x,t)dx=P\left\{C(t)=j+6,\ N(t)=n,\ x\le V^0(t)<x+dx\right\},$$

We assume that the stability condition is fulfilled in the sequel and so that we can set

$$I_0=\lim_{t\to\infty}I_0(t),\ I_n(x)=\lim_{t\to\infty}I_n(x,t),\ P_{i,n}^{(k)}(x)=\lim_{t\to\infty}P_{i,n}^{(k)}(x,t),\ Q_{i,n}^{(k)}(x,y)dy=\lim_{t\to\infty}Q_{i,n}^{(k)}(x,y,t),R_{i,n}^{(k)}(x,y)=\lim_{t\to\infty}R_{i,n}^{(k)}(x,y,t),\Omega_{j,n}(x)=\lim_{t\to\infty}\Omega_{j,n}(x,t).$$

Steady state distribution 3.1

By the method of supplementary variable technique, we obtain the following system of equations that govern the dynamics of the system behavior for (i=1,2, k=1,2 and j=1,2,…J)

$$\lambda I_0=\int_0^\infty\Omega_{J,0}(x)\gamma(x)dx \tag{3.1}$$

$$\frac{dI_n(x)}{dx}+[\lambda+\theta(x)]\psi_n(x)=0,\ n\ge 1 \tag{3.2}$$

$$\frac{dP_{i,0}^{(k)}(x)}{dx}+[\lambda+\alpha_i^{(k)}+\mu_i^{(k)}(x)]P_{i,0}^{(k)}(x)=\lambda(1-b)P_{i,0}^{(k)}(x)+\int_0^\infty\xi_i^{(k)}(y)R_{i,0}^{(k)}(x,y)dy,n=0, \tag{3.3}$$

$$\frac{dP_{i,n}^{(k)}(x)}{dx}+[\lambda+\alpha_i^{(k)}+\mu_i^{(k)}(x)]P_{i,n}^{(k)}(x)=\ \lambda(1-b)P_{i,n}^{(k)}(x)+\lambda b\sum_{k=1}^{n}\chi_k P_{i,n-k}^{(k)}(x)+\int_0^\infty\xi_i(y)R_{i,n}^{(k)}(x,y)dy,\ n\ge 1, \tag{3.4}$$

$$\frac{d\Omega_{j,0}(x)}{dx}+[\lambda+\gamma(x)]\Omega_{j,0}(x)=\lambda(1-b)\Omega_{j,0}(x),\ n=0, \tag{3.5}$$

$$\frac{d\Omega_{j,n}(x)}{dx}+[\lambda+\gamma(x)]\Omega_{j,n}(x)=\lambda(1-b)\Omega_{j,n}(x)+\lambda b\sum_{k=1}^{n}\chi_k\Omega_{j,n-k}(x),\ \ n\ge 1, \tag{3.6}$$

$$\frac{dQ_{i,0}^{(k)}(x,y)}{dy}+[\lambda+\eta_i(y)]Q_{i,0}^{(k)}(x,y)=\lambda(1-b)Q_{i,0}^{(k)}(x,y),\ n=0, \tag{3.7}$$

$$\frac{dQ_{i,n}^{(k)}(x,y)}{dy}+[\lambda+\eta_i(y)]Q_{i,n}^{(k)}(x,y)=\lambda(1-b)Q_{i,n}^{(k)}(x,y)+\lambda b\sum_{k=1}^{n}\chi_k Q_{i,n-k}^{(k)}(x,y),\quad n\geq 1, \tag{3.8}$$

$$\frac{dR_{i,0}^{(k)}(x,y)}{dy}+[\lambda+\xi_i(y)]R_{i,0}^{(k)}(x,y)=\lambda(1-b)R_{i,0}^{(k)}(x,y),\ n=0, \tag{3.9}$$

$$\frac{dR_{i,n}^{(k)}(x,y)}{dy}+[\lambda+\xi_i(y)]R_{i,n}^{(k)}(x,y)=\lambda(1-b)R_{i,n}^{(k)}(x,y)+\lambda b\sum_{k=1}^{n}\chi_k R_{i,n-k}^{(k)}(x,y),\quad n\geq 1, \tag{3.10}$$

The steady state boundary conditions at $x = 0$ and $y = 0$ are

$$I_n(0)=\sum_{j=1}^{J}\int_0^\infty\Omega_{j,n}(x)\gamma(x)dx+q\left[\int_0^\infty P_{2,n}^{(1)}(x)\mu_2(x)dx+\int_0^\infty P_{2,n}^{(2)}(x)\mu_2(x)dx\right]+p\left[\int_0^\infty P_{2,n-1}^{(1)}(x)\mu_2(x)dx+\int_0^\infty P_{2,n-1}^{(2)}(x)\mu_2(x)dx\right],\ n\geq 1 \tag{3.11}$$

$$P_{1,n}^{(k)}(0)=p_k\left[\int_0^\infty I_{n+1}(x)\theta(x)dx+\lambda r\int_0^\infty I_n(x)dx+\lambda(1-r)\int_0^\infty I_{n+1}(x)dx\right],\ n\geq 1, \tag{3.12}$$

$$P_{2,n}^{(k)}(0)=\int_0^\infty P_{1,n}^{(k)}(x)\mu_1^{(k)}(x)dx,\ n\geq 1, \tag{3.13}$$

$$\Omega_{1,n}(0)=\begin{cases} q\left(\int_0^\infty P_{2,0}^{(1)}(x)\mu_2^{(1)}(x)dx+\int_0^\infty P_{2,0}^{(2)}(x)\mu_2^{(2)}(x)dx,\right), & n=0\\ 0 & ,\quad n\geq 1\end{cases} \tag{3.14}$$

$$\Omega_{j,n}(0)=\begin{cases}\int_0^\infty\Omega_{j-1,0}(x)\gamma(x)dx, & n=0,\ j=2,3...J\\ 0 & ,\ n\geq 1,\ j=2,3....J\end{cases} \tag{3.15}$$

$$Q_{i,n}^{(k)}(x,0)=\alpha_i^{(k)}P_{i,n}^{(k)}(x),\quad n\geq 1, \tag{3.16}$$

$$R_{i,n}^{(k)}(x,0)=\int_0^\infty Q_{i,n}^{(k)}(x,y)\eta_i^{(k)}(y)dy,\ n\geq 1, \tag{3.17}$$

The normalizing condition is

$$I_0+\sum_{n=1}^{\infty}\int_0^\infty I_n(x)dx+\sum_{n=0}^{\infty}\sum_{k=1}^{2}\sum_{i=1}^{2}\int_0^\infty P_{i,n}^{(k)}(x)dx+\sum_{n=0}^{\infty}\sum_{k=1}^{2}\sum_{i=1}^{2}\int_0^\infty\int_0^\infty Q_{i,n}^{(k)}(x,y)dxdy+\sum_{n=0}^{\infty}\sum_{k=1}^{2}\sum_{i=1}^{2}\int_0^\infty\int_0^\infty R_{i,n}^{(k)}(x,y)dxdy+\sum_{j=1}^{J}\sum_{n=0}^{\infty}\Omega_{j,n}(x)dx=1 \tag{3.18}$$

The steady state solution 3.2

The probability generating function technique is used here to obtain the steady state solution of the retrial queueing model. To solve the above equations, we define the generating functions for $|z|\leq 1$, for (i=1,2 and k =1,2) as follows:

$$I(x,z)=\sum_{n=1}^{\infty}I_n(x)z^n;\ P_i^{(k)}(x,z)=\sum_{n=0}^{\infty}P_{i,n}^{(k)}(x)z^n;\ \Omega_j(x,z)=\sum_{n=0}^{\infty}\Omega_{j,n}(x)z^n;\ Q_i^{(k)}(x,y,z)=\sum_{n=0}^{\infty}Q_{i,n}^{(k)}(x,y)z^n;$$

$$R_i^{(k)}(x,y,z)=\sum_{n=0}^{\infty}R_{i,n}^{(k)}(x,y)z^n;\ X(z)=\sum_{n=1}^{\infty}\chi_n z^n$$

Now multiplying the steady state equation and steady state boundary condition Eq. (3.2) - Eq. (3.17) by z^n and summing over n, (n = 0,1,2...)

$$\frac{\partial I(x,z)}{\partial x}+[\lambda+\theta(x)]I(x,z)=0 \tag{3.19}$$

$$\frac{\partial P_i^{(k)}(x,z)}{\partial x}+[\lambda b(1-z)+\alpha_i^{(k)}+\mu_i^{(k)}(x)]P_i^{(k)}(x,z)=\int_0^{\infty}\xi_i^{(k)}(y)R_i^{(k)}(x,y,z)dy, \tag{3.20}$$

$$\frac{\partial \Omega_j(x,z)}{\partial x}+[\lambda b(1-z)+\gamma(x)]\Omega_j(x,z)=0 \tag{3.21}$$

$$\frac{\partial Q_{i,n}^{(k)}(x,y,z)}{\partial y}+[\lambda b(1-z)+\eta_i^{(k)}(y)]Q_{i,n}^{(k)}(x,y,z)=0 \tag{3.22}$$

$$\frac{\partial R_{i,n}^{(k)}(x,y,z)}{\partial y}+[\lambda b(1-z)+\xi_i^{(k)}(y)]R_{i,n}^{(k)}(x,y,z)=0 \tag{3.23}$$

$$I(0,z)=\sum_{j=1}^{J}\int_0^{\infty}\Omega_j(x,z)\gamma(x)dx-\sum_{j=1}^{J}\Omega_{j,0}(0)-\lambda I_0+(pz+q)\left[\int_0^{\infty}P_{2,n}^{(1)}(x,z)\mu_2^{(1)}(x)dx+\int_0^{\infty}P_{2,n}^{(2)}(x,z)\mu_2^{(2)}(x)dx\right] \tag{3.24}$$

$$P_1^{(k)}(0,z)=p_k\left[\frac{1}{z}\int_0^{\infty}I(x,z)\theta(x)dx+\lambda\left(\frac{(1-r)+rz}{z}\right)\int_0^{\infty}I(x,z)dx+\lambda I_0\right], \tag{3.25}$$

$$P_2^{(k)}(0,z)=\int_0^{\infty}P_1^{(k)}(x,z)\mu_1^{(k)}(x)dx, \tag{3.26}$$

$$Q_i^{(k)}(x,0,z)=\alpha_i^{(k)}P_i^{(k)}(x,z), \tag{3.27}$$

$$R_i^{(k)}(x,0,z)=\int_0^{\infty}Q_i^{(k)}(x,y,z)\eta_i^{(k)}(y)dy, \tag{3.28}$$

Solving the partial differential equations Eq. (3.19)-Eq. (3.23), it follows that

$$I(x,z)=I(0,z)[1-R(x)]e^{-\lambda x} \tag{3.29}$$

$$P_i^{(k)}(x,z)=P_i^{(k)}(0,z)[1-S_i^{(k)}(x)]e^{-A_i^{(k)}(z)x}, \tag{3.30}$$

$$\Omega_j(x,z)=\Omega_j(0,z)[1-V(x)]e^{-b(z)x},\ \text{for}(1\le j\le J) \tag{3.31}$$

$$Q_i^{(k)}(x,y,z)=Q_i^{(k)}(x,0,z)[1-D_i^{(k)}(y)]e^{-b(z)y}, \tag{3.32}$$

$$R_i^{(k)}(x,y,z)=R_i^{(k)}(x,0,z)[1-G_i^{(k)}(y)]e^{-b(z)y}, \tag{3.33}$$

where $A_i^{(k)}(z)=b(z)+\alpha_i^{(k)}\left[1-D_i^{(k)*}(b(z))G_i^{(k)*}(b(z))\right]$ *and* $b(z)=\lambda b(1-z)$.

From (3.5) we obtain, $\Omega_{j,0}(x)=\Omega_{j,0}(0)[1-V(x)]\exp^{-\lambda bx}$, $j=1,2,...J$ (3.34)

Multiplying with equation (3.34) by $\gamma(x)$ on both sides for j = J and integrating with respect to x from 0 to ∞, then from (3.1) we have, $\Omega_{J,0}(0)=\frac{\lambda P_0}{V^*(\lambda b)}$ (3.35)From the equations (3.34) and solving (3.15), (3.35) over the range *j = J-1, J-2,..1,* we get on simplification $\Omega_{j,0}(0)=\Omega_j(0,z)=\frac{\lambda P_0}{[V^*(\lambda b)]^{J-j+1}},\ j=1,2...J$ (3.36)After make calculations substitute the solved equations in (3.29)-(3.33) and makes a direct calculation, then we get the limiting probability generating functions $I(x,z),\ \Omega_j(x,z),\ P_i^{(k)}(x,z),\ Q_i^{(k)}(x,z),\ R_i^{(k)}(x,z)$.We define the partial probability generating functions as, for (i=1,2 and k=1,2)

$$I(z)=\int_0^\infty I(x,z)dx,\ P_i^{(k)}(z)=\int_0^\infty P_i^{(k)}(x,z)dx,\ \ Q_i^{(k)}(x,z)=\int_0^\infty Q_i^{(k)}(x,y,z)dy,\ \ Q_i^{(k)}(z)=\int_0^\infty Q_i^{(k)}(x,z)dx,$$

$$R_i^{(k)}(x,z)=\int_0^\infty R_i^{(k)}(x,y,z)dy,\ R_i^{(k)}(z)=\int_0^\infty R_i^{(k)}(x,z)dx,\ \ \Omega_j(z)=\int_0^\infty \Omega_j(x,z)dx,(\text{for } j=1,2...J)$$

Theorem 3.1. *Under the stability condition ρ<1, the stationary distributions of the number of customers in the system when server being idle, busy on both types, on vacation, under delaying repair on both types and under repair on both types* (for *i=1,2*, *k* = 1,2 and 1≤ *j* ≤ *J*) *are given by*

$$I(z)=\left\{z(1-R^*(\lambda))I_0\Big((N(z)-1)+(pz+q)\Big(p_1S_1^{(1)*}\Big(A_1^{(1)}(z)\Big)S_2^{(1)*}\Big(A_2^{(1)}(z)\Big)+p_2S_1^{(2)*}\Big(A_1^{(2)}(z)\Big)S_2^{(2)*}\Big(A_2^{(2)}(z)\Big)\Big)\Big)\Big/Dr(z)\right\} \quad (3.37)$$

$$Dr(z)=\left\{z-(pz+q)\Big(R^*(\lambda)+[(1-r)+rz](1-R^*(\lambda))\Big)\Big[p_1S_1^{(1)*}\Big(A_1^{(1)}(z)\Big)S_2^{(1)*}\Big(A_2^{(1)}(z)\Big)+p_2S_1^{(2)*}\Big(A_1^{(2)}(z)\Big)S_2^{(2)*}\Big(A_2^{(2)}(z)\Big)\Big]\right\}$$

$$P_1^{(k)}(z)=\left\{\lambda I_0p_k\Big(1-S_1^{(k)*}\Big(A_1^{(k)}(z)\Big)\Big)\Big\{\Big(N(z)-1\Big)\Big(R^*(\lambda)+\big[(1-r)+rz\big](1-R^*(\lambda))\Big)+z\Big\}\right\}\Big/A_1^{(k)}(z)Dr(z) \quad (3.38)$$

$$P_2^{(k)}(z)=\left\{\lambda I_0p_kS_1^{(k)*}\Big(A_1^{(k)}(z)\Big)\Big(1-S_2^{(k)*}\Big(A_2^{(k)}(z)\Big)\Big)\Big\{\Big(N(z)-1\Big)\Big(R^*(\lambda)+\big[(1-r)+rz\big](1-R^*(\lambda))\Big)+z\Big\}\right\}\Big/A_2^{(k)}(z)Dr(z) \quad (3.39)$$

$$Q_1^{(k)}(z)=\left\{\lambda I_0p_k\alpha_1^{(k)}\Big(1-S_1^{(k)*}\Big(A_1^{(k)}(z)\Big)\Big)\Big(1-D_1^{(k)*}(b(z))\Big)\Big\{\Big(N(z)-1\Big)\Big(R^*(\lambda)+\big[(1-r)+rz\big](1-R^*(\lambda))\Big)+z\Big\}\right\}\Big/A_1^{(k)}(z)b(z)Dr(z) \quad (3.40)$$

$$Q_2^{(k)}(z)=\left\{\begin{array}{l}\lambda I_0p_k\alpha_2^{(k)}\Big(1-S_2^{(k)*}\Big(A_2^{(k)}(z)\Big)\Big)\Big(1-D_2^{(k)*}(b(z))\Big)S_1^{(k)*}\Big(A_1^{(k)}(z)\Big)\\ \Big\{\Big(N(z)-1\Big)\Big(R^*(\lambda)+\big[(1-r)+rz\big](1-R^*(\lambda))\Big)+z\Big\}\end{array}\right\}\Big/A_2^{(k)}(z)b(z)Dr(z) \quad (3.41)$$

$$\Omega_j(z)=\frac{I_0[V^*(\lambda b(1-z))-1]}{b(z-1)[V^*(\lambda b)]^{J-j+1}},\ j=1,2,...J \quad (3.42)$$

$$R_1^{(k)}(z)=\left\{\begin{array}{l}\lambda I_0p_k\alpha_1^{(k)}\Big\{\Big(N(z)-1\Big)\Big(R^*(\lambda)+\big[(1-r)+rz\big](1-R^*(\lambda))\Big)+z\Big\}\\ D_1^{(k)*}\big[b(z)\big]\Big(1-S_1^{(k)*}\Big(A_1^{(k)}(z)\Big)\Big)\Big(1-G_1^{(k)*}(b(z))\Big)\end{array}\right\}\Big/A_1^{(k)}(z)b(z)Dr(z) \quad (3.43)$$

$$R_2^{(k)}(z)=\left\{\begin{array}{l}\lambda I_0p_k\alpha_2^{(k)}\Big\{\Big(N(z)-1\Big)\Big(R^*(\lambda)+\big[(1-r)+rz\big](1-R^*(\lambda))\Big)+z\Big\}\\ S_1^{(k)*}\Big[A_1^{(k)}(z)\Big]D_2^{(k)*}\big[b(z)\big]\Big(1-S_2^{(k)*}\Big(A_2^{(k)}(z)\Big)\Big)\Big(1-G_2^{(k)*}(b(z))\Big)\end{array}\right\}\Big/A_2^{(k)}(z)b(z)Dr(z) \quad (3.44)$$

Then the only point I_0 is unknown, which can be determined using the normalizing condition. $I_0+I(1)+\sum_{i=1}^{2}\Big(P_i^{(k)}(1)+Q_i^{(k)}(1)+R_i^{(k)}(1)\Big)+\sum_{j=1}^{J}\Omega_j(1)=1.$ Thus, by setting z = 1 in (3.37)–(3.44) and applying L-Hospital's rule whenever necessary, after using the normalizing condition and rearrange, we get $I_0=\left\{\frac{1}{\beta}\left(1-r(1-R^*(\lambda))-p-\varpi b\right)\right\}$ (3.45)

$$N(z)=\frac{1-[V^*(\lambda b)]^J}{[V^*(\lambda b)]^J\left(1-V^*(\lambda b)\right)}\left[V^*(\lambda b(1-z))-1\right],\ A_i^{(k)}(z)=b(z)+\alpha_i^{(k)}\left[1-D_i^{(k)*}\left(b(z)\right)G_i^{(k)*}\left(b(z)\right)\right]\ \text{and}\ b(z)=\lambda b(1-z)$$

$$N'(1)=\frac{\left\{1-\left[V^*(\lambda b)\right]^J\right\}\lambda bE(V)}{\left[V^*(\lambda b)\right]^J\left(1-V^*(\lambda b)\right)},\ \beta=\left\{\frac{N'(1)}{b}\left(1-p+(1-R^*(\lambda))(b-r)\right)+(1+\varpi)\left[1-r(1-R^*(\lambda))\right]-R^*(\lambda)(b\varpi+p)\right\}$$

Theorem 3.2. *Under the stability condition $\rho < 1$, probability generating function of number of customers in the system and orbit size distribution at stationary point of time is*

$$K(z)=I_0\begin{pmatrix} z\Big\{1-\Big(p_1S_1^{(1)*}\big[A_1^{(1)}(z)\big]S_2^{(1)*}\big[A_2^{(1)}(z)\big]+p_2S_1^{(2)*}\big[A_1^{(2)}(z)\big]S_2^{(2)*}\big[A_2^{(2)}(z)\big]\Big)\Big\}\times\Big\{\big(N(z)-1\big)\Big(R^*(\lambda)+\big[(1-r)+rz\big](1-R^*(\lambda))\Big)+z\Big\} \\ -N(z)\Big\{z-(pz+q)\Big(R^*(\lambda)+\big[(1-r)+rz\big](1-R^*(\lambda))\Big)\Big(p_1S_1^{(1)*}\big[A_1^{(1)}(z)\big]S_2^{(1)*}\big[A_2^{(1)}(z)\big]+p_2S_1^{(2)*}\big[A_1^{(2)}(z)\big]S_2^{(2)*}\big[A_2^{(2)}(z)\big]\Big)\Big\} \\ +b(1-z)\Big\{z\Big(R^*(\lambda)+N(z)(1-R^*(\lambda))\Big)+(pz+q)\Big((1-r)(z-1)(1-R^*(\lambda))-R^*(\lambda)\Big)\Big(p_1S_1^{(1)*}\big[A_1^{(1)}(z)\big]S_2^{(1)*}\big[A_2^{(1)}(z)\big] \\ +p_2S_1^{(2)*}\big[A_1^{(2)}(z)\big]S_2^{(2)*}\big[A_2^{(2)}(z)\big]\Big)\Big\} \end{pmatrix} \quad (3.46)$$

$$\times\Big\{b(1-z)\Big\{z-(pz+q)\Big(R^*(\lambda)+[(1-r)+rz](1-R^*(\lambda))\Big)\Big[p_1S_1^{(1)*}\big(A_1^{(1)}(z)\big)S_2^{(1)*}\big(A_2^{(1)}(z)\big)+p_2S_1^{(2)*}\big(A_1^{(2)}(z)\big)S_2^{(2)*}\big(A_2^{(2)}(z)\big)\Big]\Big\}\Big\}^{-1}$$

$$H(z)=I_0\begin{pmatrix} \Big\{1-\Big(p_1S_1^{(1)*}\big[A_1^{(1)}(z)\big]S_2^{(1)*}\big[A_2^{(1)}(z)\big]+p_2S_1^{(2)*}\big[A_1^{(2)}(z)\big]S_2^{(2)*}\big[A_2^{(2)}(z)\big]\Big)\Big\}\times\Big\{\big(N(z)-1\big)\Big(R^*(\lambda)+\big[(1-r)+rz\big](1-R^*(\lambda))\Big)+z\Big\} \\ -N(z)\Big\{z-(pz+q)\Big(R^*(\lambda)+\big[(1-r)+rz\big](1-R^*(\lambda))\Big)\Big(p_1S_1^{(1)*}\big[A_1^{(1)}(z)\big]S_2^{(1)*}\big[A_2^{(1)}(z)\big]+p_2S_1^{(2)*}\big[A_1^{(2)}(z)\big]S_2^{(2)*}\big[A_2^{(2)}(z)\big]\Big)\Big\} \\ +b(1-z)\Big\{z\Big(R^*(\lambda)+N(z)(1-R^*(\lambda))\Big)+(pz+q)\Big((1-r)(z-1)(1-R^*(\lambda))-R^*(\lambda)\Big)\Big(p_1S_1^{(1)*}\big[A_1^{(1)}(z)\big]S_2^{(1)*}\big[A_2^{(1)}(z)\big] \\ +p_2S_1^{(2)*}\big[A_1^{(2)}(z)\big]S_2^{(2)*}\big[A_2^{(2)}(z)\big]\Big)\Big\} \end{pmatrix} \quad (3.47)$$

$$\times\Big\{b(1-z)\Big\{z-(pz+q)\Big(R^*(\lambda)+[(1-r)+rz](1-R^*(\lambda))\Big)\Big[p_1S_1^{(1)*}\big(A_1^{(1)}(z)\big)S_2^{(1)*}\big(A_2^{(1)}(z)\big)+p_2S_1^{(2)*}\big(A_1^{(2)}(z)\big)S_2^{(2)*}\big(A_2^{(2)}(z)\big)\Big]\Big\}\Big\}^{-1}$$

Proof: The probability generating function of the number of customer in the system ($K(z)$) and the probability generating function of the number of customer in the orbit ($H(z)$) is obtained by using

$$K(z)=I_0+I(z)+z\sum_{i=1}^{2}\left(P_i^{(k)}(z)+Q_i^{(k)}(z)+R_i^{(k)}(z)\right)+\sum_{j=1}^{J}\Omega_j(z)=1 \text{ and } H(z)=I_0+I(z)+\sum_{i=1}^{2}\left(P_i^{(k)}(z)+Q_i^{(k)}(z)+R_i^{(k)}(z)\right)+\sum_{j=1}^{J}\Omega_j(z).$$

Substituting(3.37)–(3.45) in the above results (3.46) and (3.47) can be obtained by direct calculation.

3.1. Performance measures and special cases

In this section, we derive system performance measures the mean number of customers in the orbit (Lq) and system (Ls), the mean time a customer spends in the system (Ws) and orbit (Wq).

(i) The mean number of customers in the orbit (Lq) under steady state condition is obtained by differentiating (3.47) with respect to z and evaluating at z = 1

$$L_q = \frac{Nr(z)}{Dr(z)} = \lim_{z\to 1}\frac{d}{dz}H(z) = H'(1) = I_0\left[\frac{Nr'''(1)Dr''(1) - Dr'''(1)Nr''(1)}{3\left(Dr''(1)\right)^2}\right]$$

$$Nr''(1) = 2N'(1)\left[(1-R^*(\lambda))(r-b)-q\right] - 2\left(\varpi\left[1-r(1-R^*(\lambda))\right] + b\left[(1-r)(1-R^*(\lambda)) + qR^*(\lambda)\right]\right)$$

$$Nr'''(1) = -N'(1)\left[6(1-R^*(\lambda))\left(6r\varpi + pr + b\right) + 6p\varpi + 5\tau\right] - 3N''(1)\left[q + (b-r)(1-R^*(\lambda))\right] + \tau\left(3bR^*(\lambda) - 2\left[1-r(1-R^*(\lambda))\right]\right)$$
$$-6b(1-r)(1-R^*(\lambda))(p-\varpi)$$

$$Dr''(1) = -2b\left[1-\left(1-r(1-R^*(\lambda))\right) - p - \varpi\right]$$

$$Dr'''(1) = 3b\left[\tau + 2\varpi\left(r(1-R^*(\lambda)) + p\right) + 2pr(1-R^*(\lambda))\right],\ \text{where } N''(1) = \frac{(\lambda b)^2 E(V^2)\left(1-[V^*(\lambda b)]^J\right)}{\left(1-V^*(\lambda b)\right)[V^*(\lambda b)]^J};\ \tau = \sum_{i=1}^{2} p_i\beta_i$$

$$\beta_i = \Bigg(E^{(i)}(S_1^2)\lambda^2 b^2\left[1+\alpha_1^{(i)}\left(E^{(i)}(D_1)+E^{(i)}(G_1)\right)\right]^2 + E^{(i)}(S_2^2)\lambda^2 b^2\left[1+\alpha_2^{(i)}\left(E^{(i)}(D_2)+E^{(i)}(G_2)\right)\right]^2 + \alpha_1^{(i)}E^{(i)}(S_1)\lambda^2 b^2\left[\left(E^{(i)}(D_1^2)+2E^{(i)}(D_1)E^{(i)}(G_1)+E^{(i)}(G_1^2)\right)\right]$$
$$+\alpha_2^{(i)}E^{(i)}(S_2)\lambda^2 b^2\left[\left(E^{(i)}(D_2^2)+2E^{(i)}(D_2)E^{(i)}(G_2)+E^{(i)}(G_2^2)\right)\right] + 2E^{(i)}(S_1)E^{(i)}(S_2)\lambda^2 b^2\left[1+\alpha_1^{(i)}\left(E^{(i)}(D_1)+E^{(i)}(G_1)\right)\right]\left[1+\alpha_2^{(i)}\left(E^{(i)}(D_2)+E^{(i)}(G_2)\right)\right]\Bigg)$$

(ii) The mean number of customers in the system (Ls) under steady state condition is obtained by differentiating (3.46) with respect to z and evaluating at $z = 1$

$$L_s = I_0\left[\frac{Nr'''(1)Dr''(1) - Dr'''(1)Nr''(1)}{3\left(Dr''(1)\right)^2}\right]$$

$$Nr'''(1) = -N'(1)\left[6(1-R^*(\lambda))\left(6r\varpi + pr + b\right) + 6p\varpi + 5\tau\right] - 3N''(1)\left[q + (b-r)(1-R^*(\lambda))\right] + \tau\left(3bR^*(\lambda) - 2\left[1-r(1-R^*(\lambda))\right]\right)$$
$$-3\left(N'(1)+1-r(1-R^*(\lambda))\right)(\tau+2\varpi) - 6b(1-r)(1-R^*(\lambda))(p-\varpi)$$

(iii) The average time a customer spends in the system (W_s) and orbit (W_q) under steady-state condition due to Little's formula is, $L_s = \lambda W_s$ and $L_q = \lambda W_q$

Special cases 3.1.1

In this section, we analyze briefly some special cases of our model, which are consistent with the existing literature.

Case (i): Single type and phase service, No reneging, No breakdown

Let $p_1 = 1$, $Pr\,[S_2 = 0] = 1$, $r = 0$ and $\alpha_1 = \alpha_2 = 0$. Our model can be reduced to a modified vacation for an M/G/1 retrial queueing system with balking and feedback. The following result is equivalent to the result by Ke and Chang [7].

$$K(z) = I_0\left(\begin{array}{l} z\left\{1-S_1^{(1)*}\left[A_1^{(1)}(z)\right]\right\}\left\{\left(N(z)-1\right)R^*(\lambda) + z\left[R^*(\lambda)+N(z)(1-R^*(\lambda))\right]\right\} - N(z)\left\{z-(pz+q)\left(R^*(\lambda)+z(1-R^*(\lambda))\right)S_1^{(1)*}\left[A_1^{(1)}(z)\right]\right\} \\ +b(1-z)\left\{z\left(R^*(\lambda)+N(z)(1-R^*(\lambda))\right) - (pz+q)R^*(\lambda)S_1^{(1)*}\left[A_1^{(1)}(z)\right]\right\} \end{array}\right)$$
$$\times\left\{b(1-z)\left\{z-(pz+q)\left(R^*(\lambda)+z(1-R^*(\lambda))\right)S_1^{(1)*}\left(A_1^{(1)}(z)\right)\right\}\right\}^{-1}$$

Case (ii): Single type and phase service, No balking and reneging, No delaying repair.

Let p_1=1, $Pr[S_2=0]$=1, $p=0$, $b=r$=1 and $\eta_1=\eta_2=0$. Our model can be reduced to an M/G/1 retrial queueing system with a modified vacations and server breakdowns. In this case,

$K(z)$ can be simplified and the following expression is coincided with the result in Chen et. al [6].

$$K(z) = \left\{ \frac{R^*(\lambda) - \lambda E(S_1^{(1)})}{N'(1) + R^*(\lambda)} \right\} \times \frac{\left(N(z)\left\{ \left(R^*(\lambda) + z(1 - R^*(\lambda)) \right) + R^*(\lambda)(z-1) \right\} S_1^{(1)*}\left[A_1(z) \right] \right)}{\left\{ z - S_1^{(1)*}\left[A_1(z) \right] \left(R^*(\lambda) + z(1 - R^*(\lambda)) \right) \right\}}$$

3.2. Numerical illustration

In this section, we present some numerical examples using Matlab in order to illustrate the effect of various parameters in the system performance measures of our system where all retrial times, service times, vacation times and repair times are exponentially, Erlangianly and hyper-exponentially distributed, where exponential distribution is $f(x) = ve^{-vx}, x > 0$, Erlang-2 stage distribution is $f(x) = v^2 xe^{-vx}, x > 0$ and hyper-exponential distribution is $f(x) = cve^{-vx} + (1-c)v^2 e^{-v^2 x}$. We assume arbitrary values to the parameters such that the steady state condition is satisfied. The following figures computed various characteristics of our model like, probability that the server is idle I_0, the mean orbit size Lq. For the effect of the parameters θ, b, r, γ, $\eta_1^{(1)}$ and $\xi_1^{(1)}$ on the system performance measures, two and three dimensional graphs are drawn in Figure1-4. Figure1 shows that the idle probability I_0 decreases for increasing value of the non-balking probability (b). Figure2 shows that the mean orbit size L_q increases for increasing value of the non-reneging probability (r). Figure3 shows that the surface displays upwards trend as expected for increasing value of vacation rate γ and delaying repair rate on FTS $\eta_1^{(1)}$ against the idle probability I_0. The mean orbit size L_q decreases for increasing value of the retrial rate θ and repair rate on first type FPS $\xi_1^{(1)}$ is shown in Figure4.

6. Conclusion

In this paper, we have studied a single server feedback retrial queueing system with two types of service, under variant vacation policy, delaying repair, balking and reneging where the server provides each type consists of two essential phases. The probability generating functions of the number of customers in the system and orbit are found by using the supplementary variable technique. The performance measures like, the mean number of customers in the system/orbit, the average waiting time of customer in the system/orbit are obtained. The analytical results are validated with the help of numerical illustrations. This model finds potential application in Simple Mail Transfer Protocol (SMTP) to deliver the messages between mail servers and Verteiler Ensprintz Pumps manufacturing.

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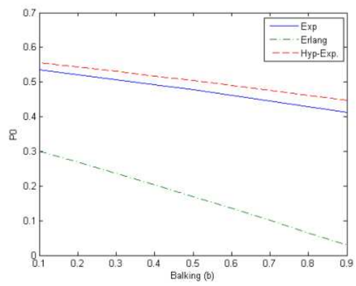

Figure 1: P_0 versus b

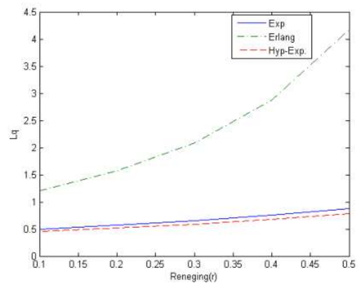

Figure 2: L_q versus r

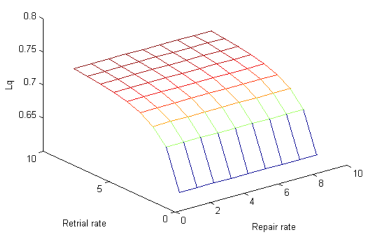

Figure 3: L_q versus θ and $\xi_1^{(1)}$

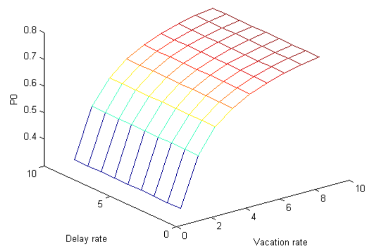

Figure 4: P_0 versus γ and $\eta_1^{(1)}$