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Range Symmetric Matrices in Indefinite Inner Product Space

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Abstract. The concept of range symmetric matrix is extended to indefinite inner product space. Equivalent characterizations of a range symmetric matrix in an indefinite inner product space in the setting of an indefinite matrix product are presented. Relations between EP and range symmetric matrices in an inner product space are discussed. Characterization of the maximal subgroups of the multiplicative semi groups of EP matrices in indefinite inner product spaces under an indefinite matrix multiplication is presented.

Keywords: Indefinite matrix product, Indefinite inner product space, Range symmetric matrix, EP matrix

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1. Introduction

An indefinite inner product in C^n is a conjugate symmetric **sesquilinear** from $[x, y]$ together with the regularity condition that $[x, y] = 0$ for all $y \in C^n$ only when $x = 0$. Any indefinite inner product is associated with a unique invertible complex matrix J (called a weight) such that $[x, y] = < x, Jy >$ where $< , >$ denotes the Euclidean inner product on C^n, we also make an additional assumption on J, that is, $J^2 = I$, to present the results with much algebraic ease. Thus an indefinite inner product space is a generalization of Minkowski space, where the weight J is a diagonal matrix of order n, with the first entry 1 and the remaining entries are all -1 (as studied by physicists). For a complex matrix A in Minkowski space, $rank(AA^*) = rank(A^*A) = rank(A^*) = rank(A)$ fails under the usual product of matrices. This motivates the study on the existence of the Moore-Penrose inverse in [7]. Further, there are two different values for dot product of vectors in indefinite inner product spaces. To overcome these difficulties, a new matrix product, called indefinite matrix multiplication is introduced and some of its properties are investigated in [9]**.** The structure of certain class of EP matrices over the complex field having the same range space have been studied by Baskett and Katz in [1] and by the author in [4]. A characterization of the maximal subgroup of complex matrices obtained in [2] was extended to matrices over an arbitrary field by the author in [5].

The aim of this manuscript is to extend the concept of range symmetric matrix to indefinite inner product space and present some interesting characterizations of range symmetric matrices similar to EP matrices in the setting of indefinite matrix product.

Then we characterize the maximal subgroup of $C^{n\times n}$ the semi group of complex matrices containing a **hermitian idempotent matrix** with respect to the indefinite matrix product in indefinite inner product spaces. The results on algebraic structure of complex EP matrices having the same range space and characterization of EP matrices over a Minkowski space [6] are deduced as special cases. In section 2, we recall the definitions and preliminary results required in characterizing the maximal subgroups of complex matrices over an indefinite inner product space. Characterizations of a range symmetric matrix in $\wp$, an indefinite inner product space in the setting of an indefinite matrix product are presented in Section 3. Characterization of the maximal subgroups of the multiplicative semi groups of EP matrices in $\wp$ are presented in Section 4. Wherever possible, we provide examples to illustrate our results.

2. Preliminaries

We first recall the notion of an indefinite multiplication of matrices.

Definition 2.1. Let $A \in C^{m\times n}, B \in C^{n\times 1}$. Let J_n be an arbitrary but fixed $n \times n$ complex matrix such that $J_n = J_n^* = J_n^{-1}$. The indefinite matrix product of A and B (relative to J) is defined as $A \circ B = AJ_nB$.

Definition 2.2. For $A \in C^{m\times n}, A^{[*]} = J_nA^*J_m$ is the adjoint of A relative to J_n and J_m, the weights in the appropriate spaces.

Remark 2.1. When J_n is the identity matrix the product reduces to the usual product of matrices and it can be easily verified that with respect to the indefinite matrix product, $rank\left(A \circ A^{[*]}\right) = rank\left(A^{[*]} \circ A\right) = rank(A)$, where as this rank property fails under the usual matrix multiplication. Thus the Moore –Penrose inverse of a complex matrix over an indefinite inner product space, with respect to the indefinite matrix product exists and this is one of its main advantages.

Definition 2.3. $A \in C^{n\times n}$, is said to be J-invertible if there exists $X \in C^{m\times n}$, such that $A \circ X = X \circ A = J_n$. Such an X is denoted as $A^{[-1]} = JA^{-1}J$..

Definition 2.4. For $A \in C^{m\times n}$, a matrix X is called the Moore-Penrose inverse if it satisfies the following equations:
$A \circ X \circ A = A, X \circ A \circ X = X, (A \circ X)^{[*]} = A \circ X$ and $(X \circ A)^{[*]} = X \circ A$.
Such an X is denoted by $A^{[\dagger]}$ and represented as $A^{[\dagger]} = J_nA^{\dagger}J_m$.

Definition 2.5. The Range space of $A \in C^{m\times n}$ is defined by $R(A) = \{y = A \circ x \in C^m / x \in C^n\}$.
The Null space of A is defined by $Nu(A) = \{ x \in C^n / A \circ x = 0\}$. It is clear that $Nu\left(A^{[*]}\right) = N(A^*)$.

Property 2.1.
(i) $(A^{[*]})^{[*]} = A$.
(ii) $(A^{[\dagger]})^{[\dagger]} = A$.

(iii) $(AB)^{[*]} = B^{[*]}A^{[*]}$
(iv) $R\left(A^{[*]}\right) = R\left(A^{[\dagger]}\right)$.
(v) $R\left(A \circ A^{[*]}\right) = R(A), R\left(A^{[*]} \circ A\right) = R(A^{[*]})$
(vi) $N\left(A \circ A^{[*]}\right) = N\left(A^{[*]}\right), N\left(A^{[*]} \circ A\right) = N(A)$.

3. Range symmetric matrices in indefinite inner product space

Let $\wp$ denotes the indefinite inner product space, with weight J, under an indefinite matrix multiplication. In this section, we shall define a range symmetric matrix in $\wp$, analogous to that of a range symmetric matrix in the unitary space. We present equivalent characterizations of a range symmetric matrix in $\wp$.

Definition 3.1. $A \in C^{n\times n}$ is range symmetric in $\wp$ if and only if $R(A) = R(A^{[*]})$.

Remark 3.1. In particular for $J = I_n$, this reduces to the definition of range symmetric matrix in Unitary space (or) equivalently to an EP matrix [1].

Theorem 3.1. For $A \in C^{n\times n}$, the following are equivalent:
(1) A is range symmetric in $\wp$.
(2) AJ is EP.
(3) JA is EP.
(4) $N(A) = N(A^{[*]})$.
(5) $N(A^*) = N(AJ)$.
(6) $A^{[*]} = AK = HA$, for some invertible matrices K and H.
(7) $R(A^*) = R(JA)$.

Proof: The proof of the equivalence of (1), (2) and (3) runs as follows:
A is range symmetric in $\wp \Leftrightarrow R(A) = R(A^{[*]})$
$\Leftrightarrow R(AJ) = R(JA^*J)$.
$\Leftrightarrow R(AJ) = R(AJ)^*$.
$\Leftrightarrow AJ$ is EP.
$\Leftrightarrow J(AJ)J^*$ is EP.
$\Leftrightarrow JA$ is EP.
Thus (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) hold.
(3) $\Leftrightarrow$ (4).
(JA) is EP $\Leftrightarrow N(JA) = N(JA)^*$.
$\Leftrightarrow N(A) = N(A^*J)$.
$\Leftrightarrow N(A) = N(JA*J)$.
$\Leftrightarrow N(A) = N(A^{[*]})$.
Thus the equivalence of (3) and (4) is proved.
(4) $\Leftrightarrow$ (5).
$N(A) = N(A^{[*]}) \Leftrightarrow N(A) = N(JA^*J)$.
$\Leftrightarrow N(A) = N(A^*J)$.
$\Leftrightarrow A^*J = A^*J.A^gA. (A^g\ is\ a\ solution\ of\ AXA = A)$.
$\Leftrightarrow A^* = A^*(J A^g)(AJ)$.
$\Leftrightarrow A^* = A^*(AJ)^g(AJ)$.
$\Leftrightarrow N(AJ) = N(A^*)$.

Thus the equivalence of (4) and (5) is proved.

(3) $\Leftrightarrow$ (6).

(JA) $is\ EP$) $\Leftrightarrow (JA)^* = JAK$ $, for\ some\ invertible\ matrix\ K.$

$\Leftrightarrow A^{[*]} = JA^*J = AK.$

$\Leftrightarrow A = (AK)^{[*]}. (By\ using\ property\ 2.1\ (i))$

$\Leftrightarrow A = (K)^{[*]}(A)^{[*]} \quad (By\ using\ property\ 2.1(iii)).$

$\Leftrightarrow A^{[*]} = HA. \quad (where\ H = ((K)^{[*]})^{-1}.$

Thus (3) $\Leftrightarrow$ (6) hold.

(1) $\Leftrightarrow$ (7).

A is range symmetric in $\wp$ $\Leftrightarrow R(A) = R(A^{[*]})$

$\Leftrightarrow R(A) = R(JA^*J)$

$\Leftrightarrow R(A) = R(JA^*)$

$\Leftrightarrow JA^* = AA^gJA^*$

$\Leftrightarrow A^* = (JA)(JA)^gA^*$

$\Leftrightarrow R(A^*) = R(JA).$

Thus (1) $\Leftrightarrow$ (7) hold.

Remark 3.2. In particular, if J is the Minkowski metric tensor, the above Theorem reduces to Theorem 2.2 of [6].

The relation between EP matrices and range symmetric matrices in $\wp$ are discussed in the following:

Theorem 3.2. For $A \in C^{n\times n}$, any two of the following conditions imply the other one.

(1) A is EP.

(2) A is range symmetric in $\wp$.

(3) $R(A) = R(JA)$.

Proof: First we prove that if (1) holds, then (2) is equivalent to (3). That is, if $R(A) = R(A^*)$, then by Theorem(3.1), A is range symmetric in $\wp \Leftrightarrow R(A) = R(A^*) = R(JA)$. Thus (2) $\Leftrightarrow$ (3).

Hence, (1) and (2) $\Rightarrow$ (3) and (1) and (3) $\Rightarrow$ (2) hold. Now let us prove (2) and (3) $\Rightarrow$ (1). By Theorem(3.1), (2) $\Rightarrow JA$ is EP. Hence, $R(JA) = R(JA)^* = R(A^*J) = R(A^*)$. By (3), it follows that, $R(A) = R(JA)$. Hence, $R(A) = R(A^*) \Rightarrow A$ is EP. Hence the Theorem.

Since the Moore Penrose inverse, $A^{[\dagger]}$ exists for $A \in C^{n\times n}$, in the indefinite inner product space $\wp$, under the indefinite matrix product, we shall derive equivalent conditions for A to be range symmetric in $\wp$ involving, $A^{[\dagger]}$, in the following:

Theorem 3.3. For $A \in C^{n\times n}$, the following are equivalent:

(1) A is range symmetric in $\wp$.

(2) $(A \circ A^{[\dagger]}) = (A^{[\dagger]} \circ A)$

(3) A is J-EP.

(4) $A^{[\dagger]}$ is a polynomial in A.

Proof: A is range symmetric in $\wp \Leftrightarrow JA$ is EP (By Theorem (3.1)

$\Leftrightarrow (JA)(JA)^+ = (JA)^+(JA).$

$\Leftrightarrow JAA^+J = A^+JJA.$

$\Leftrightarrow AA^+J = JA^+A.$

$\Leftrightarrow A \circ JA^{+}J = (JA^{+}J) \circ A$.
$\Leftrightarrow \left(A \circ A^{[\dagger]}\right) = (A^{[\dagger]} \circ A)$. (By Definition 2.4)
$\Leftrightarrow A$ is J-EP. (By Definition 3.1 of[3]
$\Leftrightarrow AJ$ is EP. (By Theorem (3.1)).
$\Leftrightarrow (AJ)^{+}$ is a polynomial in (AJ).
$\Leftrightarrow JA^{+}$ is a polynomial in (AJ).
$\Leftrightarrow A^{[\dagger]}$ is a polynomial in A.

Lemma 3.1. For $A \in C^{n\times n}$, $\left(A \circ A^{[\dagger]}\right)$ is the projection on $R(A)$ and $(A^{[\dagger]} \circ A)$ is the projection on $R(A^{*})$.
Proof: $x \in R(A) \Leftrightarrow x = A \circ y = A \circ A^{[\dagger]} \circ A \circ y = A \circ A^{[\dagger]} \circ x$, by Definition 2.4, $A \circ A^{[\dagger]}$ is the projection on $R(A)$, being symmetric and idempotent in $\wp$. Similarly, we can show that $(A^{[\dagger]} \circ A)$ is the projection on $R(A^{*})$.

Theorem 3.4. For $A \in C^{n\times n}$, the following are equivalent:
(1) A is range symmetric in $\wp$.
(2) $A^{[\dagger]}$ is range symmetric in $\wp$.
(3) There exists, $E = E^{[*]} = E^{[2]} \in C^{n\times n}$ such that $A \circ E = E \circ A$ and $R(A) = R(E)$.
Proof: (1)⇒(2): $R(A^{[\dagger]}) = R(A^{[*]}) = R(A) = R(A^{[*]})^{[*]} = R(A^{[\dagger]}))^{[*]}$. Hence, $A^{[\dagger]}$ is range symmetric in $\wp$. Thus (2) holds.
(2) ⇒ (3): Since $A^{[\dagger]}$ is range symmetric in $\wp$ and $R(A^{[\dagger]}) = R(A^{[*]}), R(A^{[\dagger]}))^{[*]} = R(A)^{[*]}$, by Lemma 3.1, $(A \circ A^{[\dagger]}) = (A^{[\dagger]} \circ A) = E \in C^{n\times n}$,,satisfy, $E = E^{[*]} = E^{[2]}$. By Definition (2.4), $A \circ E = E \circ A = A$. $R(A) \subseteq R(E)$ and $rank\ A = rank\ E$, implies $R(A) = R(E)$. Thus (3) holds.
(3) ⇒ (1): Since, $E = E^{[*]} = E^{[2]}$, it can be verified that $E^{[\dagger]} = E$. Since $R(A) = R(E)$, E is the projection on $R(A)$. Hence, $(A \circ A^{[\dagger]}) = (A^{[\dagger]} \circ A) = E$ and $A = A \circ E = E \circ A$. By taking adjoint with respect to J on $A = A \circ E = E \circ A$, we get, $E \circ A^{[*]} = A^{[*]} \circ E = A^{[*]}$. Further, $R(A^{[*]}) \subseteq R(E) = R(A)$ and $rank(A^{[*]}) = rank(A)$. Hence, $R(A^{[*]}) = R(A)$ and A is range symmetric in $\wp$. Thus (1) holds. Hence the Theorem.

4. Characterization of maximal subgroups

In this section, we shall characterize the maximal subgroups of $C^{n\times n}$. Since, A is J-EP and A is range symmetric in $\wp$ are equivalent by Theorem 3.3, hence forth we use, A is J-EP. First we shall prove certain Lemmas to simplify the proof of the main results in this section.

Lemma 4.1. (Theorem 3.7 of [3]): A matrix $A \in C^{n\times n}$, is J-EP if and only if $R(A^{[2]}) = R(A^{[*]})$ where $A^{[2]} = A \circ A$.

Lemma 4.2. If A and B are J-EP$_r$, then $A \circ B$ is J-EP$_r$ if and only if $R(A) = R(B)$.
Proof: Since A and B are J-EP$_r$, that is , J-EP of $rank\ r$, by Theorem 3.1, AJ and BJ are EP$_r$. Now by a result of Baskett and katz [1] and Theorem 3.1, $R(A) = R(B) \Leftrightarrow R(AJ) = R(BJ)$

$\Leftrightarrow (AJ)(BJ)$ is EP$_r$.

$\Leftrightarrow (A \circ B)J$ is EP_r.
$\Leftrightarrow (A \circ B)$ is J-EP_r

Hence the Lemma.

Lemma 4.3. Let E and $T \in C^{n\times n}$, such that $E^{[2]} = E$ and $T^{[2]} = T$ with $R(E) = R(T)$, then there exists an invertible matrix P of order n, such that $T = PEP^{[-1]}$.

Proof: Since $E = E \circ E$ and $T = T \circ T$, (EJ) and (TJ) are idempotents. $R(E) = R(T)$ implies $R(EJ) = R(TJ)$. Thus EJ and TJ are idempotents having the same range space. Therefore there exists an invertible matrix $P \in C^{n\times n}$, such that $TJ = PEJP^{-1}$. Which implies, $T = PE(JP^{-1}J)$
$= PEP^{[-1]}$. Hence the Lemma.

Theorem 4.1. Let $E = E^{[*]} = E^{[2]} \in C^{n\times n}$. Then, $H(E) = \{A \in C^{n\times n},/\ A$ is J-EP_r and $R(A) = R(E)\}$ is the maximal subgroup of $C^{n\times n}$, containing E as the identity element under the indefinite inner product multiplication.

Proof: Since $E = E^{[*]} = E^{[2]}$, E is J-EP follows from Lemma 4.1. From $R(A) = R(E)$, we get, $rank(A) = rank(E) = r$. Hence $E \in H(E)$. For $A, B \in H(E)$, A and B are J-EP_r with $R(A) = R(E) = R(B)$, by Lemma 4.2, $A \circ B$ is J-EP_r. $R(A \circ B) \subseteq R(A) = R(E)$ and $rank\,(A \circ B) = rank(A) = rank(E) = r$. Hence $A \circ B \in H(E)$. Thus $H(E)$is closed under the indefinite matrix product. A is J-EP_r $\Leftrightarrow$ AJ is EP_r (By Theorem 3.1) $\Leftrightarrow (AJ)^{\dagger} = J(A)^{\dagger}$ is EP_r $\Leftrightarrow$ $A^{[\dagger]} = J(A)^{\dagger}J$ is J-EP_r. (by Theorem 3.1). Since $rank(A) = rank(A^{[\dagger]})$, $R(A^{[\dagger]}) = R(A) = R(E)$. Hence $A^{[\dagger]}$ is J-EP_r with $R((A^{[\dagger]}) = R(E)$. Thus , $A^{[\dagger]} \in H(E)$ for each $A \in H(E)$.

$$A \circ A^{[\dagger]} = AJJA^{\dagger}J = AA^{\dagger}J \tag{4.1}$$
$$A^{[\dagger]} \circ A = JA^{\dagger}JJA = JA^{\dagger}A \tag{4.2}$$

Since EJ is hermitian idempotent with $R(AJ) = R(A) = R(E) = R(EJ)$, EJ is a projection on $R(AJ) = R(AJ)^{\dagger}$ .Hence, we have the following:
$EJ = (AJ)(AJ)^{\dagger} = AA^{\dagger}$ and $EJ = (AJ)^{\dagger}(AJ) = JA^{\dagger}AJ$.
Hence, $AA^{\dagger}J = E = JA^{\dagger}A$. By using (4.1) and (4.2) , we get

$$A \circ A^{[\dagger]} = E = A^{[\dagger]} \circ A \tag{4.3}$$

Then by Definition 2.4, we have, $A = A \circ A^{[\dagger]} \circ A = E \circ A$ and $A = A \circ A^{[\dagger]} \circ A = A \circ E$.

$$\text{Thus ,} A = A \circ E = E \circ A \tag{4.4}$$

From (4.3) $A^{[\dagger]}$ is the inverse of A and from (4.4) it follows that E is the identity element of $H(E)$ under the indefinite inner product multiplication. Thus $H(E)$ is a subgroup of $C^{n\times n}$ under the indefinite matrix product containing E. The maximality of $H(E)$ can be proved along the same lines as that for matrices over an arbitrary field (refer :Theorem 2.1 of [5]) and hence omitted. Hence the Theorem.

Corollary 4.1. Let A be a fixed J-EP_r matrix. Then $B_A = \{B \in C^{n\times n},/\ B$ is J-EP_r and $R(B) = R(A)\} = H(E)$,where E is the orthogonal projection on $R(A)$.

Proof: Since A is J-EP_r, by the above Theorem (4.1), $A \circ A^{[\dagger]} = E = A^{[\dagger]} \circ A$ is the orthogonal projection of $R(A)$. Further, A is J- EP_r if and only if $A \circ E = E \circ A$. Since

$R(A) = R(E)$, the result follows from the Definitions of B_A and $H(E)$. Hence the Corollary.

Remark 4.1. In particular for $0 \neq E \neq J = I_n$, E reduces to a hermitian idempotent matrix in $C^{n\times n}$ under the usual matrix multiplication and $H(E)$ in Theorem (4.1) is equivalent to $H(E) = \{A \in C^{n\times n} / AE = EA$ and $R(A) = R(E)\}$. Thus our Theorem (4.1) reduces to Theorem 2 of [2] and Corollary (4.1) reduces to Theorem 2 of [1].

Remark 4.2. For $J = I_n$, from Theorem (2.2) we have A is EP $\Leftrightarrow R(A^2) = R(A^*)$. By Theorem (2.3) of [5], for an EP matrix A over an arbitrary field with involution '$*$', $rank(A) = rank(A^2)$ is an equivalent condition for the existence of A^+. Thus our Theorem (4.1) is an analogue of Theorem (2.1) for matrices over an arbitrary field [5].

Remark 4.3. For $E = I_n$, $R(A) = R(E)$ implies each $A \in H(E)$ is invertible. Substituting for E, in equation (4.3), we get $A = A \circ I_n = I_n \circ A$, that is, $A = AJ = JA$. Hence, $J = I_n$, being the identity of the multiplicative group $H(E)$ under the usual multiplication of matrices.

Remark 4.4. We observe that, for $E = I_n$, the unique partial isometry P associated with each $A \in C^{m\times n}$ called the carrier matrix [8], reduces to I_n. Therefore, the indefinite matrix product defined as $A \circ B = AP^*B$ in [4] reduces to the usual multiplication of matrices, and the group $C(A)$ is identical with $H(E)$, the multiplicative group of inventible matrices of order n.

Remark 4.5. Let us illustrate that the condition on E is essential in Theorem 4.1 by the following example.

Example 4.1. Let us consider

$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \qquad E = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ and $J = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Clearly, $J = J^* = J^{-1}$. On computation, we see that $E \circ E = EJE = E$ and $E^{[*]} = JE^*J \neq E$. Hence E is not J-symmetric but E is J-idempotent. $AJ = A$ is EP$_1$, being symmetric of $rank$ 1. By Theorem 3.1, A is also J-EP$_1$,

Here, $A \circ E = AJE = AE = \begin{bmatrix} 2 & 0 \\ 2 & 0 \end{bmatrix} \neq A$.

and $E \circ A = EJA = EA = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = A$.

Here, E is not the identity element. However $AA^+ = EE^+$, since $R(A) = R(E)$.
Therefore the condition on E is essential in Theorem (4.1).

Theorem 4.2. G is a maximal subgroup of $C^{n\times n}$, under the indefinite matrix product if and only if $G = PH(E)P^{[-1]}$ for some $E = E^{[*]} = E^{[2]}$ and some invertible $P \in C^{n\times n}$.
Proof: Let G be a maximal subgroup of $C^{n\times n}$, with identity $T \circ T = T$. Let $E = E^{[*]} = E^{[2]}$ be the orthogonal projection on $R(T)$. Since T and E are idempotents in $\wp$ under indefinite matrix product having the same range space, by Lemma (4.3), there exists an invertible $P \in C^{n\times n}$, such that $T = PEP^{[-1]}$. Since, $PH(E)P^{[-1]}$ being the isomorphic

image of the multiplicative group under indefinite matrix product, $H(E)$ is also a group containing T. By the above Theorem 3.5, since $H(E)$ is maximal, its isomorphic image $PH(E)\,P^{[-1]}$ is also maximal containing the identity T. Thus $G = PH(E)P^{[-1]}$. Conversely, if $G = PH(E)P^{[-1]}$, then G is clearly maximal. Hence the Theorem.

Corollary 4.2. If U is unitary in $C^{n\times n}$, then $UH(E)U^{[*]} = H\,(UEU^{[*]})$ for any $E = E^{[*]} = E^{[2]} \in C^{n\times n}$.

Proof: Let T be the identity of the group $UH(E)U^{[*]}$ under indefinite matrix product. Then by the above Theorem (4.1),
$T = UEU^{[*]}$. Since, $E = E^{[*]} = E^{[2]}$, EJ is hermitian idempotent and $UU^{[*]} = U^{[*]}U = J$ is equivalent to $UU^{*} = U^{*}U = I_n$, Thus U is unitary and TJ is hermitian idempotent. Therefore $T = T^{[*]} = T^{[2]}$ is the identity element of the maximal subgroup $UH(E)U^{[*]}$. Thus $UH(E)U^{[*]} = H(T) = H(UEU^{[*]})$. Hence the corollary.

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