Annals of Pure and Applied Mathematics
Vol. 17, No. 1, 2018, 85-93
ISSN: 2279-087X (P), 2279-0888(online)
Published on 30 April 2018
www.researchmathsci.org
DOI: http://dx.doi.org/10.22457/apam.v17n1a9

Annals of Pure and Applied Mathematics

A Note on the Lattice of *L*-Closure Operators

***Pinky*[1] and *T.P.Johnson*[2]**

[1]Department of Mathematics, Cochin University of Science and Technology
Cochin-682022, India. E-mail: pritammalik90@gmail.com
[2]Applied Science and Humanities Divison, School of Engineering, Cochin University of Science and Technology, Cochin-682022, India. E-mail: tpjohnson@cusat.ac.in
[1]Corresponding author

Received 21 March 2018; accepted 28 April 2018

Abstract. In this paper, we investigate the lattice structure of the set of all L-closure operators on a given nonempty set X when membership lattice L is a bounded chain. It is proved that in this case, the lattice of all L-closure operators is distributive, modular but not atomic and not complemented. The authors disprove certain known theorems on the above lattices and the correct results are provided.

Keyword: L-topology, Lattice, Chain, L-closure operator.

AMS Mathematics Subject Classification (2010): 03G10, 54A99

1. Introduction

In 1965, the concept of fuzzy sets and fuzzy set operations were first introduced by Zadeh in his classical paper [12]. In 1968, Chang [2] applied some basic concepts from general topology to fuzzy sets and developed a theory of fuzzy topological spaces. Closure spaces which is a generalization of topological spaces were introduced by Cech [1] and then studied by many authors. They have extended many topological concepts to closure operators. Fuzzy closure spaces were first studied by Mashhour and Ghanim [7]. Fuzzy closure spaces are generalization of fuzzy topological spaces. The definition of Mashhour and Ghanim is analogue of $\breve{C}$ech closure spaces. Srivastava et al., [9] have introduced fuzzy closure spaces as analogue of Brikhoff closure spaces. Srivastava and Srivastava [10] have studied the subspace of a fuzzy closure space and introduced the notion of a T_1-fuzzy closure space. Johnson [5] has studied the lattice structure of the set $L(X)$ of all $\breve{C}$ech fuzzy closure operators on a fixed set X and proved that $L(X)$ is a complete lattice but not complemented. Zhou [11] has introduced the concepts of L-closure spaces and the convergence in L-closure spaces.

In this paper, we have investigated the lattice structure of the lattice $LC(X)$ of all L-closure operators on a given non-empty set X when membership lattice L is a bounded chain. In addition, we have identified the infra L-closure operators and their number and established a relation between ultra L-topology and ultra L-closure operator.

2. Preliminaries

Throughout this paper, X stands for a non-empty set, L for a bounded chain with the least element 0 and the greatest element 1, which is a completely distributive lattice with an order reversing involution ' $'$ ' $\big(i.e.\ \forall a,b \in L, a \le b \Rightarrow a' \ge b'$ and for every $a \in L, a'' = a\ \big)$ and

$L^X = \{ f : f : X \to L$ is a mapping $\}$. The constant function in L^X, taking value α is denoted by $\underline{\alpha}$ and x_γ, where $\gamma(\ne 0) \in L$, denotes the L - fuzzy point defined by

$$x_\gamma(y) = \begin{cases} \gamma & if \quad y = x \\ 0 & otherwise \end{cases}.$$

Any $f \in L^X$ is called as an L-subset of X and the complement of f, denoted by f' is defined by the formula $f'(x) = [f(x)]'$. The following are some important definition reported in [3,6] :

Definition 2.1. *An element of* L *is called an atom if it is a minimal element of* $L \setminus \{0\}$.

Definition 2.2. *An element of* L *is called a dual atom if it is a maximal element of* $L \setminus \{1\}$.

Definition 2.3. *Let* δ *be a nonempty subset of* L^X. *We call* δ *an* L*-topology on* X *, if* δ *satisfies the following conditions :*

(1) $\underline{0}, \underline{1} \in \delta$.

(2) if $f, g \in \delta$, then $f \wedge g \in \delta$.

(3) if $\delta_1 \subseteq \delta$, then $\bigvee_{f \in \delta_1} f \in \delta$.

The pair (L^X, δ) is called an L-topological space.

In this paper, we take the definition of L-closure operator as a generalization of fuzzy closure operator in [7].

Definition 2.4. *A* $\breve{C}ech$ L*-closure operator on a set* X *is a function* $c : L^X \to L^X$ *satisfying the following three axioms :*

(1) $c(\underline{0}) = \underline{0}$.

(2) $f \le c(f)$ for every f in L^X.

(3) $c(f \vee g) = c(f) \vee c(g)$ for all $f, g \in L^X$.

For convenience, we call it a L-closure operator on X. Also (X, c) is called L-closure space.

Definition 2.5. *In an* L*-closure space* (X, c)*, an* L*-subset* f *of* X *is said to be* L*-closed if* $c(f) = f$. *An* L*-subset* f *of* X *is* L*-open if its complement is closed in* (X, c).

The set of all open L-subsets of (X,c) forms an L-topology on X, called the L-topology associated with the L-closure operator c.
Let F be an L-topology on a set X. Then a function $c:L^X \to L^X$ defined by $c(f)=\bar{f}$ for all $f \in L^X$, where $\bar{f}$ is the closure of f in (L^X,F), is an L-closure operator on X called the L-closure operator associated with the L-topology F.
An L-closure operator on a set X is called L-topological if it is the L-closure operator associated with an L-topology on X.

Remark 2.6. *Note that the different L-closure operators can have the same associated L-topology. But different L-topologies can not have the same associated L-closure operator.*

Example 2.7. *Let $X=\{x,y,z\}$ and $L=\{0,\alpha,\beta,1\}$ be a chain with $0<\alpha<\beta<1$. Then the functions c_1, $c_2 : L^X \to L^X$ defined by :*

$$c_1(f)=\begin{cases} \underline{0} & if \quad f=\underline{0} \\ g & if \quad f=x_\alpha \\ \underline{1} & otherwise \end{cases},$$

where $g \in L^X$ is defined as $g(x)=g(y)=1$ and $g(z)=\beta$

$$\text{and } c_2(f)=\begin{cases} \underline{0} & if \quad f=\underline{0} \\ \underline{1} & otherwise \end{cases}$$

are L-closure operators. Associated L-topologies of c_1 and c_2 are same, which is the indiscrete L-topology.

3. Lattice of L-closure operators

Definition 3.1. *Let c_1 and c_2 be L-closure operators on X. Then $c_1 \le c_2$ if and only if $c_2(f) \le c_1(f), \forall f \in L^X$.*

Remark 3.2. *The set $LC(X)$ of all L-closure operators forms a lattice with this relation $\le$. If $c_1, c_2 \in LC(x)$, then the join $c_1 \vee c_2$ and the meet $c_1 \wedge c_2$ are defined respectively by the following formulas :*
$(c_1 \vee c_2)(x) =$ min $\{c_1(x), c_2(x)\}$ and
$(c_1 \wedge c_2)(x) =$ max $\{c_1(x), c_2(x)\}$.

Definition 3.3. *The L-closure operator D defined on X by $D(f)=f$ for all $f \in L^X$, is called the discrete L-closure operator.*
The L-closure operator I on X defined by

$$I(f)=\begin{cases} \underline{0} & if \quad f=\underline{0} \\ \underline{1} & otherwise \end{cases},$$

is called the indiscrete L-closure operator.

Remark 3.4. *D and I are the L-closure operators associated with the discrete and indiscrete L-topologies on X respectively. Moreover D is the unique L-closure operator whose associated L-topology is discrete. Also I and D are the smallest and the largest elements of the lattice $LC(X)$ respectively.*

Theorem 3.5. [5] *$LC(X)$ is a complete lattice.*

In [8], we find the following theorem:

Theorem 3.6. [8] *$LC(X)$ is not modular.*

But this result is not true as shown by the following theorem:

Theorem 3.7. *$LC(X)$ is a distributive lattice.*

Proof: Let c_1, c_2 and c_3 be any three elements of $LC(X)$.
Then by definition of $\leq$, we have
$c_1 \vee (c_2 \wedge c_3) =$ min $[c_1,$ max $\{c_2, c_3\}]$
and $(c_1 \vee c_2) \wedge (c_1 \vee c_3) =$ max [min $\{c_1, c_2\}$, min $\{c_1, c_3\}]$.
For any $f \in L^X$ and $x \in X$, assume that $c_1(f) = g_1$, $c_2(f) = g_2$, $c_3(f) = g_3$
and $g_1(x) = \alpha$, $g_2(x) = \beta$, $g_3(x) = \gamma$.
Since $\alpha, \beta, \gamma \in L$ and L is a chain, the following six case arise:
(1) $\gamma < \beta < \alpha$ (2) $\beta < \gamma < \alpha$ (3) $\alpha < \gamma < \beta$ (4) $\gamma < \alpha < \beta$ (5) $\beta < \alpha < \gamma$ (6) $\alpha < \beta < \gamma$.

<u>Case 1</u> : $\gamma < \beta < \alpha$.
Then $\{c_1 \vee (c_2 \wedge c_3)\}(f)(x)$
$=$ min $\{g_1(x),$ max $\{g_2(x), g_3(x)\}\}$
$=$ min $\{\alpha,$ max $\{\beta, \gamma\}\}$
$=$ min $\{\alpha, \beta\} = \beta$
and $\{(c_1 \vee c_2) \wedge (c_1 \vee c_3)\}(f)(x)$
$=$ max{ min $\{g_1(x), g_2(x)\}$, min $\{g_1(x), g_3(x)\}\}$.
$=$ max { min $\{\alpha, \beta\}$, min $\{\alpha, \gamma\}\}$.
$=$ max $\{\beta, \gamma\} = \beta$.
In the same way, it can be checked that the equality
$\{c_1 \vee (c_2 \wedge c_3)\}(f)(x) = \{(c_1 \vee c_2) \wedge (c_1 \vee c_3)\}(f)(x)$ holds good in the remaining five cases also.
Since $f \in L^X$ and $x \in X$ were arbitrary,

$\Rightarrow c_1 \vee (c_2 \wedge c_3) = (c_1 \vee c_2) \wedge (c_1 \vee c_3)$.

$\Rightarrow LC(X)$ is distributive lattice.

Corollary 1. *$LC(X)$ is a modular lattice.*

4. Infra L-closure operators

Definition 4.1. *An L-closure operator on X is called an infra L-closure operator if the only L-closure operator on X strictly smaller than it is I.*

Theorem 4.2. [5] *If $L = [0,1]$, then there is no infra L-closure operator in $LC(X)$.*

In [8], we find the following result :

Let X be any set and $a, b \in X$ such that $a \neq b$. Define $\psi_{a,b} : L^X \to L^X$ by

$$\psi_{a,b}(f) = \begin{cases} f & if \quad f = \underline{0} \\ g_{\alpha,b} & if \quad f = a_\alpha \\ \underline{1} & otherwise \end{cases},$$

where α is a dual atom in L and $g_{\alpha,b}$ is defined by

$$g_{\alpha,b}(a) = \begin{cases} 1 & if \quad a \neq b \\ \alpha & if \quad a = b \end{cases}.$$

Theorem 4.3. [8] *An L-closure operator is an infra L-closure operator if and only if it is of the form $\psi_{a,b}$ for some $a, b \in X, a \neq b$.*

But this result is not true because $\psi_{a,b}$ is not even an L-closure operator as shown below:

Let $a_\alpha, a_\eta \in L^X$, where $\eta, \alpha \in L$ such that $\eta < \alpha$ and α is a dual atom in L.

Then $\psi_{a,b}(a_\alpha \vee a_\eta) = \psi_{a,b}(a_\alpha) = g_{\alpha,b}$

and $\psi_{a,b}(a_\alpha) \vee \psi_{a,b}(a_\eta) = g_{\alpha,b} \vee \underline{1} = \underline{1}$.

$\Rightarrow \psi_{a,b}(a_\alpha \vee a_\eta) \neq \psi_{a,b}(a_\alpha) \vee \psi_{a,b}(a_\eta)$.

Remark 4.4. *Let X be any nonempty set and L be a finite chain with the atom α and the dual atom β. For any $x, y \in X$, define $c_{x,y} : L^X \to L^X$ by*

$$c_{x,y}(f) = \begin{cases} \underline{0} & if \quad f = \underline{0} \\ g_{y,\beta} & if \quad f = x_\alpha \\ \underline{1} & otherwise \end{cases},$$

where $g_{y,\beta} \in L^X$ is defined as $g_{y,\beta}(y) = \beta$ and $g_{y,\beta}(z) = 1, \forall z (\neq y) \in X$. Clearly,

$g_{y,\beta}$ is a dual atom in L^X.

It can be easily checked that $c_{x,y}$ is an L-closure operator and $g_{y,\beta}$ can be replaced by any dual atom in L^X. Therefore the number of such L-closure operators is $|X|^2$.

Theorem 4.5. *Let X be a nonempty set and L be a chain with the atom α and the dual atom β. An L-closure operator is an infra L-closure operator if and only if it is of the form $c_{x,y}$ for some $x, y \in X$.*

Proof: Let c be any L-closure operator on X such that

$$c \le c_{x,y} \Rightarrow c_{x,y}(f) \le c(f), \forall f \in L^X.$$

Therefore $c(f) = \underline{1}, \forall f(\ne x_\alpha) \in L^X$ and $g_{y,\beta} \le c(x_\alpha)$. Since $g_{y,\beta}$ is a dual atom in L^X, it follows that either $c(x_\alpha) = g_{y,\beta}$ or $c(x_\alpha) = \underline{1}$.

If $c(x_\alpha) = g_{y,\beta}$, then $c = c_{x,y}$ and if $c(x_\alpha) = \underline{1}$, then $c = I$.

Hence $c_{x,y}$ is an infra L-closure operator.

Conversely, suppose that c is any infra L-closure operator in $LC(X)$. Then c must be of the form

$$c(f) = \begin{cases} \underline{0} & \text{if} \quad f = \underline{0} \\ \ne \underline{1} & \text{if} \quad f = g \\ \underline{1} & \text{for all} \quad f(\ne \underline{0, g}) \in L^X \end{cases} \quad \text{for some } g \in L^X.$$

If $\exists$ an L-subset $h(\ne \underline{0}) \in L^X$ such that $h < g$, then $h \vee g = g$ and $c(h \vee g) = c(g)$.

$\Rightarrow c(h) \vee c(g) = c(g)$

$\Rightarrow \underline{1} \vee c(g) = c(g)$

$\Rightarrow c(g) = \underline{1}$, a contradiction.

$\Rightarrow g$ must be an atom in L^X i.e. $g = x_\alpha$ for some $x \in X$ and for the atom $\alpha \in L$.

Now, if $c(g) < f$ for some $f(\ne \underline{1}) \in L^X$, then $c_1 : L^X \to L^X$ defined by :

$$c_1(h) = \begin{cases} \underline{0} & \text{if} \quad h = \underline{0} \\ f & \text{if} \quad h = x_\alpha \\ \underline{1} & \text{otherwise} \end{cases}$$

is an L-closure operator such that $c_1 \ne I$ and $c_1 < c$, a contradiction. Therefore $c(g)$ must be a dual atom in L^X i.e. $c(g)(t) = \underline{1}, \forall t \in X$ except for some $y \in X$ and $c(g)(y) = \beta$. $\Rightarrow c = c_{x,y}$.

Thus all infra L-closure operators are of the form $c_{x,y}$ for some $x, y \in X$.

Remark 4.6. *If L is a chain with the atom α and the dual atom β, then for any nonempty set X, there are $|X|^2$ infra L-closure operators in $LC(X)$.*

Remark 4.7. *Let $X=\{x,y,z\}$ and $L=\{0,\alpha,\beta,1\}$ be a chain with $0<\alpha<\beta<1$. Then there are 9 infra L-closure operators given by*

$$c_{x,y}(f)=\begin{cases}\underline{0} & \text{if } f=\underline{0}\\ g_{y,\beta} & \text{if } f=x_\alpha\\ \underline{1} & \text{otherwise}\end{cases}$$

where $x,y\in X$ and $g_{y,\beta}\in L^X$ is defined as $g_{y,\beta}(y)=\beta$ and $g_{y,\beta}(x)=g_{y,\beta}(z)=1$. It can be easily checked that the L-closure operator $c:L^X\to L^X$ defined by

$$c(f)=\begin{cases}\underline{0} & \text{if } f=\underline{0}\\ \underline{\alpha} & \text{if } f=x_\beta\\ \underline{1} & \text{otherwise}\end{cases}$$

can not be written as the join of infra L-closure operators. Thus in general $LC(X)$ is not an atomic lattice.

5. Ultra L-closure operators

Definition 5.1. *An L-topology F on X is called an ultra L-topology if the only L-topology on X strictly finer than F is the discrete L-topology.*

Definition 5.2. *An L-closure operator on X is called an ultra L-closure operator if the only L-closure operator on X strictly larger than it, is D.*

Theorem 5.3. *Let c_1 and c_2 be two L-closure operators such that $c_1\le c_2$. If $\mathcal{F}_1$ and $\mathcal{F}_2$ are the L-topologies associated with the L-closure operators c_1 and c_2 respectively, then $\mathcal{F}_1\subseteq\mathcal{F}_2$.*

Proof: Let $g\in\mathcal{F}_1$.

$\Rightarrow c_1(g')=g'$, where g' is complement of g.

$\Rightarrow g'\le c_2(g')\le c_1(g')=g'$.

$\Rightarrow c_2(g')=g'$.

$\Rightarrow g\in\mathcal{F}_2\Rightarrow\mathcal{F}_1\subseteq\mathcal{F}_2$.

Theorem 5.4. *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two L-topologies such that $\mathcal{F}_1\subseteq\mathcal{F}_2$. If c_1 and c_2 are the L-closure operators associated with the L-topologies $\mathcal{F}_1$ and $\mathcal{F}_2$ respectively, then $c_1\le c_2$.*

Proof: For any $f\in L^X$, let $\bar{f}_1$ and $\bar{f}_2$ be the closure of f in the L-topological

spaces $(L^X, \mathcal{F}_1)$ and $(L^X, \mathcal{F}_2)$ respectively.
Since $\mathcal{F}_1 \subseteq \mathcal{F}_2$,
$\Rightarrow \bar{f}_2 \leq \bar{f}_1$.
$\Rightarrow c_2(f) \leq c_1(f)$.
$\Rightarrow c_1 \leq c_2$.

Theorem 5.5. *Let c be an ultra L-closure operator. If $\mathcal{F}$ is the L-topology associated with the L-closure operator c, then $\mathcal{F}$ is an ultra L-topology.*
Proof: If $\mathcal{F}$ is not an ultra L-topology, then there exists an L-topology $\mathcal{F}_1$ such that $\mathcal{F}_1 \neq L^X$ and $\mathcal{F} \subset \mathcal{F}_1$. Let c_1 be the L-closure operator associated with the L-topology $\mathcal{F}_1$.Then $c_1 \neq D$.
Since $\mathcal{F} \subset \mathcal{F}_1$,
$\Rightarrow c \leq c_1$ and $\exists$ an L-subset $g \in L^X$ such that $g \in \mathcal{F}_1$ but $g \notin \mathcal{F}$.
$\Rightarrow c_1(g') = g'$ and $c(g') \neq g'$
$\Rightarrow c < c_1$, a contradiction.
Hence $\mathcal{F}$ is an ultra L-topology.

Theorem 5.6. *Let $\mathcal{F}$ be an ultra L-topology. If c is the L-closure operator associated with the L-topology $\mathcal{F}$, then c is an ultra L-closure operator.*
Proof: Suppose, there exists an L-closure operator c_1 such that $c \leq c_1$. Let $\mathcal{F}_1$ be the L-topology associated with the L-closure operator c_1.
Since $c \leq c_1 \Rightarrow \mathcal{F} \subseteq \mathcal{F}_1$ and $\mathcal{F}$ is an ultra L-topology so it follows that either $\mathcal{F}_1 = \mathcal{F}$ or $\mathcal{F}_1 = L^X =$ discrete L-topology.
If $\mathcal{F}_1 = \mathcal{F}$, then $c_1 = c$ and if $\mathcal{F}_1 = L^X$, then $c_1 = D =$ discrete L-closure operator.
Hence c is an ultra L-closure operator.

Theorem 5.7. *Let X be a nonempty set and L be a bounded chain. Then an L-closure operator is an ultra L-closure operator if and only if it is the L-closure operator associated with some ultra L-topology on X.*

Remark 5.8. *If $L = [0,1]$, then there is no ultra L-topology in L^X and hence no ultra L-closure operator in $LC(X)$* [4].

Remark 5.9. *Let $X = \{x, y, z\}$ and $L = \{0, \alpha, \beta, 1\}$ be a chain with* $0 < \alpha < \beta < 1$.
Then the L-closure operator $c : L^X \rightarrow L^X$ defined by

$$c(f)=\begin{cases}\underline{0} & if \quad f=\underline{0}\\ \underline{\alpha} & if \quad f=x_{\beta}\\ \underline{1} & otherwise\end{cases}$$

has no complement $\Rightarrow LC(X)$ is not complemented in general.

Remark 5.10. [5] *If* $L=[0,1]$, *then* $LC(X)$ *is not complemented.*

6. Conclusion

In this paper, we have identified infra L-closure operators and established a relation between ultra L-closure operators and ultra L-topologies. Also it is proved that $LC(X)$ is a distributive lattice when L is a bounded chain. Lattice of L-closure operators when L is a bounded lattice other than a chain, will be discussed in future papers.

Acknowledgement. The first author wishes to thank the University Grants Commission, India, for giving financial support.

The authors are thankful to the referees for their valuable suggestions.

REFERENCES

1. E Cech, *Topological Spaces*, Interscience Publishers, John Wiley and Sons New York, 1996.
2. C.L.Chang, Fuzzy topological spaces, *Journal of Mathematical Analysis and Applications*, 24 (1968) 182-190.
3. B.A.Davey, H.A.Priestley, *Introduction to Lattices and Order*, second edition, Cambridge University Press, New York, 2002.
4. T.P.Johnson, On the lattice of fuzzy topologies I, *Fuzzy Sets and Systems*, 48 (1992) 133-135.
5. T.P.Johnson, Completely homogeneous fuzzy closure spaces and lattice of fuzzy closure operators, *Fuzzy Sets and Systems*, 52 (1992) 89-91.
6. Y.-M. Liu, M.-K. Luo, *Fuzzy Topology*, World Scientific Publishing Co., River Edge, New Jersey, 1997.
7. A.S.Mashhour and M.H.Ghanim, Fuzzy closure spaces, *Journal of Mathematical Analysis and Applications*, 106(1) (1985) 154-170.
8. R.George , T.P.Johnson, On the Lattice of L-closure Operators, *International Journal of Science and Research*, 2(3) (2013) 409-412.
9. R.Srivastava, A.K.Srivastava and A.Choubey, Fuzzy Closure spaces, *Journal of Fuzzy Mathematics*, 2 (1994) 525-534.
10. R.Srivastava and M.Srivastava, On T_0 and T_1 fuzzy closure spaces, *Fuzzy Sets and Systems*, 109(2) (2000) 263-269.
11. W.-N.Zhou, Generalization of L-closure spaces, *Fuzzy Sets and Systems*, 149(3) (2005) 415-432.
12. L.A.Zadeh, Fuzzy sets, *Inform and Control*, 8 (1965) 338-353.