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Some Uniqueness Results for Langevin Equations Involving Two Fractional Orders

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Abstract. The purpose of this paper is to investigate the existence and uniqueness of solutions for Langevin equations with two fractional orders:

$$\begin{cases} {}_0^cD_t^{\beta}({}_0^cD_t^{\alpha} - \gamma)x(t) = f(t, x(t)), & 0 < t < 1, \\ x^{(k)}(0) = \mu_k, & 0 \le k < l, \\ x^{(\alpha+k)}(0) = \nu_k & 0 \le k < n, \end{cases}$$

where ${}_0^cD_t^{\alpha}$ and ${}_0^cD_t^{\beta}$ denote the Caputo fractional derivatives, $f:[0,1]\times\Re\to\Re$ is a continuous function and $m-1<\alpha\le m$, $n-1<\beta\le n$, $\gamma>0$, $l=\max\{m,n\}$, $n, m\in$ N, $\mu_k, \nu_k \in \Re$. By using e-positive operators and Altman fixed point theory, several existence and uniqueness results of solutions are obtained. Moreover, an example is given to illustrate the main results.

Keywords: e-positive operator; initial value problem; fractional Langevin equation; the first eigenvalue.

AMS Mathematics Subject Classification (2010): 26A33, 34A08, 34A12, 34B15

1. Introduction

In 1908, Paul Langevin proposed the Langevin equation by studying the Brownian motion and analyzing the trajectory of Brownian particles, see [1]. For a long time, Langevin equations have been widely used to describe many of stochastic problems in fluctuating environments. However, in complex medium dynamic systems, integer order Langevin equations can not correctly describe dynamics. Thus, Kube gave a generalized Langevin equation for modeling anomalous diffusive processes in complex and viscoelastic environment in 1966 [2,3].

With the development of fractional differential equations, it is a natural generalization that the integral derivative of Langevin equation is replaced by fractional derivative. Since then, fractional Langevin equations were proposed by Mainardi and his collaborators in early 1990s, see [4,5]. Moreover, fractional Langevin equations have wide applications such as fractional Langevin equations for modeling of single-file diffusion [6] and for a free particle driven by power law type of noise [7]. So fractional Langevin equations have been paid more and more attention and the existence results of solutions have been widely studied by a great number of scholars, see [8-23] for instance. Recently, there are many papers considered fractional Langevin equations involving two fractional orders, see the works [8-13, 15, 20-23] and the references. Most of these articles are concerned with the existence and uniqueness of solutions of boundary value problems for Langevin equations involving two fractional orders, and many results have been obtained by using different methods such as Banach contraction principle, Krasnoselskii fixed point theorem, Schauder fixed point theorem, Leray-Schauder nonlinear alternative and Leray-Schauder degree. However, we can find that there are still few papers devoted to the study of solutions of initial value problems for Langevin equations involving two fractional orders. In [23], the authors studied the following initial value problem of Langevin equations with two fractional orders:

$$\begin{cases} {}_0^cD_t^{\beta}({}_0^cD_t^{\alpha}-\gamma)x(t)=f(t,x(t)), & 0<t<1, \\ x^{(k)}(0)=\mu_k, & 0\le k<l, \\ x^{(\alpha+k)}(0)=\nu_k & 0\le k<n, \end{cases}$$

where ${}_0^cD_t^{\alpha}$ and ${}_0^cD_t^{\beta}$ denote the Caputo fractional derivatives, $f:[0,1]\times\Re\to\Re$ is a continuous, differential function and $\gamma\in\Re$, $n,m\in \mathrm{N}^+$, $m-1<\alpha\le m$, $n-1<\beta\le n$, $l=\max\{m,n\}$. The existence of solutions was gave by using Leray-Schauder nonlinear alternative. Further, the uniqueness of solutions was also obtained by using Banach contraction principle. Recently, the author [11] studied this problem by introduced a new Banach space $L_{p,\alpha}([0,1],\Re^n)$ with the norm

$$\| f \|_{p,\alpha} = \sup_{t\in[0,1]} \left(\int_0^t \frac{| f(s)|^p}{(t-s)^{\alpha}} ds \right)^{\frac{1}{p}}, \quad \alpha \in (0,1), \quad p \geq 1$$

for Lebesgue measurable function $f:[0,1]\times\Re\to\Re$, and get the existence and uniqueness of solutions for this problem via the Banach contraction principle.

Different from the above papers mentioned, in this paper, we will use e-positive operators and Altman fixed point theory to consider the following existence and uniqueness of solutions for Langevin equations with two fractional orders:

$$\begin{cases} {}_0^cD_t^{\beta}({}_0^cD_t^{\alpha}-\gamma)x(t) = f(t,x(t)), & 0<t<1, \\ x^k(0)=\mu_k, & 0\leq k<l, \\ x^{(\alpha+k)}(0)=\nu_k, & 0\leq k<n, \end{cases} \tag{1}$$

where ${}_0^cD_t^{\alpha}$ and ${}_0^cD_t^{\beta}$ denote the Caputo fractional derivatives, and

(H_1) $f:[0,1]\times\Re\to\Re$ is a continuous function, $m-1<\alpha\leq m$, $n-1<\beta\leq n$, $\mu_k,\nu_k\in\Re$, $\gamma>0$, $l=\max\{m,n\}$, $n,m\in$ N,

We will establish the existence and uniqueness of solutions for problem (1), which are new results on initial value problems for Langevin equations.

This paper is organized as follows. In Section 2, we list some necessary results. In Section 3, we present the existence and uniqueness of solutions for problem (1). In Section 4, we give an example to illustrate our main results.

2. **Preliminaries**

In order to obtain our results, we first list necessary definitions, lemmas and basic results.

Definition 2.1. [24,29,30,31] For a function $x(t)$, the Riemann-Liouville fractional integral of order $\alpha>0$ is

$${}_aI_t^{\alpha}x(t)=\int_a^t \frac{(t-u)^{\alpha-1}}{\Gamma(\alpha)}x(u)du.$$

Definition 2.2([24,29,30,31]). For a continuous function $x(t)$, the Caputo fractional derivativeof order $\alpha>0$ is

$${}_a^cD_t^{\alpha}x(t)=\int_a^t \frac{(t-u)^{n-\alpha-1}}{\Gamma(n-\alpha)}x^{(n)}(u)du, \quad n=[\alpha]+1.$$

Definition 2.2([25]). Let P be a cone in a Banach space E ,and K_1 be a cone in a Banach space E_1 . Let $e\in K_1\setminus\{\theta\}$. A linear operator $T:P\to K_1$ is called e -positive,if for every $x\in P\setminus\{\theta\}$ there exist two positive number $c(x), d(x)$ such that

$$c(x)e\le Tx\le d(x)e.$$

Lemma 2.1 (Altman fixed point theory [26]). Let Ω be an open bounded subset of a Banach space E with $\theta\in\Omega$, and $T:\overline{\Omega}\to E$ be a completely continuous operator such that

$$\| \mathrm{Tx} - \mathrm{x} \|^2 \ge \| Tx\|^2 - \| x\|^2, \quad \forall\, x\in\partial\Omega.$$

Then T has a fixed point in $\overline{\Omega}$..

Lemma 2.2 (Krein-Rutmann theorem [27,28]).Let P be a cone in a Banach space E. Let $S:E\to E$ is a completely continuous linear operator and S（P）$\subset$ P.If there exist $\psi\in E\setminus(-P)$ and $c>0$ such that $cS\psi\ge\psi$, then the spectral radius $r(S)>0$ and S has a positiveeigenfunction $\varphi(t)$ corresponding to its first eigenvalue $\lambda_1=(r(S))^{-1}$, *i.e.* $\varphi=\lambda_1 S\varphi$.

Lemma 2.3 ([23]).Let (H_1) be satisfied. Then $x(t)$ is a solution of problem (1) if and only if $x(t)$ is a solution of the integralequation

$$x(t)=\int_0^t\frac{(t-u)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}f(u,x(u))du+\gamma\int_0^t\frac{(t-u)^{\alpha-1}}{\Gamma(\alpha)}x(u)du+\phi(t), \tag{2}$$

where

$$\phi(t)=\sum_{i=0}^{n-1}\frac{\nu_i-\gamma\mu_i}{\Gamma(\alpha+i+1)}t^{\alpha+i}+\sum_{j=0}^{m-1}\frac{\mu_j}{\Gamma(j+1)}t^j. \tag{3}$$

Define operator T and $A:C[0,1]\to C[0,1]$ by

$$Tx(t)=\int_0^1\left(\frac{t^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}+\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right)x(u)du, \tag{4}$$

$$Ax(t)=\int_0^t\frac{(t-u)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}f(u,x(u))du+\gamma\int_0^t\frac{(t-u)^{\alpha-1}}{\Gamma(\alpha)}x(u)du+\phi(t). \tag{5}$$

Let $E=C[0,1]$ be the Banach space with norm $\|x\|=\max_{t\in[0,1]}|x(t)|$. Denote the usual normal cone $P=\{x\in E:x(t)\ge 0,\ \forall\, t\in[0,1]\}$. In this paper, the partial ordering is always given by P. Clearly, $T:P\to P$ is linear completely continuous, and from Lemma 2.3, we can see that $x(t)$ is a solution of problem (1) if and only if x is a fixed point of A.

Remark 2.1. $A: E \to E$ is completely continuous, a detailed proof is given in Appendix of literature [23].

Lemma 2.4. T is e-positive with $e(t) = t^{\alpha-1},\ t \in [0,1]$.

Proof. For any $x \in P \setminus \{\theta\}$, by (4),

$$Tx(t) = \int_0^1 \left(\frac{t^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} + \frac{t^{\alpha-1}}{\Gamma(\alpha)} \right) x(u)du \le \int_0^1 \left(\frac{1}{\Gamma(\alpha+\beta)} + \frac{1}{\Gamma(\alpha)} \right) x(u)du \cdot t^{\alpha-1}.$$

On the other hand, we have

$$Tx(t) = \int_0^1 \left(\frac{t^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} + \frac{t^{\alpha-1}}{\Gamma(\alpha)} \right) x(u)du \ \ge \int_0^1 \frac{1}{\Gamma(\alpha)} x(u)du \cdot t^{\alpha-1}.$$

So T is e-positive with $e(t) = t^{\alpha-1}$. □

Lemma 2.5. Let T be given by (4), then the spectral radius $r(T) > 0$ and T has a positive eigenfunction $\varphi^*(t)$ corresponding to its first eigenvalue $\lambda_1 = (r(T))^{-1}$.

Proof. Take $\psi(t) = t^{\alpha+\beta-1},\ t \in [0,1]$. Then $\psi \in E \setminus (-P)$ and

$$T\psi(t) = \int_0^1 \left(\frac{t^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} + \frac{t^{\alpha-1}}{\Gamma(\alpha)} \right) \psi(u)du = \int_0^1 \left(\frac{t^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} + \frac{t^{\alpha-1}}{\Gamma(\alpha)} \right) u^{\alpha+\beta-1}du$$
$$\ge \int_0^1 \frac{t^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} u^{\alpha+\beta-1}du \ = \frac{1}{\Gamma(\alpha+\beta+1)} \psi(t).$$

So $\Gamma(\alpha+\beta+1)T\psi(t) \ge \psi(t),\ t \in [0,1]$. Thus, from Lemma 2.2, we know that the spectral radius $r(T) > 0$ and T has a positive eigenfunction $\varphi^*(t)$ corresponding to its first eigenvalue $\lambda_1 = (r(S))^{-1}$, *i.e.* $\varphi^* = \lambda_1 T\varphi^*$. □

Remark 2.2. From Lemma 2.4 and Definition 2.3, there exist $a(\varphi^*), b(\varphi^*) > 0$ such that

$$a(\varphi^*)e \le T\varphi^* = \frac{1}{\lambda_1}\varphi^* \le b(\varphi^*)e. \tag{6}$$

3. Main results

In this section, we apply e-positive operators and Altman fixed point theory to study problem (1) and we obtain some new results on the existence results of solutions.

Let $L_1 = \max\{|f(t,0)| : t \in [0,1]\}$ and

$$C_1 = \frac{L_1}{\Gamma(\alpha+\beta+1)}, \ C = \max_{t\in[0,1]} |\phi(t)|,$$

where $\phi(t)$ is given in (3). Then $L_1, C_1, C \geq 0$.

Theorem 3.1. Let (H_1) be satisfied and there exists a constant $\sigma > 0$, such that

$$|f(t,y) - f(t,x)| \leq \sigma |y - x|, \quad \forall\ t \in [0,1],\ x, y \in \Re.$$

If $\tau := \frac{\sigma}{\Gamma(\alpha+\beta+1)} + \frac{\gamma}{\Gamma(\alpha+1)} < 1$. Then problem (1) has at least one solution in $\overline{\Omega}$, where

$\Omega = \{x \in \mathrm{E} : \|x\| < R\}$ with $R \geq \frac{C+C_1}{1-\tau}$.

Proof. We consider operator A defined by (5). From Remark 2.2, we know that $A : \overline{\Omega} \to \overline{\Omega}$ is completely continuous. From Lemma 2.1, we only need to prove that

$$\|Ax\| \leq \|x\|, \ \forall\ x \in \partial\Omega.$$

For $x \in \partial\Omega = \{x \in E : \|x\| = R\}$, we have

$$\begin{aligned}
\|Ax\| &\leq \|Ax - A\theta\| + \|A\theta\| \\
&= \max_{t\in[0,1]} \left| \int_0^t \frac{(t-u)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} (f(u,x(u)) - f(u,0)) du + \gamma \int_0^t \frac{(t-u)^{\alpha-1}}{\Gamma(\alpha)} x(u) du \right| \\
&\quad + \max_{t\in[0,1]} \left| \int_0^t \frac{(t-u)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} f(u,0) du + \phi(t) \right| \\
&\leq \max_{t\in[0,1]} \left(\int_0^t \frac{(t-u)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} |(f(u,x(u)) - f(u,0))| du + \gamma \int_0^t \frac{(t-u)^{\alpha-1}}{\Gamma(\alpha)} |x(u)| du \right) \\
&\quad + \max_{t\in[0,1]} \left(\int_0^t \frac{(t-u)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} |f(u,0)| du + |\phi(t)| \right) \\
&\leq \max_{t\in[0,1]} \left(\int_0^t \frac{(t-u)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \sigma |x(u)| du + \gamma \int_0^t \frac{(t-u)^{\alpha-1}}{\Gamma(\alpha)} |x(u)| du \right) \\
&\quad + \max_{t\in[0,1]} \left(\int_0^t \frac{(t-u)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} |f(u,0)| du + |\phi(t)| \right) \\
&\leq \left(\frac{\sigma}{\Gamma(\alpha+\beta+1)} + \frac{\gamma}{\Gamma(\alpha+1)} \right) \|x\| + \frac{L_1}{\Gamma(\alpha+\beta+1)} + C \\
&= \tau R + C_1 + C \leq R = \|x\|.
\end{aligned}$$

So by (2), we know that A has a fixed point. That is, problem (1) has a solution in $\overline{\Omega}$. □

Theorem 3.2. Let (H_1) be satisfied and there exists $b \in (0, \lambda_1)$ such that

$$|f(t,y) - f(t,x)| \leq b|y - x|, \quad \forall\ t \in [0,1],\ x, y \in \Re.$$

If $\gamma \in (0, \lambda_1)$, where λ_1 is the first eigenvalue of T. Then problem (1) has a unique solution $x^* \in E$, and for any $x_0 \in E$, the iterative sequence $x_n = Ax_{n-1}$ $(n = 1, 2, \cdots)$ converges to x^*.

Proof. For any given $x_0 \in E$, let $x_n = Ax_{n-1}$ $(n = 1, 2, \cdots)$. We can get the iterative sequence $\{x_n\} \subset E$. If $x_1 = x_0$, *i.e.* $Ax_0 = x_0$, so x_0 is a solution of problem (1). If $x_1 \neq x_0$, so $\left|x_1(t) - x_0(t)\right| \in P \setminus \{\theta\}$. From Lemma 2.2 and Remark 2.2, there exists $\beta_1 > 0$ such that

$$T(|x_1 - x_0|)(t) \leq \beta_1 \varphi^*(t), \quad \forall t \in [0,1].$$

Let $\varepsilon = \max\{b, \gamma\}$, then $\varepsilon \in (0, \lambda_1)$. Hence, we have

$$\begin{aligned}
|Ay(t) - Ax(t)| &\leq \int_0^t \frac{(t-u)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} |f(u, y(u)) - f(u, x(u))| du \\
&\quad + \gamma \int_0^t \frac{(t-u)^{\alpha-1}}{\Gamma(\alpha)} |y(u) - x(u)| du \\
&\leq b \int_0^t \frac{(t-u)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} |y(u) - x(u)| du \\
&\quad + \gamma \int_0^t \frac{(t-u)^{\alpha-1}}{\Gamma(\alpha)} |y(u) - x(u)| du \\
&\leq \varepsilon \int_0^t \left(\frac{(t-u)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} + \frac{(t-u)^{\alpha-1}}{\Gamma(\alpha)} \right) |y(u) - x(u)| du \\
&\leq \varepsilon \int_0^1 \left(\frac{t^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} + \frac{t^{\alpha-1}}{\Gamma(\alpha)} \right) |y(u) - x(u)| du \\
&= \varepsilon T(|y - x|)(t). \qquad (7)
\end{aligned}$$

Further, by (6), (7),

$$\begin{aligned}
|x_{n+1}(t) - x_n(\mathrm{t})| &= |Ax_n(t) - Ax_{n-1}(t)| \leq \varepsilon T(|x_n - x_{n-1}|)(t) \\
&\leq \cdots \leq \varepsilon^n T^n(|x_1 - x_0|)(t) \leq \varepsilon^n T^{n-1}(\beta_1 \varphi^*(t)) \\
&= \varepsilon^n \beta_1 \frac{1}{\lambda_1^{\,n-1}} \varphi^*(t) = \left(\tfrac{\varepsilon}{\lambda_1}\right)^n \beta_1 \lambda_1 \varphi^*(t), \qquad (n = 1, 2, \cdots). \qquad (8)
\end{aligned}$$

Thus, from (8),

$$\begin{aligned} |x_{n+p}(t)-x_n(t)| &= |x_{n+p}(t)-x_{n+p-1}(t)+\cdots+x_{n+1}(t)-x_n(t)| \\ &\le |x_{n+p}(t)-x_{n+p-1}(t)|+\cdots+|x_{n+1}(t)-x_n(t)| \\ &\le \beta_1\lambda_1\left(\left(\tfrac{\varepsilon}{\lambda_1}\right)^{n+p-1}+\cdots+\left(\tfrac{\varepsilon}{\lambda_1}\right)^n\right)\varphi^*(t) \\ &= \beta_1\lambda_1\frac{\left(\tfrac{\varepsilon}{\lambda_1}\right)^n\left(1-\left(\tfrac{\varepsilon}{\lambda_1}\right)^p\right)}{1-\tfrac{\tau}{\lambda_1}}\varphi^*(t) \\ &\le \beta_1\lambda_1\frac{\left(\tfrac{\varepsilon}{\lambda_1}\right)^n}{1-\tfrac{\tau}{\lambda_1}}\varphi^*(t), \qquad (n,\ p=1,\ 2,\cdots), \end{aligned} \tag{9}$$

and thus

$$\|x_{n+p}-x_n\|\le\beta_1\lambda_1\frac{\left(\tfrac{\varepsilon}{\lambda_1}\right)^n}{1-\tfrac{\varepsilon}{\lambda_1}}\|\varphi^*\|,\qquad (n,\ p=1,\ 2,\cdots).$$

Since $\frac{\varepsilon}{\lambda_1}\in(0,1)$, $\{x_n\}$ is a Cauchy sequence in $E.$ Because E is complete. Hence, there exists $x^*\in E$ such that $x_n\to x^*$ as $n\to\infty.$ Passing to the limit into $x_n=Ax_{n-1}$ and using the fact that A is continuous, we have $x^*=Ax^*$. That is, x^* is the fixed point of $A.$ Therefore, x^* is the solution of problem (1).

In the following, we show that the solution x^* of problem (1) is a unique solution in $E.$ Suppose that $\bar{x}\in E$ is the other solution of problem (1). Then $\bar{x}$ is a fixed point of A in $E.$ From Lemma 2.4 and Remark 2.2, there exists $\beta_2>0$ such that

$$T(|\bar{x}-x^*|)(t)\le\beta_2\varphi^*(t),\quad \forall\, t\in[0,1].$$

Thus, for any $n\in N,$ by (7),

$$\begin{aligned} |\bar{x}(t)-x_n(\mathrm{t})| &= |A\bar{x}(t)-Ax_{n-1}(t)|\le\varepsilon T(|\bar{x}-x_{n-1}|)(t) \\ &\le \cdots\le\varepsilon^nT^n(|\bar{x}-x_0|)(t)\le\varepsilon^nT^{n-1}(\beta_2\varphi^*(t)) \\ &= \varepsilon^n\beta_2\frac{1}{\lambda_1^{n-1}}\varphi^*(t)=\left(\tfrac{\varepsilon}{\lambda_1}\right)^n\beta_2\lambda_1\varphi^*(t),\qquad (n=1,\ 2,\cdots). \end{aligned}$$

And thus

$$\|\bar{x}-x_n\|\le\left(\frac{\varepsilon}{\lambda_1}\right)^n\beta_2\lambda_1\|\varphi^*\|,\qquad (n=1,\ 2,\cdots).$$

Because $x_n \to x^*$ as $n \to \infty$, we get $\| \bar{x} - x^* \| = 0$, and thus $\bar{x} = x^*$. □

Theorem 3.3. Let (H_1) be satisfied and

$$\begin{array}{ll} {}_0^cD_t^{\beta}({}_0^cD_t^{\alpha} - \gamma)x(t) \leq f(t, x(t)), & 0 < t < 1, \\ x^k(0) = \mu_k, & 0 \leq k < l, \\ x^{(\alpha+k)}(0) \leq \nu_k, & 0 \leq k < n; \end{array}$$

(H_2) there exists $x_0 \in E$ satisfying the following conditions:

(H_3) there exists $b \in (0, \lambda_1)$ such that

$$0 \leq f(t, y(t)) - f(t, x(t)) \leq b(y(t) - x(t)), \quad y(t) \geq x(t), \quad \forall t \in [0,1], \ x, \ y \in \Omega_1.$$

where $\Omega_1 = \{x \in E : x(t) \geq x_0(t), \ t \in [0,1]\}$. If $\gamma \in (0, \lambda_1)$, where λ_1 is the first eigenvalue of T. Then problem (1) has a unique solution x^* in Ω_1.

Proof. From Lemma 2.3 and (H_2), we know that $x_0(t) \leq Ax_0(t), \ \forall t \in [0,1]$. For $x, y \in \Omega_1$ with $x \leq y$, we have $y(t) \geq x(t), \ t \in [0,1]$, from (H_3),

$$\begin{aligned} Ay(t) &= \int_0^t \frac{(t-u)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} f(u, y(u))du + \gamma \int_0^t \frac{(t-u)^{\alpha-1}}{\Gamma(\alpha)} y(u)du + \phi(t) \\ &\geq \int_0^t \frac{(t-u)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} f(u, x(u))du + \gamma \int_0^t \frac{(t-u)^{\alpha-1}}{\Gamma(\alpha)} x(u)du + \phi(t) \\ &= Ax(t), \end{aligned}$$

which means that A is increasing in Ω_1. For any $x \in \Omega_1$, we have $Ax(t) \geq Ax_0(t) \geq x_0(t)$, that is $A(\Omega_1) \subset \Omega_1$. Let $x_n = Ax_{n-1}$ $(n = 1, 2, \cdots)$, then we have

$$x_0 \leq x_1 \leq \cdots \leq x_n \leq \cdots.$$

Similar to the discussion in the proof of Theorem 3.2, we assume $x_1 \neq x_0$, By Lemma 2.1 and Remark 2.2, there exists $\beta_1 > 0$ such that

$$T(|x_1 - x_0|)(t) \leq \beta_1 \varphi^*(t), \quad \forall t \in [0,1].$$

From (7), (8) and (9), we have

$$x_{n+1}(t)-x_n(t)\le\left(\frac{\varepsilon}{\lambda_1}\right)^n\beta_1\lambda_1\varphi^*(t),\qquad (n=1,\,2,\cdots).$$

And

$$x_{n+p}(t)-x_n(t)\le\beta_1\lambda_1\frac{\left(\frac{\varepsilon}{\lambda_1}\right)^n}{1-\frac{\varepsilon}{\lambda_1}}\varphi^*(t),\qquad (n,\,p=1,\,2,\cdots).$$

Since $\frac{\varepsilon}{\lambda_1}\in(0,1)$, $\{x_n\}$ is a Cauchy sequence in Ω_1. Because Ω_1 is complete. Hence, there exists $x^*\in\Omega_1$ such that $x_n\to x^*$ as $n\to\infty$. Passing to the limit into $x_n=Ax_{n-1}$ and using the fact that A is continuous, we have $x^*=Ax^*$. That is, x^* is the fixed point of A in Ω_1. Therefore, x^* is the solution of problem (1).

In the following, we show that the solution x^* of problem (1) is a unique solution in Ω_1. Suppose that $\bar{x}\in\Omega_1$ is the other solution of problem (1). Then $\bar{x}$ is a fixed point of A in Ω_1. From Lemma 2.4 and Remark 2.2, there exists $\beta_2>0$ such that

$$T(|\bar{x}-x^*|)(t)\le\beta_2\varphi^*(t),\quad \forall\, t\in[0,1].$$

And for any $n\in N$, we have

$$\bar{x}\ge x_n\ge x_0.$$

Then,

$$\bar{x}\ge x^*\ge x_n\ge x_0.$$

Hence, for any $n\in N$ and $t\in[0,1]$, we have

$$\begin{aligned}|\bar{x}(t)-x^*(t)| &\le |\bar{x}(t)-x_n(t)|+|x^*(t)-x_n(t)|\\ &\le |A^n\bar{x}(t)-A^nx_0(t)|+|A^nx^*(t)-A^nx_0(t)|\\ &\le 2|A^n\bar{x}(t)-A^nx_0(t)|\\ &\le 2\beta_2\left(\tfrac{\varepsilon}{\lambda_1}\right)^n\lambda_1\varphi^*(t).\end{aligned}$$

And thus

$$\|\bar{x}-x_n\|\le 2\left(\frac{\varepsilon}{\lambda_1}\right)^n\beta_2\lambda_1\|\varphi^*\|,\qquad (n=1,\,2,\cdots).$$

Thus, we obtain $\bar{x} = x^*$. □

Corollary 3.4. Let (H_1), (H_3) be satisfied and $f:[0,1]\times[0,\infty]\to[0,\infty]$. Assume $\gamma\in(0,\lambda_1)$ where λ_1 is the first eigenvalue of T. Then problem (1) has a unique solution x^* in Ω_1.

Proof. The proof of this theorem is similar to the proof of Theorem 3.3. We just need to take $x_0(t)=\phi(t)$. □

Theorem 3.5. Let (H_1), (H_3) be satisfied and
(H_5) there exists $x_0\in E$ satisfing the following conditions:

$$\begin{aligned} &{}_0^cD_t^{\beta}({}_0^cD_t^{\alpha}-\gamma)x(t)\ge f(t,x(t)), && 0<t<1,\\ &x^k(0)=\mu_k, && 0\le k<l,\\ &x^{(\alpha+k)}(0)\ge \nu_k, && 0\le k<n; \end{aligned}$$

where Ω_1 is (H_3) replaced by $\Omega_2=\{x\in E: x(t)\le x_0(t),\ t\in[0,1]\}$. Assume $\gamma\in(0,\lambda_1)$, where λ_1 is the first eigenvalue of T. Then problem (1) has a unique solution x^* in Ω_2.

Proof. The proof of this theorem is exactly similar to the proof of Theorem 3.3. □

4. Example

We present an example to better illustrate our main results.

Example 4.1. Consider the following initial value problem

$$\begin{cases} {}_0^cD_t^{1/2}\left({}_0^cD_t^{3/2}-\frac{1}{8}\Gamma\left(\frac{1}{2}\right)\right)x(t)=\dfrac{1}{(t+2)^2}\dfrac{|x(t)|}{1+|x(t)|}+e^{-t}, & 0<t<1,\\ x^{(0)}(0)=x^{(1)}(0)=1, & \\ x^{(3/2)}(0)=\frac{1}{4}\Gamma\left(\frac{1}{2}\right). & \end{cases} \tag{10}$$

In this example, $\alpha=\frac{3}{2}$, $\beta=\frac{1}{2}$, $\mu_0=\mu_1=1$, $\nu_0=\frac{1}{4}\Gamma(\frac{1}{2})$, $\gamma=\frac{1}{8}\Gamma(\frac{1}{2})$, $m=2$, $n=1$, $l=2$ and $f(t,x)=\frac{1}{(t+2)^2}\frac{|x|}{1+|x|}+e^{-t}$. For $x, y\in\Re$ and $t\in[0,1]$, we have

$$|f(t,y)-f(t,x)|=\frac{1}{(t+2)^2}\left|\frac{|y|}{1+|y|}-\frac{|x|}{1+|x|}\right|\le\frac{|y-x|}{4(1+|x|)(1+|y|)}\le\frac{1}{4}|y-x|.$$

Choosing $\sigma = \frac{1}{4}$ Further,

$$\tau := \frac{\sigma}{\Gamma(\alpha+\beta+1)} + \frac{\gamma}{\Gamma(\alpha+1)} = \frac{1}{8} + \frac{1}{6} = \frac{7}{24} < 1.$$

$$L_1 = \max\{| f(t,0) |: t \in [0,1]\} = \max\{e^{-t} : t \in [0,1]\} = 1,$$

$$C_1 = \frac{L_1}{\Gamma(\alpha+\beta+1)} = \frac{1}{2},\ C = \max_{t\in[0,1]} |\phi(t)| = \max_{t\in[0,1]} \left| \frac{1}{6} t^{\frac{3}{2}} + t + 1 \right| = \frac{13}{6},$$

and take

$$R \geq \frac{C + C_1}{1-\tau} = \frac{64}{17}.$$

Therefore, from Theorem 3.1, we know that problem (10) has a solution in $\Omega_1 = \{x \in E : \| x \| \leq R\}$.

In addition, take $b = \frac{1}{4}$. By (4), we have

$$\begin{aligned} r(T) \quad \leq \quad \| T \| \quad & \leq \max_{t\in[0,1]} \int_0^1 \frac{t^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} + \frac{t^{\alpha-1}}{\Gamma(\alpha)} du \\ & = \frac{1}{\Gamma(\alpha+\beta)} + \frac{1}{\Gamma(\alpha)} = 1 + \frac{1}{\Gamma(\frac{3}{2})} \end{aligned}$$

Thus $\lambda_1 = \frac{1}{r(T)} \geq \frac{\Gamma(\frac{3}{2})}{1+\Gamma(\frac{3}{2})} \approx 0.469841$, and thus $b, \gamma \in (0, \lambda_1)$. Therefore, from Theorem 3.2, we know that problem (10) has a unique solution $x^* \in E$, and for any $x_0 \in E$ the iterative sequence

$$x_n(t) = \int_0^t (t-u) f(u, x_{n-1}(u)) du + \frac{1}{4} \int_0^t (t-u)^{\frac{1}{2}} x_{n-1}(u) du + \frac{1}{6} t^{\frac{3}{2}} + t + 1,\ \ n = 1, 2, \cdots,$$

converges to

$$x^*(t) = \int_0^t (t-u) f(u, x^*(u)) du + \frac{1}{4} \int_0^t (t-u)^{\frac{1}{2}} x^*(u) du + \frac{1}{6} t^{\frac{3}{2}} + t + 1,\ \ t \in [0,1]. \square$$

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