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A Note on Posner's Theorems

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Abstract. A well-known theorem of Posner [1] states that: if iterate of derivations of 2-torsion free prime ring is a derivation, then one of them must be zero, which is called Posner's first theorem. In this paper, we intend to prove this result in more general settings. Moreover, we give a deduction of Posner's second theorem from the first on Lie ideals.

Keywords: Prime ring, Lie ideal, Skew-derivation

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1. Introduction

All through this paper our ring R will be an associative ring with center Z(R). Recall that a ring R is called a prime ring if for all x, y ϵ R, xRy=(0) implies x= 0 or y=0 and is called semiprime if xRx=(0) implies x=0. Clearly, every prime ring is semiprime but the converse is not true, for instance Z×Z, where Z denotes the ring of integers. An additive group U of a ring R is called Lie ideal of R if [U, R] is contained in R, where the symbol [U, R] stands for a set of commutators i.e. {[u, r]= ur-ru : for all u in U and r in R}. We shall use frequently the basic commutator identities: [xy, z] = x[y,z]+[x,z]y, [x, yz]=y[x,z]+[x,y]z. By a derivation we mean an additive function d of R into itself satisfying the Leibnitz rule i.e. d(xy)=d(x)y + xd(y) for all x, y in R. The notion of a derivation has been generalized in many ways for the last five decades. The notion of skew derivation is one of them. By a skew derivation of a ring R we mean an additive map μ of R into itself associated with an automorphism σ of R such that μ(xy)=μ(x)y+ σ(x)μ(y) for all x, y in R. For convenience, we shall denote a skew derivation of R as an order pair (μ, σ). Note that if we take σ as the identity map then the skew derivation (μ, σ) is merely the ordinary derivation of R.

In the mid of the twentieth century, after the development of the general structure theory for rings, a large amount of work was done that showed that under certain type of restrictions a ring had to be commutative. In 1957, Posner [1] initiated the study of derivations in associative rings. Precisely, he proved two very striking results that got fame as Posner's first theorem and Posner's second theorem respectively. Posner's first theorem states that; *If R is a 2-torsion free prime ring and d_1, d_2 are derivations of R such that the iterate d_1d_2 is also a derivation of R, then either $d_1=0$ or $d_2=0$.* In the sequel, Posner's second theorem gives a criterion for commutativity of prime rings involving nonzero derivations, states that; *If R is a prime ring and d is a derivation of R such that [d(x), x] is central, then either d=0 or R is commutative*. In 1992, Mathieu [2] proved that Posner's second theorem can be deduced from Posner's first theorem that made Posner theorems more reliable. Very recently, Ashraf and Siddeeque [3] extended Posner's first theorem for prime rings with involution. They proved: *Let R be a 2-torsion free *-prime ring, I a nonzero *-ideal of R and d_1, d_2: I → R be the derivations such that the iterate d_1d_2: I → R is also a derivation. If at least one of d_1 and d_2 commutes with '*', then d_1= 0 or d_2= 0.*

Inspired by Ashraf and Siddeeque [3], in the present note, we focus on Posner's first theorem and prove it for the class of skew derivation and derivations of Lie ideals. Also, we deduce Posner's second theorem on Lie ideals from the first.

2. Main results

The following theorem is a direct generalization of Posner's first theorem.

Theorem 2.1. Let (μ_1, σ_1) and (μ_2, σ_2) be the skew-derivations of 2-torsion free prime ring R with $\mu_2\sigma_2 = \sigma_2\mu_2$. If the iterate $(\mu_1\mu_2, \sigma_1\sigma_2)$ is a skew derivation of R, then either $(\mu_1, \sigma_1) = 0$ or $(\mu_2, \sigma_2) = 0$.

Proof : By hypothesis, we have

$(\mu_1\mu_2)(xy) = (\mu_1\mu_2)(x)y+(\sigma_1\sigma_2)(x)(\mu_1\mu_2)(y)$ for all x, y in R. (1)

On the other hand, we find

$(\mu_1\mu_2)(xy) = \mu_1(\mu_2(xy))$

$= \mu_1(\mu_2(x)y+ \sigma_2(x)\mu_2(y))$

$= \mu_1(\mu_2(x)y)+ \mu_1(\sigma_2(x)\mu_2(y))$

$= \mu_1(\mu_2(x))y+\sigma_1(\mu_2(x))\mu_1(y)+\mu_1(\sigma_2(x))\mu_2(y)+\sigma_1(\sigma_2(x))\mu_1(\mu_2(y))$ (2)

On combining (1) and (2), we get

$(\sigma_1\mu_2)(x)\mu_1(y)+(\mu_1\sigma_2)(x)\mu_2(y)=0$ for all x, y in R. (3)

Replacing y by $\mu_2(y)z$ in Eq. (3), we find that $(\sigma_1\mu_2)(x)\mu_1(\mu_2(y))z+(\sigma_1\mu_2)(x)\sigma_1(\mu_2(y))\,\mu_1 z +(\mu_1\sigma_2)(x)\mu_2(\mu_2(y))z+(\mu_1\sigma_2)(x)\sigma_2(\mu_2(y))\mu_2(z)=0$ for all x, y, z in R. We re-write this expression as

$A(x, y, z) + B(x, y, z) = 0$ (4)

where $A(x, y, z) = (\sigma_1\mu_2)(x)(\mu_1\mu_2)(y)z + (\mu_1\sigma_2)(x)(\mu_2)^2(y)z$

and $B(x, y, z) = (\sigma_1\mu_2)(x)(\sigma_1\mu_2)(y)\mu_1(z) + (\mu_1\sigma_2)(x)(\sigma_2\mu_2)(y)\mu_2(z)$.

Note that $A(x, y, z) = \{(\sigma_1\mu_2)(x)\mu_1(\mu_2(y))+(\mu_1\sigma_2)(x)\mu_2(\mu_2(y))\}z$, which is zero in view of Eq. (3). Thus from (4), we left with $B(x, y, z) = 0$, that is

$(\sigma_1\mu_2)(x)(\sigma_1\mu_2)(y)\mu_1(z) + (\mu_1\sigma_2)(x)(\sigma_2\mu_2)(y)\mu_2(z) = 0$ (5)

A repeated application of Eq. (3) yields

$\mu_1(\sigma_2(x))\,(\mu_2(\sigma_2(y)) + \sigma_2(\mu_2(y)))\mu_2(z) = 0$ for all x, y, z in R.

Since, we have $\mu_2\sigma_2 = \sigma_2\mu_2$, the above relation becomes

$\mu_1(\sigma_2(x))\,(\mu_2(\sigma_2(y)) + \mu_2(\sigma_2(y)))\mu_2(z) = 0$ for all x, y, z in R.

i.e. $2\,\mu_1(\sigma_2(x))\,\mu_2(\sigma_2(y))\,\mu_2(z) = 0$ for all x, y, z in R.

Since R is 2-torsion free prime ring and σ_2 is an automorphism of R, we may infer that $\mu_1(r)\mu_2(s)\mu_2(z) = 0$ for all r, s, z in R. Replacing r by rp, where p is any element of R, we obtain $\mu_1(r)R\mu_2(s)\mu_2(z) = (0)$ for all r, s, z in R. Primeness of R forces that either $\mu_1(r) = 0$ or $\mu_2(s)\mu_2(z) = 0$. That gives, either $\mu_1 = 0$ or $\mu_2 = 0$ as desired.

Corollary 2.2. Iterate of two nonzero Jordan derivations cannot be a Jordan derivation.
Proof: By taking $\sigma_1 = \sigma_2 = I_d$ (identity mapping) in Theorem 2.1 together with Theorem 3.1 of Herstein [4] one may conclude the result.

Let U be a Lie ideal of a prime ring R. An additive mapping d: U→R is called a derivation of U if d(uv)=d(u)v+ud(v) for all u, v in U.

Theorem 2.3. Let d_1, d_2: U → R be two derivations of a square-closed non-central Lie ideal of U of a 2-torsion free prime ring R. If the iterate d_1d_2 is also a derivation of U, then either d_1=0 or d_2=0.
Proof: Let us assume that d_1d_2 is a derivation of U. Now, we observe that $d_2(U)$ is a subset of U, since d_1 is a derivation of U. By opting the same technique as in Theorem 2.1, we can obtain

$$d_1(u)d_2(v)+d_2(u)d_1(v)=0 \text{ for all u, v in U.} \quad (6)$$

Since U is square-closed and $d_2(v)$ is in U for all v in U, we replace v by $2d_2(v)w$ in (6) in order to find

$$\{d_1(u)d_2(d_2(v))+d_2(u)d_1(d_2(v))\}w+d_1(u)d_2(v)d_2(w)+d_2(u)d_2(v)d_1(w)=0.$$

Using (6), it reduces to $d_1(u)d_2(v)d_2(w)+d_2(u)d_2(v)d_1(w)=0$ for all u, v, w in U. Again utilization of (6) yields $d_2(u)d_2(v)d_1(w)=0$ for all u, v, w in U. Replacing w by wz in the last relation and using it, we get $d_2(u)d_2(v)Ld_1(z)=0$ for all u, v, z in U. Applying Lemma 4 of [5], we obtain that either $d_2(u)d_2(v)=0$ for all u, v in U or $d_1(U)=(0)$. If $d_2(u)d_2(v)=0$ for all u, v in U, again using the same argument, we get $d_2(U)=(0)$. It completes the proof.

Theorem 2.4. Let R be a 2-torsion free prime ring and U be a square-closed Lie ideal of R. If d be a derivation of U such that $[d(u), u] \in Z(R)$ for all u in U, then either d=0 or U is central.
Proof: If possible assume that U is not central. By hypothesis, we have $[d(u), u] \in Z(R)$ for all u in U. Linearizing, we get $[d(u), v] + [d(v), u] \in Z(R)$ for all u, v in U. In particular, we have

$$[d(u), u^2] + [d(u^2), u] \in Z(R) \text{ for all u, v in U.} \quad (7)$$

Further, we note that

$$[d(u), u^2] - [d(u^2), u] = 0 \text{ for all u, v in U.} \quad (8)$$

Combining (7) and (8), we may infer that $[d(u), u^2] \in Z(R)$ for all u in U. Also, we have $[d(u), u] \in Z(R)$ for all u in U, by hypothesis. It yields that $0=[d(u), [d(u), u^2]] = [d(u), 2u[d(u), u]]=2[d(u),u][d(u), u]$ for all u in U. Since R is 2-torsion free, we get $[d(u),u][d(u), u] = 0$ for all u in U, which is not possible, as center of a prime ring contains no zero-divisor. Hence, $[d(u), u] = 0$ for all u in U. Again linearizing, we find that $[d(u), v] + [d(v), u] = 0$ for all u, v in U.

$$[d(u), v] + [d(v), u] = 0 \text{ for all u, v in U.} \quad (9)$$

Let λ_a: U→R denotes the inner derivation of L associated with a fixed element a in R. In view of (9), it follows that $\lambda_u d(v)= \lambda_{d(u)}(v)$ for all u, v in L. With the aid of Theorem 2.3, we find that either $\lambda_u(U)=(0)$ for all u in U or d(U)=(0). That means, either U is commutative or d=0. If U is commutative, then by Lemma 2.6 of [6] we get U is in Z(R), which is a contradiction. It completes the proof.
Replacing U by R in Theorem 2.4, consequently we get Posner's second theorem as following:

Corollary 2.5. [[1], Theorem 2] Let d be a derivation of a 2-torsion free prime ring R such that $[d(x), x] \in Z(R)$ for all x in R. then either d =0 or R is commutative.

We conclude with the following example, which shows that Posner's first theorem (and hence Theorem 2.1) cannot be extended to the class of semiprime rings:

Example 2.6. Let $R= Z_2[x]\times Z_2[x]$, where $Z_2[x]$ stands for the ring of polynomials in the indeterminate x over the field of integers module 2. Note that R is a semiprime ring but not prime. For any element (p(x), q(x)) of R, we define mappings d_1, d_2: $R\rightarrow R$ as $d_1((p(x), q(x))) = (p'(x), q'(x))$, where ' denotes the usual differential operator and $d_2((p(x), q(x)))= (p'(x), 0)$. Then, one can easily check that d_1, d_2 and the iterate d_1d_2 are derivations of R. But neither $d_1=0$ nor $d_2=0$.

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