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The Infinitude of Solutions to the Diophantine Equation $p^3 + q = z^3$ when p, q are Primes

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Abstract. In this paper, we consider the Diophantine equation $p^3 + q = z^3$ where $p \geq 2$ and q are primes. We determine the value z, and the form of q for which q may be prime. The equation then has infinitely many solutions. The first five numerical solutions in which q is prime are also exhibited.

Keywords: Diophantine equations

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1. Introduction

The field of Diophantine equations is ancient, vast, and no general method exists to decide whether a given Diophantine equation has any solutions, or how many solutions. In most cases, we are reduced to study individual equations, rather than classes of equations.

The literature contains a very large number of articles on non-linear such individual equations involving primes and powers of all kinds. Among them are for example [1, 2, 5, 6]. The title equation stems from the equation $p^x + q^y = z^2$.

Whereas in most articles, the values x, y are investigated for solutions of the equation, in this paper these values are fixed positive integers. In the equation

$$p^3 + q^1 = z^3, \tag{1}$$

we consider all primes $p \geq 2$ and q prime. Our objective is to find solutions to equation (1). This is done in Section 2.

2. The main result

In Theorem 2.1, we establish the values q and z in equation (1).

Theorem 2.1. Suppose that $p^3 + q^1 = z^3$ and $p \geq 2$ is prime. For every prime q for which the equation has a solution, then

$$q = 3p^2 + 3p + 1 \qquad \text{and} \qquad z = p + 1.$$

Proof: In equation (1) $p < z$. Denote $z = p + A$ where $A \geq 1$ is an integer. The equation $p^3 + q = z^3$ then yields

$$p^3 + q = (p + A)^3$$

or

$$q = (p + A)^3 - p^3 = 3p^2A + 3pA^2 + A^3 = A(3p^2 + 3pA + A^2). \quad (2)$$

Since q is prime, therefore $A = 1$ in (2), and hence

$$q = 3p^2 + 3p + 1. \quad (3)$$

Thus, in (3) we have determined the value of each prime q in terms of the prime p. The value z is then $z = p + 1$. One now obtains

$$p^3 + (3p^2 + 3p + 1) = (p + 1)^3. \quad (4)$$

The identity (4) is valid for each and every prime $p \geq 2$. The value q has been determined, and yields q prime or q composite.

This completes the proof of Theorem 2.1. □

The first five solutions mentioned earlier are now presented as follows.

Solution 1. $2^3 + 19 = 3^3$.
Solution 2. $3^3 + 37 = 4^3$.
Solution 3. $11^3 + 397 = 12^3$.
Solution 4. $13^3 + 547 = 14^3$.
Solution 5. $17^3 + 919 = 18^3$.

The primes $p = 5$ and $p = 7$ yield composite values of $q = 3p^2 + 3p + 1$.

Remark 2.1. Every prime $p > 2$ is either of the form $4N + 1$ or $4N + 3$. One can easily verify for each prime $4N + 1/4N + 3$ that $q = 3p^2 + 3p + 1$ is of the form $4U + 3/4V + 1$.

Final remark. All primes $p > 3$ are also of the form $p = 6M + 1$ and $p = 6M + 5$. The prime $q = 3p^2 + 3p + 1 = 3p(p + 1) + 1$ is of the form $6M + 1$. There are infinitely many primes of the form $6M + 1$. Therefore, there are infinitely many primes $q = 3p^2 + 3p + 1$. Hence, when p, q are primes, the equation $p^3 + q = z^3$ has infinitely many solutions.

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