Annals of Pure and Applied Mathematics
Vol. 17, No. 1, 2018, 113-122
ISSN: 2279-087X (P), 2279-0888(online)
Published on 11 May 2018
www.researchmathsci.org
DOI: http://dx.doi.org/10.22457/apam.v17n1a13

Annals of Pure and Applied Mathematics

Fuzzy *M*-solid Subpseudovarieties

Atthchai Chada

Department of Mathematics, Faculty of Science and Technology, Rajabhat Mahasarakham University, Mahasarakham-44000, Thailand
E-mail: Valentine_at@hotmail.co.th

Received 8 April 2018; accepted 9 May 2018

Abstract. In this paper, we define the concept of fuzzy *M*-solid subpseudovarieties of a given *M*-solid pseudovariety and show that the lattice of all fuzzy *M*-solid subpseudovarieties is a complete sublattice of the lattice of all fuzzy subpseudovarieties of the same *M*-solid pseudovariety, where *M* is a submonoid of hypersubstitutions. Moreover, for submonoids M_1, M_2 of hypersubstitutions with $M_1 \subseteq M_2$, we show that the lattice of all fuzzy M_2-solid subpseudovarieties is a complete sublattice of the lattice of all fuzzy M_1-solid subpseudovarieties.

Keywords: Pseudovarieties; Fuzzy subalgebras; Fuzzy subpseudovarieties; *M*-solid pseudovarieties, Fuzzy *M*-solid subpseudovarieties

AMS Mathematics Subject Classification (2010): 03E72, 06D72, 16Y30, 20N10

1. Introduction

The notion of fuzzy set was first introduced by Zadeh [14] in 1965. The first inspiration application for many algebraic structures was the concept of fuzzy group introduced by Rosenfeld [11]. The fuzzification was applied to subclass of algebras by many authors (see e.g., [10,12,13]). In [6], Murali defined fuzzy subalgebras of a given algebra in universal algebra sense, studied its lattice and showed that the lattice of all fuzzy subalgebras forms complete lattice. In 2011, Pibaljommee [8] defined the concept of fuzzy subvarieties of a variety *V* in universal algebra sense and found that the lattice of all subvarieties of the variety can be embedded into the lattice of all fuzzy subvarieties of the varieties as complete lattices. In 2011, Pibaljommee [9] defined the concept of fuzzy *M*-solid subvarieties in universal algebra. In 2011, Patchakhieo and Pibaljomme [7] gave a connection between *L*-fuzzy fully invariant congruence relations and varieties. Where *L* is a complete lattice. In 2013, Chada and Pibaljommee [1] defined the concept of fuzzy subpseudovarieties of a subpseudovariety *V* in universal algebra. In universal algebra many authors investigated structure of solid, *M*-solid pseudovarieties (see e.g., [2,3,4,5]) which play important role in computer science. It is natural to consi-der the structure of fuzzy *M*-solid subpseudovarieties. So, the purpose of this work is to define the concept of fuzzy *M*-solid subpseudovarieties of a given *M*-solid pseudovariety extending from the concept of fuzzy subpseudovarieties and show that the lattice of all fuzzy *M*-solid subpseudovarieties of an *M*-solid pseudovariety is a complete sublattice of the lattice of all fuzzy subpseudovarieties of the same *M*-solid pseudo-variety. Moreover, for submonoids M_1, M_2 of hypersubstitutions with $M_1 \subseteq M_2$, we show that the lattice of all fuzzy M_2-solid subpseudovarieties is a complete sublattice of the lattice of all fuzzy M_1-solid subpseudovarieties.

2. Preliminaries

Let $\tau = (n_i)_{i\in I}$ be a type of algebras with operation symbols $(f_i)_{i\in I}$ where f_i is an n_i*-ary* operation. An *algebra* of type τ is an ordered pair $\mathrm{A} := (A, (f_i^A)_{i\in I})$, where *A* is a non-empty set and $(f_i^A)_{i\in I}$ is a sequence of operations on *A* indexed by a non-empty index set *I* such that to each n_i*-ary* operation symbol f_i there is a corresponding n_i*-ary* operation f_i^A on *A*. The set *A* is called the *universe* of A. We often write A instead of $\mathrm{A} := (A, (f_i^A)_{i\in I})$. We denote by $Alg(\tau)$ the class of all algebras of type τ and $Alg_f(\tau)$ the class of all finite algebras of type τ. The concept of a pseudovariety was first introduced by Eilenberg and Schutzenberger in [4] as a class of finite algebras of the same type τ closed under the formations of homomorphic images (**H**), subalgebras (**S**) and finite direct products (**P**$_\mathbf{f}$). A class *V* of algebras of type τ is called a *variety* if it is closed under taking of the formations (**H**), (**S**) and direct product (**P**), then its finite part, i.e., the class V^{fin} of all finite algebras contained in *V* is a pseudovariety. Let $K \subseteq Alg(\tau)$. We denote by **H**(*K*) the class of all homomorphic images of algebras in *K*, by **S**(*K*) the class of all subalgebras of algebras in *K*, by **P**(*K*) the class of all direct products of algebras in *K* and by **P**$_\mathbf{f}$(*K*) the class of all finite direct products of algebras in *K*. Instead of **H**({ A }), **S**({ A }), **P**({ A }) and **P**$_\mathbf{f}$({ A }), we write **H**(A), **S**(A), **P**(A) and **P**$_\mathbf{f}$(A), respectively for every algebra A in $Alg(\tau)$. Clearly, $Alg_f(\tau)$ and the class of all one-element algebras $\mathrm{T}(\tau)$ are pseudovarieties which are called *trivial pseudo-varieties*.

Let $X_n := \{x_1, \ldots, x_n\}$ be a finite set of variables, $W_\tau(X_n)$ be the set of all *n-ary* terms of type τ and let $W_\tau(X) := \bigcup_{n=1}^{\infty} W_\tau(X_n)$ with $X := \{x_1, \ldots, x_n, \ldots\}$ be the set of all terms of type τ. Then we denote by $F_\tau(X)$ the absolutely free algebra; $F_\tau(X) := (W_\tau(X_n); (\bar{f}_i)_{i\in I})$ with $(\bar{f}_i) : (t_1, \ldots, t_{n_i}) \to f_i(t_1, \ldots, t_{n_i})$. An *equation* of type τ is a pair (s,t) from $W_\tau(X)$ such pairs are commonly written $s \approx t$. An equation $s \approx t$ is an *identity* of algebra A, denote by $\mathrm{A} \models s \approx t$ if $s^{\mathrm{A}} \approx t^{\mathrm{A}}$, where s^{A} and t^{A} are the term operations induced by terms s and t on A.

Let *V* be a pseudovariety of type τ. We denote by $Sub(V)$ the class of all pseudovarieties of *V*. It is well-known [2] the lattice $L(V) := (Sub(V); \wedge, \vee)$ is a complete lattice, where

$\wedge\{W_i \in Sub(V) \mid i \in I\} = \bigcap_{i\in I} W_i$ and

$\vee\{W_i \in Sub(V) \mid i \in I\} = \bigcap\{W \in Sub(V) \mid \bigcup_{i\in I} W_i \subseteq W\}$

for every family $\{W_i \mid i \in I\}$ of subpseudovarieties of V. Let K be a class of algebras of type τ, the class $\mathbf{HSP_f}(K)$ the smallest pseudovariety containing K. Let A be a non-empty set and $P(A)$ be the power set of A. A mapping $\gamma : P(A) \to P(A)$ is called a *closure operator* on A if for any $X, Y \in P(A)$, the following conditions hold:

1. $X \subseteq \gamma(X)$ (extensivity),
2. $X \subseteq Y \Rightarrow \gamma(X) \subseteq \gamma(y)$ (monotonicity),
3. $\gamma(\gamma(X)) \subseteq \gamma(X)$ (idempotency).

3. Fuzzy subpseudovarieties

In this section, we present the concept of fuzzy subpseudovarieties of a pseudovariety V and some results which shown in [1].

Definition 3.1. [1] Let V be a pseudovariety of type τ. A mapping $\lambda : V \to [0,1]$ is called *fuzzy subpseudovarieties* of V if the following conditions hold:

(1.) $\forall \mathrm{B} \in V \;\; \forall \mathrm{A} \in \mathbf{H}(\mathrm{B})$, $\lambda(\mathrm{A}) \geq \lambda(\mathrm{B})$,

(2.) $\forall \mathrm{B} \in V \;\; \forall \mathrm{A} \in \mathbf{S}(\mathrm{B})$, $\lambda(\mathrm{A}) \geq \lambda(\mathrm{B})$ and

(3.) $\lambda(\prod_{i=1}^{n} \mathrm{A}_i) \geq \inf\{\lambda(\mathrm{A}_i) \mid 1 \leq i \leq n\}$ where $\{\mathrm{A}_i \mid 1 \leq i \leq n\} \subseteq Alg_f(\tau)$.

We denote by $FPV(\tau)$ the set of all fuzzy subpseudovarieties of V. By the definition above, we note that if $\mathrm{A}, \mathrm{B} \in V$ such that A is isomorphic to B, then $\lambda(\mathrm{A}) = \lambda(\mathrm{B})$. Since every one element algebra of type τ is contained in every pseudovariety of type τ.

A fuzzy subset λ of a pseudovariety V of type τ is a function $\lambda : V \to [0,1]$ and for $t \in [0,1]$ the set $\lambda_t = \{\mathrm{A} \in V \mid \lambda(\mathrm{A}) \geq t\}$ is called *level subset* of V.

Next proposition is correspondence to the definition of fuzzy subpseudovarieties.

Proposition 3.1. [1] Let V be a pseudovariety of type τ, $\lambda : V \to [0,1]$ be a fuzzy subset of V. Then λ is a fuzzy subpseudovarieties of V if and only if for all $t \in [0,1]$ the level subset $\lambda_t \neq \phi$ is a subpseudovariety of V.

Let V be a pseudovariety of type τ and $\lambda, \nu \in FPV(\tau)$. We define the order on $FPV(\tau)$ by $\lambda \leq \nu$ (sometime written by $\lambda \subseteq \nu$) which $\lambda \leq \nu$ mean $\lambda(\mathrm{A}) \leq \nu(\mathrm{A})$ for all $\mathrm{A} \in V$. We review the notions of unions and intersections of fuzzy subset of V. Let $\{\lambda_i \mid i \in I\}$ be a family of fuzzy subsets of V. Arbitrary intersections and unions are defined by

$$(\bigcap_{i\in I}\lambda_i)(\mathrm{A}) := \inf\{\lambda_i(\mathrm{A}) \mid i \in \mathrm{I}\} \text{ and}$$

$$(\bigcup_{i\in I}\lambda_i)(\mathrm{A}) := \sup\{\lambda_i(\mathrm{A}) \mid i \in \mathrm{I}\}.$$

Then it is easy the verify that arbitrary intersection of fuzzy subpseudovarieties of V is again a fuzzy subpseudovarieties of V. In general, the union of fuzzy subpseudovarieties of V need not to be a fuzzy subpseudovarieties of V (see in [1]). For a fuzzy subset λ of V, we define the fuzzy subpseudovarieties generated by λ by

$$\langle\lambda\rangle_F := \bigcap\{\nu \in FPV(\tau) \mid \lambda \subseteq \nu\}.$$

Next, we consider the lattice of all fuzzy subpseudovarieties of V. Let $\{\lambda_i \mid i \in I\}$ be a family of fuzzy subpseudovarieties of V. We define the meet $\wedge$ and the join $\vee$ on $FPV(\tau)$ as follow:

$$\bigwedge_{i\in I}\lambda_i := \bigcap_{i\in I}\lambda_i \text{ and } \bigvee_{i\in I}\lambda_i := \bigcap\{\lambda \in FPV(\tau) \mid \bigcup_{i\in I}\lambda_i \subseteq \lambda\}.$$

Then the following theorem which is easy to verify.

Theorem 3.1. [1] The lattice of all fuzzy subpseudovarieties of V denoted by **FPV(** τ **)** $:= (FPV(\tau);\wedge,\vee)$ forms a complete lattice which the least and the greatest elements, say **0, 1,** respectively where $\mathbf{0}(\mathrm{A}) = 0$**,** $\mathbf{1}(\mathrm{A}) = 1$ for all $\mathrm{A} \in V$.

4. Fuzzy *M*-solid subpseudovarieties

In this section, we assume that V is an *M*-solid pseudovariety of type τ and then we give the notion of fuzzy *M*-solid subpseudovarieties. First of all we start with the concept of hypersubstitutions.

A mapping $\sigma : \{f_i \mid i \in I\} \to W_\tau(X)$ which maps each n_i*-ary* operation symbol to an n_i*-ary* term is called a *hypersubstitution* of type τ **.** Each hypersubstitution σ induces a map $\hat{\sigma}$ on the set of all terms which is defined by

(1) $\hat{\sigma}(x) := x$ if $x \in X$ is a variable,

(2) $\hat{\sigma}[f_i(t_1,...,t_{n_i})] := \sigma(f_i)(\hat{\sigma}[t_1],...,\hat{\sigma}[t_{n_i}])$ for composite terms $f_i(t_1,...,t_{n_i})$.

On the set $Hyp(\tau)$ of all hypersubstitutions of type τ, we define a binary operation $\circ_h : Hyp(\tau) \to Hyp(\tau)$ by $\sigma_1 \circ_h \sigma_2 = \hat{\sigma}_1 \circ \sigma_2$ where $\circ$ is the usual composition of functions. Then together with the identity element σ_{id} mapping each f_i to term $f_i(x_1,...,x_{n_i})$, we obtain a monoid $(Hyp(\tau);\circ_h,\sigma_{id})$. Let $\mathrm{A} = (A;(f_i^A)_{i\in I})$ be an algebra of type τ and $\sigma \in Hyp(\tau)$. The algebra $\mathrm{A} = (A;\sigma(f_i^A)_{i\in I})$ is called *derived algebra* determined by A and σ. For the class $K \subseteq Alg_f\ (\tau)$ and for submonoid $M \subseteq Hyp(\tau)$ we define

$$\chi_M^A[K] := \{\sigma(\mathrm{A}) \mid \sigma \in M, \mathrm{A} \in K\}.$$

We note that χ_M^A is a closure operator on $Alg_f(\tau)$. A pseudovarity V of type τ is called *M –solid* if $\chi_M^A[V] = V$. Now we give the notion of fuzzy *M*-solid subpseudovarity of a given *M*-solid pseudovariety.

Definition 4.1. Let M be a submonoid of $Hyp(\tau)$ and let V be an *M*-solid pseudovariety of type τ. A fuzzy subset λ of V is called a *fuzzy M-solid subpseudovarity* if

(1) λ is a fuzzy subpseudovarity of V,

(2) $\forall \sigma \in M \quad \forall \mathrm{A} \in V,\ \lambda(\sigma(\mathrm{A})) \geq \lambda(\mathrm{A})$.

We denote by $FMP(V)$ the set of all fuzzy *M*-solid subpseudovarities of V.

Proposition 4.1. Let V be an *M*-solid pseudovariety of type τ, $\lambda : V \to [0,1]$ be a fuzzy subset of V. Then λ is a fuzzy *M*-solid subpseudovarity of V iff for all $t \in [0,1]$ the level subset $\lambda_t \neq \phi$ is an *M*-solid subpseudovarity of V.

Proof: Assume that λ is a fuzzy *M*-solid subpseudovarity of V and let $t \in [0,1]$. Since V is a pseudovariety and by Proposition 3.1**,** we have λ_t is a subpseudovariety of V. Let $\sigma \in M$ and $\mathrm{A} \in \lambda_t$. By assumption, we have $\lambda(\sigma(\mathrm{A})) \geq \lambda(\mathrm{A}) \geq t$. Hence, $\sigma(\mathrm{A}) \in \lambda_t$. Altogether, λ_t is an *M*-solid subpseudovarity of V.

Conversely, assume that for every $t \in [0,1]$, $\lambda_t \neq \phi$ is an M –solid subpseudovarity of V. Since V is a pseudovariety and by Proposition 3.1**,** we have λ is a fuzzy subpseudovariety of V. Let $\sigma \in M$ and $\mathrm{A} \in \lambda_t$. We choose $t = \lambda(\mathrm{A})$ and by assumption, we have $\sigma(\mathrm{A}) \in \lambda_t$. So $\lambda(\sigma(\mathrm{A})) \geq t = \lambda(\mathrm{A})$. Therefore, λ is a fuzzy *M*-solid subpseudovarity of V.

The following proposition is a consequence of Proposition 4.1.

Proposition 4.2. Let *W* be a subclass of an *M*-solid pseudovariety V. Then the characteristic function λ_W is a fuzzy *M*-solid subpseudovarity of V iff W is an *M*-solid subpseudovarity of V.

For fuzzy subset μ of an *M*-solid pseudovarity of V, we define the fuzzy *M*-solid subpseudovarity of V generated by μ by

$$\langle \mu \rangle_F := \bigcap \{\lambda \in FMPV(\tau) \mid \mu \subseteq \lambda\}.$$

Let K be subclass of *M*-solid pseudovarity of V, the class **HSP$_\mathbf{f}$**$(\chi_M^A[K])$ is the *M*-solid subpseudovarity of V generated by K.

Lemma 4.1. Let V be a pseudovariety of type τ and λ be a fuzzy subset of V. Define a mapping $\nu : V \to [0,1]$ by for every $\mathrm{A} \in V$,

$$\nu(\mathrm{A}) := \sup\{t \in [0,1] \mid \mathrm{A} \in \mathbf{HSP_f}(\lambda_t)\}.$$

Then $\nu = \langle \lambda \rangle_F$.

Proof: Let $\mathrm{A} \in V$ and let $I_\mathrm{A} = \{t \in [0,1] \mid \mathrm{A} \in \mathbf{HSP_f}(\lambda_t)\}$. First, we prove that ν is a fuzzy subpseudovarity of V. It sufficient to prove that for all $t \in \mathrm{Im}(\nu)$, ν_t is an sub-pseudovarity of V. Let $t \in \mathrm{Im}(\nu)$ and $t_n = t - \dfrac{1}{n}$, $n \in \mathbb{N} \setminus \{0\}$. Let $\mathrm{A} \in \nu_t$. Then $\nu(\mathrm{A}) \ge t$, so $\nu(\mathrm{A}) \ge t_n$, for all $n \in \mathbb{N} \setminus \{0\}$. Hence, there exists $s \in I_\mathrm{A}$ such that $s \ge t_n$, since if for all $s \in I_\mathrm{A}$, $s \le t_n$, then $\nu(\mathrm{A}) = \sup I_\mathrm{A} \le t_n$. This give a contradiction. Thus, $\lambda_s \subseteq \lambda_{t_n}$ and so $\mathrm{A} \in \mathbf{HSP_f}(\lambda_s) \subseteq \mathbf{HSP_f}(\lambda_{t_n})$ for all $n \in \mathbb{N} \setminus \{0\}$. Therefore, $\mathrm{A} \in \bigcap_{n \in \mathrm{N}\setminus\{0\}} \mathbf{HSP_f}(\lambda_{t_n})$. Conversely, let $\mathrm{A} \in \bigcap_{n \in \mathrm{N}\setminus\{0\}} \mathbf{HSP_f}(\lambda_{t_n})$, then $t_n \in I_\mathrm{A}$ for all $n \in \mathbb{N} \setminus \{0\}$. Then $t_n = t - \dfrac{1}{n} \le \sup I_\mathrm{A} = \nu(\mathrm{A})$ for all $n \in \mathbb{N} \setminus \{0\}$ Hence, $\nu(\mathrm{A}) \ge t$, i.e., $\mathrm{A} \in \nu_t$. Then $\nu_t = \bigcap_{n \in \mathrm{N}\setminus\{0\}} \mathbf{HSP_f}(\lambda_{t_n})$, which mean that ν_t is an subpseudovarity of V and by Proposition 3.1, we have ν is a fuzzy subpseudovariety of V. Next step is to show that $\lambda \subseteq \nu$. Let $\mathrm{A} \in V$. Since $\lambda(\mathrm{A}) \in I_\mathrm{A}$, we have $\nu(\mathrm{A}) \ge \lambda(\mathrm{A})$. This mean that $\lambda \subseteq \nu$. Finally, we want to prove that for any fuzzy subpseudovariety δ of V containing λ, we have $\nu \subseteq \delta$. it is clear that for all $t \in [0,1]$, we have that $\mathbf{HSP_f}(\lambda_t)$ is a subpseudovariety of δ_t, since for all $\mathrm{A} \in V$ and $\mathrm{A} \in \lambda_t$ implies that $t \le \lambda(\mathrm{A}) \ge \delta(\mathrm{A})$, i.e., $\mathrm{A} \in \delta_t$ and so $\mathbf{HSP_f}(\lambda_t)$ is a subpseudovariety of δ_t. Now, we prove that for every $\mathrm{A} \in V$, $\nu(\mathrm{A}) \le \delta(\mathrm{A})$. Let $\mathrm{A} \in V$ and $t \in I_\mathrm{A}$. Then $\mathrm{A} \in \mathbf{HSP_f}(\lambda_t) \subseteq \delta_t$ implies $\delta(\mathrm{A}) \ge t$. Therefore, $\nu(\mathrm{A}) = \sup I_\mathrm{A} \le \delta(\mathrm{A})$, i.e., $\nu \subseteq \delta$.

Lemma 4.2. Let V be an *M*-solid pseudovariety of type τ and μ be a fuzzy subset of V. Define a mapping $\lambda : V \to [0,1]$ by for every $\mathrm{A} \in V$,

$$\lambda(\mathrm{A}) := \sup\{t \in [0,1] \mid \mathrm{A} \in \mathbf{HSP_f}(\chi_M^A[\mu_t])\}.$$

Then $\lambda = \langle \mu \rangle_F$.

Proof: Let $A \in V$ and let $I_A = \{t \in [0,1] \mid A \in \mathbf{HSP_f}(\chi_M^A[\mu_t])\}$. First, we prove that λ is a fuzzy *M*-solid subpseudovarity of V. It sufficient to prove that for all $t \in \mathrm{Im}(\lambda)$, λ_t is an *M*-solid subpseudovarity of V. Let $t \in \mathrm{Im}(\lambda)$ and $t_n = t - \frac{1}{n}$, $n \in \mathbb{N} \setminus \{0\}$. Let $A \in \lambda_t$. Then $\lambda(A) \geq t$, so $\lambda(A) \geq t_n$, for all $n \in \mathbb{N} \setminus \{0\}$. Hence, there exists $s \in I_A$ such that $s \geq t_n$, since if for all $s \in I_A$, $s \leq t_n$, then $\lambda(A) = \sup I_A \leq t_n$. This give a contradiction. Thus, $\mu_s \subseteq \mu_{t_n}$ and so $A \in \mathbf{HSP_f}(\chi_M^A[\mu_s]) \subseteq \mathbf{HSP_f}(\chi_M^A[\mu_{t_n}])$ for all $n \in \mathbb{N} \setminus \{0\}$. since χ_M^A is a closure operator. Therefore, $A \in \bigcap_{n \in \mathrm{N} \setminus \{0\}} \mathbf{HSP_f}(\chi_M^A[\mu_{t_n}])$. Conversely, let $A \in \bigcap_{n \in \mathrm{N} \setminus \{0\}} \mathbf{HSP_f}(\chi_M^A[\mu_{t_n}])$, then $t_n \in I_A$ for all $n \in \mathrm{N} \setminus \{0\}$. Then $t_n = t - \frac{1}{n} \leq \sup I_A = \lambda(A)$ for all $n \in \mathbb{N} \setminus \{0\}$. Hence, $\lambda(A) \geq t$, i.e., $A \in \lambda_t$. Then $\lambda_t = \bigcap_{n \in \mathrm{N} \setminus \{0\}} \mathbf{HSP_f}(\chi_M^A[\mu_{t_n}])$, which mean that λ_t is an *M*-solid subpseudovarity of V and by Proposition 4.1**,** we have λ is a fuzzy *M*-solid subpseudovariety of V. Next step is to show that $\mu \subseteq \lambda$. Let $A \in V$. Since $\mu(A) \in I_A$, we have $\lambda(A) \geq \mu(A)$. This mean that $\mu \subseteq \lambda$. Finally, we want to prove that for any fuzzy *M*-solid subpseudovariety α of V containing μ, we have $\lambda \subseteq \alpha$. it is clear that for all $t \in [0,1]$, we have $\mathbf{HSP_f}(\chi_M^A[\mu_t])$ is an *M*-solid subpseudovariety of α_t, since for al $A \in V$ and $A \in \mu_t$ implies that $t \leq \mu(A) \geq \alpha(A)$, i.e., $A \in \alpha_t$ and so $\mathbf{HSP_f}(\chi_M^A[\mu_t])$ is an *M*-solid subpseudovariety of α_t. Now, we prove that for every $A \in V$, $\lambda(A) \leq \alpha(A)$. Let $A \in V$ and $t \in I_A$. Then $A \in \mathbf{HSP_f}(\chi_M^A[\mu_t]) \subseteq \alpha_t$ implies that $\alpha(A) \geq t$. Therefore, $\lambda(A) = \sup I_A \leq \alpha(A)$, i.e., $\lambda \subseteq \alpha$.

By Theorem 3.1, we obtain the following lemma.

Lemma 4.3. The lattice $\mathbf{FMP}(V) := (FMP(V); \wedge, \vee)$ of all fuzzy M-solid subpseudovarieties of V is a complete lattice.

As the same result in [1], we obtain that the lattice of all M-solid subpseudovarieties of V can be embedded into the lattice **FMP**(V).

Lemma 4.4. Let M be a submonoid of $Hyp(\tau)$, V be an *M*-solid pseudovariety of type τ and $\{\lambda_j \mid j \in J\} \subseteq FMP(V)$. Then $(\bigcup_{j \in J} \lambda_j)_t$ is closed under taking the operator χ_M^A, i.e., $\chi_M^A[(\bigcup_{j\in J}\lambda_j)_t] = (\bigcup_{j\in J}\lambda_j)_t$.

Proof: Let $\{\lambda_j \mid j\in J\}\subseteq FMP(V)$. Since χ_M^A is a closure operator on $Alg_f(\tau)$. Then $\chi_M^A[(\bigcup_{j\in J}\lambda_j)_t] \supseteq (\bigcup_{j\in J}\lambda_j)_t$. For another inclusion, let $\mathrm{A}\in(\bigcup_{j\in J}\lambda_j)_t$ and $\sigma\in M$. We have $(\bigcup_{j\in J}\lambda_j)(\mathrm{A}) = \sup\{\lambda_j(\mathrm{A}) \mid j\in J\}\geq t$. Since $\lambda_j(\sigma(\mathrm{A}))\geq\lambda_j(\mathrm{A})$, for all $j\in J$, we have $\sup\{\lambda_j(\sigma(\mathrm{A})) \mid j\in J\}\geq\sup\{\lambda_j(\mathrm{A}) \mid j\in J\}\geq t$. Then $(\bigcup_{j\in J}\lambda_j)(\sigma(\mathrm{A}))\geq t$ for all $\sigma\in M$, i.e., $\chi_M^A[(\bigcup_{j\in J}\lambda_j)_t]\subseteq(\bigcup_{j\in J}\lambda_j)_t$. Therefore, $\chi_M^A[(\bigcup_{j\in J}\lambda_j)_t] = (\bigcup_{j\in J}\lambda_j)_t$.

Theorem 4.1. The lattice **FMP**(V) $:= (FMP(V);\wedge,\vee)$ of all fuzzy M-solid subpseudovarieties of V is a complete sublattice of **FPV**$(\tau) := (FPV(\tau);\wedge,\vee)$ of all fuzzy subpseudovarieties of V.

Proof: Obviously, $FMP(V)\subseteq FPV(\tau)$. Let $\{\lambda_j \mid j\in J\}\subseteq FMP(V)$. It is clear that if $\underset{j\in J}{\wedge}\lambda_j\in FPV(\tau)$, then $\underset{j\in J}{\wedge}\lambda_j\in FMP(V)$. Now, we want to show if $\underset{j\in J}{\vee}\lambda_j\in FPV(\tau)$, then $\underset{j\in J}{\vee}\lambda_j\in FMP(V)$. Let $\mathrm{A}\in V$. By Lemma 4.1 and Lemma 4.4, we obtain

$$
\begin{aligned}
(\underset{j\in J}{\vee}\lambda_j)(\mathrm{A}) &= \sup\{t\in[0,1] \mid \mathrm{A}\in \mathbf{HSP_f}(\bigcup_{j\in J}\lambda_j)_t\} \\
&= \sup\{t\in[0,1] \mid \mathrm{A}\in \mathbf{HSP_f}\ \chi_M^{\mathrm{A}}(\bigcup_{j\in J}\lambda_j)_t\} \text{ for all } \mathrm{A}\in V.
\end{aligned}
$$

By Lemma 4.2, we have $\underset{j\in J}{\vee}\lambda_j\in FMP(V)$. This show that the lattice **FMP**(V) is a complete sublattice of **FPV**(τ).

Let V be an M-solid pseudovariety of type τ. It is natural to ask for the relationship between the complete lattices **FM**$_1$**P**(V) $:= (FM_1P(V);\wedge,\vee)$ and **FM**$_2$**P**(V) $:= (FM_2P(V);\wedge,\vee)$ when M_1 and M_2 are both submonoids of $Hyp(\tau)$, and M_1 is a submonoid of M_2.

Theorem 4.2. For any two submonoids M_1 and M_2 of the monoids $Hyp(\tau)$ with $M_1\subseteq M_2$ and M_1, M_2 solid pseudovariety V, the lattice **FM**$_2$**P**(V) is a complete sublattice of the lattice **FM**$_1$**P**(V).

Proof: First, we show that $FM_2P(V)\subseteq FM_1P(V)$. Let $\lambda\in FM_2P(V)$, $\sigma\in M_2$ and $\mathrm{A}\in V$. We have $\lambda(\sigma(\mathrm{A}))\geq\lambda(\mathrm{A})$. Since $M_1\subseteq M_2$, we have $\lambda(\sigma(\mathrm{A}))\geq\lambda(\mathrm{A})$ for all

$\sigma \in M_1$. Hence, $\lambda \in FM_1P(V)$. Let $\{\lambda_j \mid j \in J\} \subseteq FM_2P(V)$. It is clear that if $\underset{j\in J}{\wedge} \lambda_j \in FM_1P(V)$, then $\underset{j\in J}{\wedge} \lambda_j \in FM_2P(V)$. Next, we want to show that if $\underset{j\in J}{\vee} \lambda_j \in FM_1P(V)$, then $\underset{j\in J}{\vee} \lambda_j \in FM_2P(V)$. By Lemma 4.4 and $FM_2P(V) \subseteq FM_1P(V)$, we have $\chi^A_{M_2}[(\bigcup_{j\in J}\lambda_j)_t] = (\bigcup_{j\in J}\lambda_j)_t. = \chi^A_{M_1}[(\bigcup_{j\in J}\lambda_j)_t]$. Let $\mathrm{A} \in V$. Assume that $\underset{j\in J}{\vee} \lambda_j \in FM_1P(V)$. By Lemma 4.2, we have

$$(\underset{j\in J}{\vee} \lambda_j)(\mathrm{A}) = \sup\{t \in [0,1] \mid \mathrm{A} \in \mathbf{HSP_f}\chi^A_{M_1}[(\bigcup_{j\in J}\lambda_j)_t]\}$$

$$= \sup\{t \in [0,1] \mid \mathrm{A} \in \mathbf{HSP_f}\chi^A_{M_2}[(\bigcup_{j\in J}\lambda_j)_t]\}.$$

This mean that $\underset{j\in J}{\vee} \lambda_j \in FM_2P(V)$. Therefore, the lattice $\mathbf{FM_2P}(V)$ is a complete sublattice of the lattice $\mathbf{FM_1P}(V)$.

5. Conclusion

We have defined the notion of fuzzy *M*-solid subpseudovariety of a given *M*-solid pseudovariety and shown that the lattice of all fuzzy *M*-solid subpseudovarieties of a given *M*-solid pseudovariety is a complete sublattice of the lattice of all fuzzy subpseudo-varieties of the *M*-solid pseudovariety. Moreover, for submonoids M_1, M_2 of hypersub-stitutions with $M_1 \subseteq M_2$, we show that the lattice of all fuzzy M_2-solid subpseudovarie-ties is a complete sublattice of the lattice of all fuzzy M_1-solid subpseudovarieties. It is known that every *M*-solid pseudova-rieties can be defined by a set of *M*-hyperidentities. But the connection between the lattice of all fuzzy *M*-solid subpseudovarieties of a given *M*-solid pseudovariety and the lattice of all fuzzy *M*-hyperidentities of the pseudovariety is still open.

Acknowledgements. The authors would like to thank the referees for valuable suggestions on the paper and thank the Faculty of Science and Technology, Rajabhat Mahasarakham University, Mahasarakham, Thailand for financial support .

REFERENCES

1. A.Chada and B.Pibaljommee, Fuzzy identity filters and fuzzy subpseudovarieties, *Far East journal of Mathematical Science*, 73(2) (2013) 347-367.
2. K.Denecke and B.Pibaljommee, Lattice of *M*-solid generalized varieties and *M*-solid pseudovarieties, *Thai Journal of Mathematics*,1 (2003) 1-19.
3. K.Denecke and B.Pibaljommee, *M*-solid pseudovarieties and galois-connections, *Advances in Algebra*, 40(3) (2010) 124-129.
4. S.Eilenberg and M.P.Schützenberger, On pseudovarieties, *Advance in Mathematics*, 19 (1976), 413-418.
5. E.Graczyn'ska, Birkhoff' theorem for *M*-solid varieties, *Algebra Universalis*, 40 (1989) 109-117.

6. V.Murali, Lattice of fuzzy subalgebras and closure systems in T^X, *Fuzzy set and systems*, 41 (1991) 101-111.
7. S. Patchakhieo and B. Pibaljommee, On fuzzy fully invariant congruence relations, *Annals of Fuzzy Mathematics and informatics*, 3 (2011) 187-201.
8. B.Pibaljommee, Fuzzy subvarieties in universal algebra, *Advances Fuzzy sets and systems*, 9(1) (2011) 1-16.
9. B.Pibaljommee, Fuzzy *M*-solid subvarieties, *International Journal of Algebra*, 5(1) (2011) 1195-1205.
10. T.Priya and T.Ramachandran, A note on anti Q-fuzzy R-closed PS-ideals in PS-algebras, *Annals of Pure and Applied Mathematics*, 6(2) (2014) 250-259.
11. A.Rosenfeld, Fuzzy groups, *Journal of Mathematical Analysis and Applications*, 35 (1971) 512-517.
12. P.K.Sharma, Modal operator $F_{\alpha,\beta}$ in Intuitionistic fuzzy groups, *Annals of Pure and Applied Mathematics*, 7(1) (2014) 19-28.
13. P.M.Sithar Selvam and K.T.Nagalakshmi, Fuzzy PMS ideals in PMS algebras, *Annals of Pure and Applied Mathematics*, 12(2) (2016) 153-159.
14. L.A.Zadeh, Fuzzy sets, *Information and Control*, 8 (1965) 338-353.