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Solutions of the Diophantine Equation $p^x + (p+6)^y = z^2$ when p, $(p + 6)$ are Primes and $x + y = 2, 3, 4$

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Abstract. In this paper we consider the Diophantine equation $p^x + (p+6)^y = z^2$ when p, $(p + 6)$ are primes, and x, y, z are positive integers. All the six possibilities of $x + y = 2, 3, 4$ are examined. We establish that: (i) For the first 10000 primes p and $x = y = 1$, the equation has exactly seven solutions. (ii) When $x = 2$ and $y = 1$, the equation has exactly one solution. (iii) For the other four possibilities, the equation has no solutions. All the solutions are exhibited.

Keywords: Diophantine equations, Sexy primes

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1. Introduction

A prime gap is the difference between two consecutive primes. Numerous articles have been written on prime gaps, a very minute fraction of which is brought [5, 6] here. In 1849, A.de Polignac conjectured that for every positive integer k, there are infinitely many primes p such that $p + 2k$ is prime too. Many questions and conjectures on the above still remain unanswered and unsolved.

When $k = 1$, the pairs $(p, p + 2)$ are known as Twin primes. The first four such pairs are: (3, 5), (5, 7), (11, 13), (17, 19). The Twin prime conjecture stating that there are infinitely many such pairs remains unproved. When $k = 2$, the pairs $(p, p + 4)$ are called Cousin primes. The first four pairs are: (3, 7), (7, 11), (13, 17), (19, 23). The author [2] showed that the equation $p^x + (p+4)^y = z^2$ when $x + y = 2, 3, 4$ has the unique solution $3^2 + 7^1 = 4^2$. Moreover, the author [1] established for Cousin primes $p > 3$ and $p + 4$, that $p^x + (p+4)^y = z^2$ is insolvable in positive integers x, y, z.

In this paper, we concern ourselves with the case $k = 3$, i.e., pairs of primes of the form $(p, p + 6)$. These pairs are named in the literature as "Sexy primes" since "*sex*" in *Latin* means "six". The first four such pairs are: (5, 11), (7, 13), (11, 17), (13, 19). As of today, it is not known weather or not there exist infinitely many Sexy pairs.

We investigate the equation

$$p^x + (p+6)^y = z^2, \qquad (1)$$

when p, $p + 6$ are Sexy primes, and x, y, z are positive integers. We examine all the possibilities of $x + y = 2, 3, 4$ for solutions of equation (1). This is done in Section 2.

2. Solutions of the equation $p^x + (p+6)^y = z^2$ when $x + y = 2, 3, 4$
In this section we prove the following result.

Theorem 2.1. Suppose that p, $(p + 6)$ are any two primes, and x, y, z are positive integers. If $x + y = 2, 3, 4$, then the equation $p^x + (p + 6)^y = z^2$ has:
(i) For the first 10000 primes p, exactly seven solutions when $x = y = 1$.
(ii) Exactly one solution when $x = 2$ and $y = 1$.
(iii) No solutions for all other four possibilities.
Proof: For $x + y = 2, 3, 4$, we examine all possible values x, y. These are:
Case 1. $x + y = 2$ $x = 1$, $y = 1$.
Case 2. $x + y = 3$ $x = 1$, $y = 2$.
Case 3. $x + y = 3$ $x = 2$, $y = 1$.
Case 4. $x + y = 4$ $x = 1$, $y = 3$.
Case 5. $x + y = 4$ $x = 2$, $y = 2$.
Case 6. $x + y = 4$ $x = 3$, $y = 1$.

Each case is self-contained, and considered separately.

Case 1. Suppose in equation (1) $x = 1$, $y = 1$. We obtain

$$p^1 + (p + 6)^1 = z^2, \qquad z = 2T. \tag{2}$$

Then from (2)

$$p + (p + 2) = 2(p + 1) = z^2 - 4 = (z - 2)(z + 2). \tag{3}$$

The value z is even, therefore 2 divides $z - 2$ and also $z + 2$. Since $2 \mid (z - 2)$ or $2 \mid 2(T - 1)$, denote $T - 1 = \alpha$, and $z + 2 = 2(T + 1) = 2(\alpha + 2)$. Thus, from (3) $2(p + 1) = (2\alpha)2(\alpha + 2)$ or

$$p + 1 = 2\alpha(\alpha + 2). \tag{4}$$

Evidently, when $p = 4N + 3$ then $p + 6 = 4N + 9 = 4(N + 2) + 1$ is of the form $4M + 1$, whereas when $p = 4N + 1$ then $p + 6 = 4N + 7 = 4(N + 1) + 3$ is of the form $4V + 3$. Both possibilities are investigated.

Suppose that $p = 4N + 3$. From (4) $4(N + 1) = 2\alpha(\alpha + 2)$ or $2(N + 1) = \alpha(\alpha + 2)$ implying that α is even. Denote $\alpha = 2R$, hence $N = 2R(R + 1) - 1$. Thus, if p and $p + 6$ are primes, then

$$p = 8R(R + 1) - 1, \qquad p + 6 = 8R(R + 1) + 5. \tag{5}$$

When $p + 6$ is prime in (5), it clearly follows that $R \neq 5A$ and $R \neq 5A + 4$. If $R = 5A + 1$ and $R = 5A + 3$, the value $8R(R + 1) - 1$ is a multiple of 5 and is not a prime p. Hence, $R \neq 5A + 1$ and $R \neq 5A + 3$. Therefore, when p and $p + 6$ are primes in (5), then $R = 5A + 2$.

If $R = 5A + 2$, the conditions for a solution of equation (2) are

$$\begin{cases} p = 8R(R + 1) - 1 = 200A(A + 1) + 47, \\ p + 6 = 8R(R + 1) + 5 = 200A(A + 1) + 53, \\ z = 2(2R + 1) = 10(2A + 1). \end{cases} \tag{6}$$

The twenty-three values $A = 0, 1, 2, \ldots, 22$ in (6) have been verified for all primes $p < 104729$ (the 10000th prime is $104729 = 4N + 1$). When $A = 0, 5, 12, 20$, four solutions of equation (2) have been established. For all other nineteen values A, p and $(p + 6)$ are not simultaneously primes. The four solutions are demonstrated as follows.

Solution 1. $47 + 53 = 10^2$.

Solution 2. $6047 + 6053 = 110^2$.

Solution 3. $31247 + 31253 = 250^2$.

Solution 4. $84047 + 84053 = 410^2$.

This concludes the case of equation (2) when $p = 4N + 3$ and $p < 104729$.

Suppose that $p = 4N + 1$. Then from (4) we have $4N + 2 = 2(2N + 1) = 2\alpha(\alpha + 2)$ or $2N + 1 = \alpha(\alpha + 2)$ implying that α is odd. Denote $\alpha = 2Q + 1$, hence $N = 2Q(Q + 2) + 1$. Thus, if p and $p + 6$ are primes, then

$$p = 8Q(Q + 2) + 5, \qquad p + 6 = 8Q(Q + 2) + 11. \tag{7}$$

If $Q = 0$ ($\alpha = N = 1$), then $p = 5$ and $p + 6 = 11$ are primes. Hence

Solution 5. $5 + 11 = 4^2$.

Suppose that $Q > 0$. When $Q = 5B$ and $Q = 5B + 3$, the value $8Q(Q + 2) + 5$ is a multiple of 5 and is not a prime p. Hence $Q \neq 5B$ and $Q \neq 5B + 3$. Furthermore, when $Q = 5B + 1$ and $Q = 5B + 2$, the value $8Q(Q + 2) + 11$ is a multiple of 5 and is not a prime $(p + 6)$. Thus, $Q \neq 5B + 1$ and $Q \neq 5B + 2$. Therefore, if p and $p + 6$ are primes in (7), then $Q = 5B + 4$.

When $Q = 5B + 4$, the conditions for a solution of equation (2) are

$$\begin{cases} p = 8Q(Q + 2) + 5 = 200B(B + 2) + 197, \\ p + 6 = 8Q(Q+ 2) + 11 = 200B(B + 2) + 203, \\ z = 4(Q + 1) = 20(B + 1). \end{cases} \tag{8}$$

The twenty-two values $B = 0, 1, 2, \ldots, 21$ in (8) have been verified for the first 10000 primes p. When $B = 10, 15$, two additional solutions of equation (2) are established. For all other twenty values B, p and $(p + 6)$ are not simultaneously primes. The two solutions are:

Solution 6. $24197 + 24203 = 220^2$.

Solution 7. $51197 + 51203 = 320^2$.

This completes the case of equation (2) when $p = 4N + 1$ and $p \leq 104729$.

We now sum up **Case 1** with the following two Remarks.

Remark 2.1. For $p = 4N + 3/p = 4N + 1$ a prime, and $p + 6$ a prime, it is easily verified that equation (2) has a solution if respectively $2N + 3/2N + 2$ is a square. The condition for a solution of equation (2) is therefore established.

Remark 2.2. In sets (6) and (8), whenever p and $p + 6$ are primes, then the sum $p + (p + 6)$ yields a solution of equation (2).
If in sets (6) and (8), the total number of Sexy pairs is finite, then the number of solutions of equation (2) is also finite. If one of sets (6), (8) has infinitely many Sexy pairs, then equation (2) has infinitely many solutions.

Case 2. Suppose in equation (1) $x = 1$, $y = 2$. We have

$$p^1 + (p + 6)^2 = z^2. \quad (9)$$

From (9) we have

$$p = z^2 - (p + 6)^2 = (z - (p + 6))(z + (p + 6)). \quad (10)$$

Thus, p divides at least one of the values $(z - (p + 6))$, $(z + (p + 6))$.

If $p \mid (z - (p + 6))$, then $p\alpha = z - (p + 6)$ or $z = p\alpha + (p + 6)$ implying that $z + (p + 6) = p\alpha + 2(p + 6)$. Substituting these values in (10) yields
$p = (p\alpha)(p\alpha + 2(p + 6))$ or $1 = \alpha(p\alpha + 2(p + 6))$ which is impossible.
If $p \mid (z + (p + 6))$, then $p\beta = z + (p + 6)$ and $z - (p + 6) = p\beta - 2(p + 6)$. These two values yield in (10)

$$p = (p\beta - 2(p + 6))(p\beta) \quad \text{or} \quad 1 = (p\beta - 2(p + 6))\beta$$

which is impossible. Equation (9) has no solutions for all primes p.
Case 2 does not contribute solutions to equation (1).

Case 3. Suppose in equation (1) $x = 2$, $y = 1$. We obtain

$$p^2 + (p + 6)^1 = z^2. \quad (11)$$

From (11) $p + 6 = z^2 - p^2 = (z – p)(z + p)$. Since $(p + 6)$ is prime, it follows that

$$z – p = 1 \quad \text{and} \quad z + p = p + 6.$$

Then $z + p = p + 6$ implies that $z = 6$. Hence $p = 5$, and $p + 6 = 11$.
Thus, **Case 3** yields a unique solution of equation (1), namely:

Solution 8. $5^2 + 11 = 6^2$.

Case 4. Suppose in equation (1) $x = 1$, $y = 3$. We have

$$p^1 + (p + 6)^3 = z^2. \quad (12)$$

If $p = 4N + 3$, then from (12)

$$(4N + 3) + (4N + 9)^3 = z^2 \qquad z = 2T$$

or

$$64N^3 + 432N^2 + 976N + 732 = 4T^2.$$

Thus

$$16N^3 + 108N^2 + 244N + 183 = T^2. \quad (13)$$

The left-hand side of (13) is odd, therefore T^2 is odd. Denote $T = 2A + 1$ and $T^2 = 4A(A + 1) + 1$. Then from (13) we obtain

$$2(8N^3 + 54N^2 + 122N + 91) = 4A(A + 1). \quad (14)$$

The two sides of (14) now contradict each other, and hence (14) is impossible.
Thus, when $p = 4N + 3$, equation (12) has no solutions.

If $p = 4N + 1$, then from (12)

$$(4N + 1) + (4N + 7)^3 = z^2 \qquad z = 2T$$

or

$$64N^3 + 336N^2 + 592N + 344 = 4T^2.$$

Hence

$$16N^3 + 84N^2 + 148N + 86 = T^2. \quad (15)$$

The left-hand side of (15) is even, therefore T^2 is even. Denote $T = 2B$ and $T^2 = 4B^2$. Then from (15) it follows that

$$2(8N^3 + 42N^2 + 74N + 43) = 4B^2. \quad (16)$$

The two sides of (16) now contradict each other, implying that (16) is impossible. Thus, when $p = 4N + 1$, equation (12) has no solutions.

Case 4 does not yield any solutions of equation (1).

Case 5. Suppose in equation (1) $x = 2$, $y = 2$. We obtain

$$p^2 + (p + 6)^2 = z^2. \quad (17)$$

From (17) it follows that

$$(p + 6)^2 = z^2 - p^2 = (z - p)(z + p). \quad (18)$$

Since $(p + 6)$ is prime, the only possibility that (18) exists is when

$$z - p = 1, \qquad z + p = (p + 6)^2.$$

Then $z = p + 1$, and hence $z + p = 2p + 1 = (p + 6)^2$ which is impossible. Equation (17) has no solutions for all primes p.

Case 5 does not contribute solutions to equation (1).

Case 6. Suppose in equation (1) $x = 3$, $y = 1$. We obtain

$$p^3 + (p + 6)^1 = z^2, \qquad z = 2T. \quad (19)$$

If $p = 4N + 3$, then from (19)

$$(4N + 3)^3 + (4N + 9) = z^2$$

or

$$64N^3 + 144N^2 + 112N + 36 = 4T^2.$$

Thus

$$16N^3 + 36N^2 + 28N + 9 = T^2. \quad (20)$$

In (20), the left-hand side is odd. Hence, T is odd, and denote $T = 2\alpha + 1$. Then (20) yields

$$16N^3 + 36N^2 + 28N + 9 = 4\alpha^2 + 4\alpha + 1$$

and after simplifications

$$4N^3 + 9N^2 + 7N + 2 = \alpha(\alpha + 1).$$

The even term $\alpha(\alpha + 1)$ is the product of two consecutive integers, and it is seen that the equality does not hold.

Therefore, when $p = 4N + 3$ equation (19) has no solutions.

If $p = 4N + 1$, then from (19)

$$(4N + 1)^3 + (4N + 7) = z^2$$

or

$$64N^3 + 48N^2 + 16N + 8 = 4T^2.$$

Hence, after simplification

$$2(8N^3 + 6N^2 + 2N + 1) = T^2 \qquad (21)$$

implying that T is even, and T^2 is a multiple of 4. The two sides of (21) now contradict each other, and therefore (21) is impossible. When $p = 4N + 1$, equation (19) has no solutions.

Case 6 does not produce solutions to equation (1).

This completes the proof of Theorem 2.1. □

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