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The Diophantine Equation $p^x + (p+4)^y = z^2$ when $p>3$, $p+4$ are Primes is Insolvable in Positive Integers x, y, z

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Abstract. In this paper we consider the Diophantine equation $p^x + (p+4)^y = z^2$ when $p > 3$, $(p + 4)$ are primes, and x, y, z are positive integers. All the possibilities of x, y are examined, and it is established that the equation has no solutions for each and every prime $p > 3$. When $p = 3$, the solution obtained in [1] is also exhibited.

Keywords: Diophantine equations, Cousin primes

AMS Mathematics Subject Classification (2010): 11D61

1. Introduction

A prime gap is the difference between two consecutive primes. Numerous articles have been written on prime gaps, a very minute fraction of which is brought [4, 5] here. In 1849, A. de Polignac conjectured that for every positive integer k, there are infinitely many primes p such that $p + 2k$ is prime too. Many questions and conjectures on the above still remain unanswered and unsolved.

When $k = 1$, the pairs $(p, p + 2)$ are known as Twin Primes. The first four such pairs are: (3, 5), (5, 7), (11, 13), (17, 19). The Twin Prime conjecture stating that there are infinitely many such pairs remains unproved. When $k = 2$, the pairs $(p, p + 4)$ are called Cousin Primes. The first six pairs are: (3, 7), (7, 11), (13, 17), (19, 23), (37, 41), (43, 47).

In this paper, the known Diophantine equation $p^x + q^y = z^2$ [see 1, 3, 6, 7, 8] is considered when p and q are Cousin Primes i.e.,

$$p^x + (p+4)^y = z^2, \tag{1}$$

and x, y, z are positive integers. We examine all the possibilities of x, y for solutions of equation (1). This is done in Section 2.

2. On solutions of the equation $p^x + (p + 4)^y = z^2$ with $p, (p + 4)$ primes

In this section, we first show for all primes of the form $p = 4N + 1$, that the equation $p^x + (p + 4)^y = z^2$ has no solutions in positive integers x, y, z. This is done in Lemma 2.1 and Theorem 2.1.

Secondly, in Theorems 2.2, 2.3 and 2.4, we consider all primes of the form $p = 4N + 3$ $(N > 0)$. In Theorem 2.2 we show that the equation with even values

$x = 2T$ has no solutions. In Theorems 2.3 and 2.4, odd values $x = 2T + 1$ are considered. It is respectively shown for all values R when $y = 2R + 1$ is odd and when $y = 2R$ is even, that the equation has no solutions.

Each of the theorems is self-contained.

Lemma 2.1. Suppose that $x \geq 1$ and $y \geq 1$ are integers. If p is any prime of the form $p = 4N + 1$, then $4 \nmid (p^x + p^y)$.

Proof: If $p = 4N + 1$, then evidently for each value $x \geq 1$ as well as for each value $y \geq 1$ with $x = y$ inclusive, the values p^x and p^y are respectively of the form $4U + 1$ and $4V + 1$. Hence,

$$p^x + p^y = (4U + 1) + (4V + 1) = 2(2U + 2V + 1)$$

implying that $4 \nmid (p^x + p^y)$ as asserted. □

Theorem 2.1. Let p be any prime of the form $p = 4N + 1$. Then the equation $p^x + (p + 4)^y = z^2$ has no solutions in positive integers x, y and z.

Proof: The value z^2 is even, hence z is even. Thus, z^2 is a multiple of 4.

Consider the Binomial Theorem

$$(a + b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!} a^{n-2}b^2 + \cdots + \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!} a^{n-k}b^k + \cdots + b^n. \tag{2}$$

In (2), we substitute p for a, 4 for b and y for n to obtain the required term $(p + 4)^y$ in equation (1). It then follows from (2) that all terms beginning with $na^{n-1}b + \cdots + b^n$ are now equal to

$$yp^{y-1} \cdot 4 + \cdots + 4^y$$

each term of which is a multiple of at least 4. Since the value z^2 is a multiple of 4, hence from (1) and (2) we have that

$$4 \mid (p^x + p^y). \tag{3}$$

But $p = 4N + 1$, therefore by Lemma 2.1 (3) is impossible. Thus, when $p = 4N + 1$ equation (1) has no solutions in positive integers x, y and z. □

Theorem 2.2. Suppose that $p = 4N + 3$ $(N > 0)$ and $(p + 4)$ are any two primes. Let $T > 1$ be an integer. If $x = 2T$, then the equation $p^x + (p + 4)^y = z^2$ has no solutions in positive integers x, y and z.

Proof: The equation $p^{2T} + (p + 4)^y = z^2$ yields

$$(p + 4)^y = z^2 - p^{2T} = (z - p^T)(z + p^T). \tag{4}$$

Let α, β be non-negative integers. In (4) denote

$$z - p^T = (p + 4)^\alpha, \qquad z + p^T = (p + 4)^\beta, \qquad \alpha < \beta, \qquad \alpha + \beta = y. \tag{5}$$

From (5) we have

$$2p^T = (p + 4)^\beta - (p + 4)^\alpha = (p + 4)^\alpha((p + 4)^{\beta-\alpha} - 1). \tag{6}$$

Since $(p + 4)^\alpha$ divides the right-hand side of (6), but not the left-hand side, it follows that $(p + 4)^\alpha = 1$ and $\alpha = 0$. The value $\alpha = 0$ yields in (5) that $z = p^T + 1$ and $\beta = y$. Hence, from (6) we have

$$2p^T = (p + 4)^y - 1. \tag{7}$$

For any prime $p > 3$, one can easily verify that equation (7) does not hold when $T > 1$. Thus, the equation $p^x + (p + 4)^y = z^2$ with $x = 2T$ has no solutions.

This concludes the proof of Theorem 2.2. □

Remark 2.1. In (7), when $T = 1$ ($x = 2$), it follows that $y = 1$ implying that $p = 3$ and $z = 4$. Thus, $(p, x, y, z) = (3, 2, 1, 4)$ is a solution of $p^x + (p + 4)^y = z^2$**.** Equation (7) is valid for every prime $p \geq 3$, but invalid for all values $T > 1$. Hence, for $p = 3$ and any even value $x > 2$, no solutions of $p^x + (p + 4)^y = z^2$ exist. In this case, the above solution is therefore unique.

This solution has also been established [1], but in a different manner.

Theorem 2.3. Suppose that $p = 4N + 3$ ($N > 0$) and $(p + 4)$ are any two primes. Let T, R be non-negative integers. If $x = 2T + 1$ and $y = 2R + 1$, then the equation $p^x + (p + 4)^y = z^2$ has no solutions in positive integers x, y and z.

Proof: Consider the equation

$$p^{2T+1} + (p + 4)^{2R+1} = z^2$$

For each value T, the value p^{2T+1} is of the form $4A + 3$, whereas for every value R, the value $(p + 4)^{2R+1}$ has the form $4B + 3$. Thus, for all values A, B

$$p^{2T+1} + (p + 4)^{2R+1} = (4A + 3) + (4B + 3) = 4(A + B + 1) + 2 = z^2$$

is impossible since z^2 is a multiple of 4.

Hence, when $x = 2T + 1$ and $y = 2R + 1$, the equation $p^x + (p + 4)^y = z^2$ has no solutions in positive integers x, y and z. □

Theorem 2.4. Suppose that $p = 4N + 3$ ($N > 0$) and $(p + 4)$ are any two primes. Let T, R be non-negative integers. If $x = 2T + 1$ and $y = 2R$, then the equation $p^x + (p + 4)^y = z^2$ has no solutions in positive integers x, y and z.

Proof: The equation $p^{2T+1} + (p + 4)^{2R} = z^2$ yields

$$p^{2T+1} = z^2 - (p + 4)^{2R} = z^2 - ((p + 4)^R)^2 = (z - (p + 4)^R)(z + (p + 4)^R). \qquad (8)$$

Let α, β be non-negative integers. In (8) denote

$$z - (p + 4)^R = p^\alpha, \qquad z + (p + 4)^R = p^\beta, \qquad \alpha < \beta, \qquad \alpha + \beta = 2T + 1. \qquad (9)$$

From (9) we obtain

$$2(p + 4)^R = p^\beta - p^\alpha = p^\alpha(p^{\beta-\alpha} - 1). \qquad (10)$$

In (10) $p \nmid (p + 4)$ implying that $p^\alpha = 1$ and $\alpha = 0$. This yields in (9) that $\beta = 2T + 1$. Thus, from (10) we have $2(p + 4)^R = p^{2T+1} - 1$, and hence

$$2(p + 4)^R = p^{2T+1} - 1 = p^{2T+1} - 1^{2T+1} = (p - 1)(p^{2T} + p^{2T-1} + \cdots + p^1 + 1). \qquad (11)$$

The factor $(p - 1)$ divides the right-hand side of (11). Since $p > 3$, it follows that $(p - 1) \neq 2$. Furthermore, $(p - 1) \nmid (p + 4)$. Therefore, for all primes $p > 3$ (11) is impossible.

Thus, when $x = 2T + 1$ and $y = 2R$, the equation $p^x + (p + 4)^y = z^2$ has no solutions in positive integers x, y and z.

This completes the proof of Theorem 2.4. □

Remark 2.2. In Theorem 2.4, suppose the condition $N > 0$ is omitted. If $p = 3$

($N = 0$), then (11) results in $2 \cdot 7^R = 3^{2T+1} - 1 = 2(3^{2T} + 3^{2T-1} + \cdots + 3^1 + 1)$ or

$$7^R = 3^{2T} + 3^{2T-1} + \cdots + 3^1 + 1.$$

This equality is not pursued here, but rather examined for each value $R = 1, 2,\ldots,9$ where $7^9 < 10^8$. No value T satisfies the above equality, and we presume that values R and T do not exist.

If this is indeed true, then for all primes $p \geq 3$, the equation $p^x + (p + 4)^y = z^2$ has one and only one solution when $p = 3$. The solution mentioned earlier, namely: $(p, x, y, z) = (3, 2, 1, 4)$.

Final Remark. In [1 – Question 1], the author raised the question whether equation (1) has solutions when $x + y > 4$. He presumed that the answer is negative. In this paper, it has been shown for all primes $p > 3$ that the answer is indeed negative.

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