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A New Class of Exact Solutions to a Generalized form of Charap's Nonlinear Chiral Field Equations of Field Theory

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Abstract. Exact solutions to a generalized form of the Charap's nonlinear Chiral field equations have been presented for different values of the coupling constants. In most of the situations the solutions could not be obtained as the equations led to non-integrable expressions. In one situation the solutions were exact and complex. In the rest of the situations the solutions were well behaved.

Keywords: Nonlinear partial differential equations, exact solutions and Charap's Chiral field theory.

AMS Mathematics Subject Classification (2010): 37K40, 74H65, 34G20

1. Introduction

This paper describes a new class of solutions for a generalized form of the celebrated Chiral nonlinear equations of field theory due to Charap [1]. The generalized form was first proposed by Saha and Chanda [2]. Some of the recent studies which deal with nonlinear equations are due to Das [3], Kiwne, Waghmare and Avhale [4], Avhale and Kiwne [5], Roshid, Rahman and Akbar [6], Mitra and Chanda [7]. It has the interesting observations [1] that the equations admit infinite number of solutions and have similarities with the equations originating from other areas of study (e.g. equations of two-dimensional Heisenberg ferromagnets [8,9], Ginzberg Pitaevski equations in super fluids [10], stationary wave envelope in non-linear optics [11]). Moreover, in many occasions presented in the literature previously and some of these occasions presented here, the solutions are solitary in nature. The importance of such type of study is the opportunity that may be available to a theoretician in dealing with nonlinear equations.

2. The equations under study

The generalized form of Charap's equation that has been dealt with here in this paper reads like the following:

$$\Box'\phi = k''\eta^{\mu\nu}\frac{\partial\phi}{\partial x^{\mu}}\frac{\partial\beta}{\partial x^{\nu}} \tag{2.1a}$$

$$\Box'\psi = k''\eta^{\mu\nu}\frac{\partial\psi}{\partial x^{\mu}}\frac{\partial\beta}{\partial x^{\nu}} \tag{2.1b}$$

$$\Box'\chi = k''\eta^{\mu\nu}\frac{\partial\chi}{\partial x^{\mu}}\frac{\partial\beta}{\partial x^{\nu}} \tag{2.1c}$$

where

$\Box'\phi = \phi_{11} + \phi_{22} + \phi_{33} + \varepsilon\phi_{44}$ and

$\phi_1 = \frac{\partial\phi}{\partial x^1}, \phi_{11} = \frac{\partial^2\phi}{\partial x^{1^2}}$

where

$\eta^{\mu\nu} = 0 \; for \; \mu \neq \nu$
$\quad = 1 \; for \; \mu = \nu \neq 4$
$\quad = \epsilon \; for \; \mu = \nu = 4$

$\epsilon = +1 \; or -1$

k'' = arbitrary constant.

$\beta = \ln(f_\pi^2 + \phi^2 + \psi^2 + \chi^2)$

f_π = constant

Equations (2.1) can be written explicitly as

$$\phi_{11} + \phi_{22} + \phi_{33} + \varepsilon\phi_{44} = k''(\phi_1\beta_1 + \phi_2\beta_2 + \phi_3\beta_3 + \varepsilon\phi_4\beta_4) \tag{2.2a}$$

$$\psi_{11} + \psi_{22} + \psi_{33} + \varepsilon\psi_{44} = k''(\psi_1\beta_1 + \psi_2\beta_2 + \psi_3\beta_3 + \varepsilon\psi_4\beta_4) \tag{2.2b}$$

$$\chi_{11} + \chi_{22} + \chi_{33} + \varepsilon\chi_{44} = k''(\chi_1\beta_1 + \chi_2\beta_2 + \chi_3\beta_3 + \varepsilon\chi_4\beta_4) \tag{2.2c}$$

where $\beta = \ln(f_\pi^2 + \phi^2 + \psi^2 + \chi^2)$

f_π = constant

The equations (2.1) and (2.2) with $k'' = 1$ and $\epsilon = -1$ represent the celebrated Charap's equation. The equation was first written by Charap to describe a Chiral field [12].

3. The solutions

Solutions of (2.2) as obtained by the present authors previously [7] are given by

$$\phi = \int (f_\pi^2 + \phi^2 + \alpha^2)^{k''} dX, \; \phi = \phi(X), \; \alpha = \alpha(X) \tag{3.1a}$$

$$\alpha_{\phi\phi} = \frac{A^2}{\alpha^3} + \frac{B^2\alpha}{\left(f_\pi^2 + \phi^2 + \alpha^2\right)^{2k''}} \tag{3.1b}$$

$$\psi = \alpha\cos\theta \tag{3.1c}$$

$$\chi = \alpha\sin\theta \tag{3.1d}$$

where

$$\theta = A\int\left(\frac{\left(f_\pi^2 + \phi^2 + \alpha^2\right)^{k''}}{\alpha^2}\right)dX + BY + C \tag{3.1e}$$

A, B *and* C are constants of integration which are again functions of $(x^3 - x^4)$ and

$$X_{11} + X_{22} = 0 \tag{3.1f}$$

$$Y_{11} + Y_{22} = 0 \tag{3.1g}$$

$$X_1 = Y_2 \tag{3.1h}$$

$$X_2 = -Y_1 \tag{3.1i}$$

i.e. X *and* Y are mutually conjugate Laplace solutions.

The procedure for obtaining (3.1) is the same as that used from the equation (5) to the equation (17) of the work of Chanda, Ray and De [12].

In our previous communication [7] we had reported the particular case represented by $k'' = \frac{3}{2},\ A \neq 0, B \neq 0$ and the ansatz given by

$$f_\pi^2 + \phi^2 = \alpha^2 \tag{3.2}$$

The situation resulted in complex solutions for all of ϕ, ψ and χ. It was intimated there that a similar situation occurs [7] for another set of celebrated equations, namely Yang's R-gauge equations [13].

Other solutions that are available for different values of k'' (including $k'' = \frac{3}{2}$) in (3.1) and the same ansatz given by (3.2) has been explored. It may be noted that the exponent of the denominator of the second term of (3.1b) is $2k''$. For obvious convenience one can rewrite k'' as $\frac{n}{2}$ where n is an integer. With this one gets back the original Charap equation for $n = 2$. In the above it has been already mentioned that the solution for $k^{"}=\frac{3}{2}$, i.e. $n=3$, is complex. In the table given below (Table 1) the results for the particular values of $k^{"}$ in (2.1) are given.

Table 1: Nature of solutions to the equations (2.1) for different values of $k^{"}$ which have been presented in this paper

Values of k'' in (1.3)	Values of A and B in (1.3)	Nature of solutions obtained	Case No
$k''=\frac{1}{2}$	$A\neq 0, B\neq 0$	×	I(a)
$k''=-\frac{1}{2}$		×	I(b)
$k''=\frac{2}{2}=1$		√	I(c)
$k''=-\frac{2}{2}=-1$		×	I(d)
$k''=\frac{3}{2}$		√(complex)	I(e)
$k''=-\frac{3}{2}$		×	I(f)
$k''=\frac{1}{2}$	$A\neq 0, B=0$	√	II(a)
$k''=-\frac{1}{2}$		×	II(b)
$k''=\frac{2}{2}=1$		√	II(c)
$k''=-\frac{2}{2}=-1$		×	II(d)
$k''=\frac{3}{2}$		√	II(e)
$k''=-\frac{3}{2}$		×	II(f)

×: Explicit solutions could not be obtained
√: Explicit solutions could be obtained
√ (complex): Explicit solutions could be obtained and those are complex

It may be noted that Chanda and Ray [14] were the first to investigate the class of solutions to the equations (2.1) for $k^{"}=1$. There they had indicated that the situation B=0 might indicate important results. The details of the solutions indicated in Table 1 are given below.

Case I(a): $k^{"}=\frac{1}{2}$

With the ansatz (3.2) one gets from (3.1a)

$$\phi = \sqrt{2}\int \alpha dX \tag{3.3a}$$

and, from (3.1b) and (3.3a)

$$X = \int \frac{d\alpha}{\sqrt{D\alpha^2 + B^2\alpha^2 \ln\alpha - A^2}} \tag{3.3b}$$

which could not be integrated.

Case I(b): $k^{"} = -\dfrac{1}{2}$

With the ansatz (3.2) one gets from (3.1a)

$$\phi = \sqrt{2}\int \alpha dX \tag{3.4a}$$

and, from (3.1b) and (3.4a)

$$X = \int \frac{d\alpha}{\sqrt{B^2\alpha^6 + D\alpha^2 - A^2}} \tag{3.4b}$$

which could not be integrated.

Case I(c): $k^{"} = 1$

With the ansatz (3.2) one gets

$$\phi = f_\pi \tan(2 f_\pi X - G) \tag{3.5a}$$

$$\psi = f_\pi \sec\left(2 f_\pi - G\right) Cos\left(2AX + BY + C\right) \tag{3.5b}$$

$$\chi = f_\pi \sec\left(2 f_\pi - G\right) Sin\left(2AX + BY + C\right) \tag{3.5c}$$

where $A^2 + B^2 = 4 f_\pi^2$

This set of solutions can be indentified to be particular form of the solutions obtained by Chanda and Chakraborty [15] previously. The solutions obtained by Chanda and Chakraborty could represent the dependence on x^3 and x^4 other than $(x^3 - x^4)$. Here the dependence of x^3 and x^4 can be represented only in terms of $(x^3 - x^4)$ through A, B and C which are functions of $(x^3 - x^4)$.

Case I(d): $k^{"} = -1$

With the ansatz (3.2) one arrives at

$$X = \int \frac{2\sqrt{3}\alpha^3 dX}{\sqrt{4B^2\alpha^8 + 3D\alpha^2 - 3A^2}}$$

which could not be integrated. This becomes a hindrance towards getting explicit solutions of ϕ, ψ and χ in terms of X and Y.

Case I(e): $k^{"}=\frac{3}{2}$

With the ansatz (3.2) one gets complex solutions which are quite involved and are available in a previous work of the authors [7]

$$\phi=-\frac{B^2 i}{16A^3}\left[-\mathrm{cosec}\left\{\frac{2^{3/2}B(E-X)}{4}+\frac{\pi}{4}\right\}\cot\left\{\frac{2^{3/2}B(E-X)}{4}+\frac{\pi}{4}\right\}+\log\left|\tan\left\{\frac{\frac{2^{3/2}B(E-X)}{4}+\frac{\pi}{4}}{2}\right\}\right|\right] \tag{3.7a}$$

$$\psi=\frac{B}{4A}\mathrm{cosec}\left\{\frac{2^{3/2}B(E-X)}{4}+\frac{\pi}{4}\right\}[-\sinh\left\{\log\left|\tan\left\{\frac{\frac{2^{3/2}B(E-X)}{4}+\frac{\pi}{4}}{2}\right\}\right|\right\}\sin(BY+C)$$

$$+i\cosh\left\{\log\left|\tan\left\{\frac{\frac{2^{3/2}B(E-X)}{4}+\frac{\pi}{4}}{2}\right\}\right|\right\}\cos(BY+C)] \tag{3.7b}$$

$$\chi=\frac{B}{4A}\mathrm{cosec}\left\{\frac{2^{3/2}B(E-X)}{4}+\frac{\pi}{4}\right\}[\sinh\left\{\log\left|\tan\left\{\frac{\frac{2^{3/2}B(E-X)}{4}+\frac{\pi}{4}}{2}\right\}\right|\right\}\cos(BY+C)$$

$$-i\cosh\left\{\log\left|\tan\left\{\frac{\frac{2^{3/2}B(E-X)}{4}+\frac{\pi}{4}}{2}\right\}\right|\right\}\sin(BY+C)] \tag{3.7c}$$

Case I(f): $\mathrm{k}^{"}=-\frac{3}{2}$

With the ansatz (3.2) one gets from (3.1a)

$$\phi=2^{-3/2}\int\frac{1}{\alpha^3}dX \tag{3.8a}$$

and, from (3.1b) and (3.8a)

$$X = 2^{5/2}\int \frac{\alpha^4 d\alpha}{\sqrt{4D\alpha^2 + \sqrt{8}B\alpha^{10} - 4A^2}}$$

which cannot be integrated.

Case II(a): $k^{''} = \frac{1}{2}$

With the ansatz (3.2) one gets the explicit solutions given by

$$\phi = \frac{1}{2\sqrt{D}}\left[e^{\sqrt{2D}X} - \left(\frac{A}{\sqrt{D}}\right)^2 e^{-\sqrt{2D}X}\right] \qquad (3.9a)$$

$$\psi = \left\{\frac{e^{\sqrt{2D}X} + \left(\frac{A}{\sqrt{D}}\right)^2 e^{-\sqrt{2D}X}}{2}\right\} Cos\left[2Tan^{-1}\left(\frac{\sqrt{D}}{A}e^{\sqrt{2D}X}\right) + C\right] \qquad (3.9b)$$

$$\chi = \left\{\frac{e^{\sqrt{2D}X} + \left(\frac{A}{\sqrt{D}}\right)^2 e^{-\sqrt{2D}X}}{2}\right\} Sin\left[2Tan^{-1}\left(\frac{\sqrt{D}}{A}e^{\sqrt{2D}X}\right) + C\right] \qquad (3.9c)$$

$D =$ Arbitrary constant of integration which can be a function of $\left(x^3 - x^4\right)$

$f_\pi = A$

Case II(b): $k^{''} = -\frac{1}{2}$

With the ansatz (3.2) one gets from (3.1a)

$$\phi = \frac{1}{\sqrt{2}}\int \frac{dX}{\alpha} \qquad (3.10a)$$

And from (3.1b) and (3.10a)

$$X = \frac{\alpha}{2}\sqrt{\frac{2}{D}}\sqrt{\alpha^2 - \left(\frac{A}{\sqrt{D}}\right)^2} + \frac{1}{2}\left(\frac{A}{\sqrt{D}}\right)^2\sqrt{\frac{2}{D}}\log\left|\alpha + \sqrt{\alpha^2 - \left(\frac{A}{\sqrt{D}}\right)^2}\right| \qquad (3.10b)$$

where A and D are arbitrary constants which can be arbitrary functions of $\left(x^3 - x^4\right)$.

However, from (3.10b) α could not be written explicitly in terms of X .

Case II(c): $k^{''} = 1$

With ansatz (3.2) here one arrives at the solutions given by

$$\phi = A\tan\left[2AX - G\right] \tag{3.11a}$$

$$\psi = A\sec\left[2AX - G\right]Cos[2AX + C] \tag{3.11b}$$

$$\chi = A\sec\left[2AX - G\right]Sin[2AX + C] \tag{3.11c}$$

where A, G, C are arbitrary constants and can be arbitrary functions of $\left(x^3 - x^4\right)$.

Case II(d): $k^{"} = -1$

With the ansatz (3.2) here one arrives at

$$X = \frac{2}{D^2}\left[\frac{\left(D\alpha^2 - A^2\right)^{3/2}}{3} + A^2\left(D\alpha^2 - A^2\right)^{1/2}\right] \tag{3.12}$$

From this α could not be written explicitly in terms of X. This becomes a hindrance towards getting explicit solutions of ϕ, ψ and χ in terms of X and Y..

Case II(e): $k^{"} = \frac{3}{2}$

With the ansatz (3.2) one arrives at

$$\phi = \frac{\sqrt{8}A^3}{D}\frac{X}{\sqrt{D - 8A^4X^2}} \tag{3.13a}$$

$$\psi = \frac{A}{\sqrt{D - 8A^4X^2}}Cos\left\{Sin^{-1}\left(\frac{\sqrt{8}A^2X}{\sqrt{D}}\right) + C\right\} \tag{3.13b}$$

$$\chi = \frac{A}{\sqrt{D - 8A^4X^2}}Sin\left\{Sin^{-1}\left(\frac{\sqrt{8}A^2X}{\sqrt{D}}\right) + C\right\} \tag{3.13c}$$

with $8A^4 = D$

Case II(f): $k^{"} = -\frac{3}{2}$

With the ansatz (3.2) one arrives at

$$X = \frac{2^{3/2}A^4}{D^{5/2}}\int \sec^5\gamma d\gamma \tag{3.14}$$

where $\alpha = \frac{A}{\sqrt{D}}\sec\gamma$

The integration in (3.14) was not possible and this restricted the calculations for obtaining explicit solutions ϕ, ψ and χ.

4. Summary and conclusion

This paper has described a new class of exact solutions for different values of the coupling constants in a generalized form (Equations 2.1a to 2.1c) of the celebrated Chiral equations of field theory due to Charap [1]. The generalized form was first proposed by Saha and Chanda [2]. In most of the situations the solutions could not be arrived at as the equations led to non-integrable expressions. In one situation the solutions were exact and complex. In the rest of the situations the solutions are explicit. The importance of such type of study is the opportunity that may be available to a theoretician in dealing with coupled partial differential equations which relate to her/his area of interest.

REFERENCES

1. J.M.Charap, Complete solutions for a model with Chiral dynamics, *J. Phys A, Math & Gen*, 6(7) (1973) 987-998.
2. R.K.Saha, Thesis: On some aspects of nonlinearity and role of the parameters in some non-linear PDE's from physical origin, *unpublished* (2009).
3. P.K.Das, Nonlinear statistical model and its applications to diffusion of mobile telephony in India, *Annals of Pure and Applied Mathematics*, 5(1) (2013) 64-70.
4. S.B.Kiwne, R.V.Waghmare and P.S.Avhale, Mathematical modeling of pollutant uptake by the plant root from the soil, *Annals of Pure and Applied Mathematics*, 9(1) (2015) 1-7.
5. P.S.Avhale and S.B.Kiwne, Application of green function method for solution advection diffusion equation of nutrient concentration in groundwater, *Annals of Pure and Applied Mathematics*, 7(2) (2014) 1-5.
6. Harun-Or-Roshid, Nizhum Rahman and M. Ali Akbar, Traveling wave solutions of nonlinear Klein-Gordon equation by extended (G/G')-expansion method, *Annals of Pure and Applied Mathematics*, 3(1) (2013) 10-16.
7. I.Mitra and P.K.Chanda, On some complex exact solutions for Charap's equations in a generalized form, *Annals of Pure and Applied Mathematics*, 10(2) (2015) 199-206.
8. S.Trimper, Two-dimensional Heisenberg model and pseudoparticle solutions, *Phys Lett A*, 70(2) (1979) 114-116.
9. D.Ray, Solutions for two-dimensional Heisenberg ferromagnets, *Phys Lett A*, 76 (1980) 274-276.
10. V.L.Ginzburg and L.V.Pitaevski, Zh. Eksp. Theor. Fiz. 34, 1240 (1958), [*Sov.Phys. JETP* 7,858 (1959)].
11. P.K.Chanda and D.Ray, Stationary wave envelope in non-linear optics, *J. Math. Phys.,* 24 (1983) 1923-1926.
12. P.K.Chanda, D.Ray and U.K.De, Some exact solutions of non-linear chiral field equations, *J. Math. Phys.*, 25(8) (1984) 2557-2562.

13. S.Chakraborty and P.K.Chanda, On a revisit to the painleve test for integrability and exact solutions for Yang's self-dual equation for SU(2) gauge field, *Pramana- J. Phys*., 65(6) (2006) 971-983.
14. C.N.Yang, Condition of self duality for SU(2) gauge fields on Euclidean four dimensional space, *Phys. Rev. Lett.*, 38(24) (1977) 1377-1382.
15. S.Chakraborty and P.K.Chanda, Exact solutions of the field equations for Charap's Chiral invariant model of the pion dynamics, *Pramana-J. Phys*., 52(3) (1999) 245-256.