On the Spectrum and Energy of Concatenated Singular Graphs

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Received 5 November 2017; accepted 22 November 2017

Abstract. The roots of the characteristic polynomial of the adjacency matrix A(G) of a graph G are called eigenvalues. The eigenvalues together with their multiplicities constitute the spectrum of G. Graphs having zero as an eigenvalue are called singular graphs. Nullity η of G is the multiplicity of the eigenvalue zero. The null spread of the edge e is defined as η_e(G) = η(G) − η(G−e). Null spread of the edges of singular graphs depends on the null spread of its pendant vertices. The concatenation or edge gluing of two graphs G₁ and G₂ is the graph obtained by identifying two edges of G₁ and G₂. In this paper we study on the spectrum of the concatenation of two graphs. The effect of concatenation on energy is also a part of the investigation.

Keywords: Singular graph, spectrum, nullity, concatenation, energy of graph.

AMS Mathematics Subject Classification (2010): 05C50, 05C76

1. Introduction

Let G = (V(G), E(G)) be a finite, undirected simple graph of order n with vertex set V(G) and edge set E(G). The adjacency matrix A(G) of the graph G is a n x n matrix whose entries a_{ij} are the number of edges from vertex v_i to the vertex v_j. The characteristic polynomial of the adjacency matrix A(G) of the graph G is the characteristic polynomial of G and is denoted by ϕ(G, x). The roots of the equation ϕ(G, x) = 0 are called the eigenvalues of the graph G. The collection of the eigenvalues together with their multiplicities constitute the spectrum of G denoted by spec(G). Graphs having zero as an eigenvalue are called singular graphs. The nullity η(G) of the graph G is the multiplicity of zero in the graph’s spectrum.

Definition 1.1. [3] Let G − u be the induced sub graph of the graph G obtained on deleting the vertex u. The null spread of the vertex u is η_u(G) = η(G) − η(G − u).

Obviously the null spread satisfies −1 ≤ η_u(G) ≤ 1. If u is a core vertex, then η_u(G) = 1. There are vertices with η_u(G) = 0 and η_u(G) = -1. Such vertices are called noncore
vertices of null spread zero and noncore vertices of null spread \(-1\) respectively (See Figure 1).

![Figure 1: Three types of vertices](image)

**Definition 1.2.** [3] Let \(G - e\) be the induced subgraph of the graph \(G\) obtained on deleting an edge \(e\) from \(G\). The null spread of the edge \(e\) is defined as \(\eta_e(G) = \eta(G) - \eta(G - e)\).

If \(G\) is any nonempty graph, then for each \(e \in E(G)\), \(|\eta(G) - \eta(G - e)| \leq 2\). In Figure 2, the graph \(G\) has nullity two and \(G - e\) has nullity zero. Thus \(\eta_e(G) = 2\). Deletion of edges with positive null spread decreases the nullity of the graph.

**Definition 1.3.** [9] Let \(G_1\) and \(G_2\) be two graphs with disjoint vertex sets. If a vertex \(u \in G_1\) is identified with a vertex \(v \in G_2\), then the graph \(G_1 \circ G_2\) obtained of order \(|G_1| + |G_2| - 1\), is said to be the coalescence of \(G_1\) and \(G_2\) with respect to \(u\) and \(v\).

The characteristic polynomial \(\varphi(G, x)\) of the graph \(G = G_1 \circ G_2\) is given by the following theorem.
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**Theorem 1.1.** [9] The characteristic polynomial of the coalescence $G_1 \circ G_2$ of two rooted graphs $(G_1, u)$ and $(G_2, w)$ obtained by identifying the vertices $u$ and $w$ so that the vertex $v = u = w$ become a cut vertex of $G_1 \circ G_2$ is given by
\[
\phi(G_1 \circ G_2) = \phi(G_1) \phi(G_2 - w) + \phi(G_1 - u) \phi(G_2) - x \phi(G_1 - u) \phi(G_2 - w)(1.1)
\]
We have the following results about the coalescence of graphs:

**Theorem 1.2.** [6] The coalescence of two singular graphs with nullity $\eta_1$ and $\eta_2$ coalesced at a core vertex yield a singular graph of nullity $\eta_1 + \eta_2 - 1$.

**Theorem 1.3.** [7] Let $G_1$ be a nonsingular graph and $G_2$ be a singular graph with nullity $\eta_2$. If $G_1$ and $G_2$ are coalesced at a vertex $u$ of $G_1$ and a core vertex $v$ of $G_2$, then the nullity of $G_1 \circ G_2$ is $\eta_2 - 1$.

**Theorem 1.4.** [7] Let $G_1$ and $G_2$ be two singular graphs of order $n_1$ and $n_2$ respectively. If $G_1 \circ G_2$ is the coalescence of $G_1$ and $G_2$ at noncore vertices of null spread $-1$, then $\eta(G_1 \circ G_2) = \eta_1 + \eta_2 + 1$.

**Theorem 1.5.** [7] Let $G_1$ and $G_2$ be two singular graphs with nullity $\eta_1$ and $\eta_2$ respectively. The nullity of the coalescence of $G_1$ and $G_2$ at noncore vertices of null spread zero is $\eta_1 + \eta_2$.

**Theorem 1.6.** [7] Let $G_1$ and $G_2$ be two singular graphs with nullity $\eta_1$ and $\eta_2$ respectively. The coalescence of $G_1$ and $G_2$ at a core vertex of $G_1$ and at a noncore vertex of null spread 0 or -1 of $G_2$ or vice versa yield a singular graph of nullity $\eta_1 + \eta_2 - 1$.

**Theorem 1.7.** [7] Let $G_1$ and $G_2$ be two singular graphs with nullity $\eta_1$ and $\eta_2$ respectively. The coalescence of $G_1$ and $G_2$ at a noncore vertex of null spread zero of $G_1$ and at a noncore vertex of null spread $-1$ of $G_2$ or vice versa yield a singular graph of nullity $\eta_1 + \eta_2$.

**Theorem 1.8.** [7] Let $G_1$ be a non singular graph and $G_2$ be a singular graph with nullity $\eta_2$. Then the nullity of the coalescence of $G_1$ and $G_2$ with respect to any vertex of $G_1$ and a noncore vertex of zero null spread of $G_2$ is $\eta_2$.

**Theorem 1.9.** [7] Let $G_1$ be a nonsingular graph and $G_2$ be a singular graph of nullity $\eta_2$. Then the nullity of the coalescence of $G_1$ and $G_2$ with respect to any vertex $u$ of $G_1$ and a noncore vertex $w$ of $G_2$ of null spread $-1$ is
1. $\eta_2 + 1$, if $G_1 - u$ is singular.
2. $\eta_2$, if $G_1 - u$ is nonsingular.

**Theorem 1.10.** [8] Let $G_1$ and $G_2$ be two nonsingular graphs and $G$ be the coalescence of $G_1$ and $G_2$ with respect to a vertex $u$ of $G_1$ and $w$ of $G_2$. If $G_1 - u$ and $G_2 - w$ are singular, then $G$ is singular.

**Theorem 1.11.** [8] A singular graph with noncore vertices always satisfies the following conditions.
Theorem 1.12. [7] Let $G_1$ be a non-singular graph and $G_2$ be a singular graph with nullity $\eta_2$. Let $G$ be the coalescence of $G_1$ and $G_2$ with respect to any vertex $u \in G_1$ and a core vertex $w$ of $G_2$. Then in $G$ the coalesced vertex and its neighbours in $G_1$ will be non-core vertices of null spread zero or $-1$ according as $G_1 - u$ is non-singular or singular.

Theorem 1.13. [7] Let $G_1$ be a non-singular graph and $G_2$ be a singular graph with nullity $\eta_2$. Let $G$ be the coalescence of $G_1$ and $G_2$ with respect to any vertex $u \in G_1$ and a non-core vertex $w$ of $G_2$. If $G_1 - u$ is non-singular, then in $G$ the coalesced vertex and its neighbours in $G_1$ will be non-core vertices of null spread zero or $-1$ according as $w$ is of null spread zero or $-1$.

Theorem 1.14. [7] Let $G_1$ be a non-singular graph and $G_2$ be a singular graph with nullity $\eta_2$. Let $G$ be the coalescence of $G_1$ and $G_2$ with respect to any vertex $u \in G_1$ and a non-core vertex $w$ of $G_2$ of null spread $-1$. If $G_1 - u$ is singular, then in $G$ the coalesced vertex is a non-core vertex of null spread $-1$ and its neighbours in $G_1$ will be core vertices.

Theorem 1.15. [7] Let $G_1$ be a non-singular graph and $G_2$ be a singular graph with nullity $\eta_2$. Let $G$ be the coalescence of $G_1$ and $G_2$ with respect to any vertex $u \in G_1$ and a non-core vertex $w$ of $G_2$ of null spread zero. If $G_1 - u$ is singular, then in $G$ the coalesced vertex is a non-core vertex of null spread $-1$ and its neighbours corresponding to $G_1$ will be non-core vertices of null spread zero.

Theorem 1.16. [7] Let $G_1$ and $G_2$ be two singular graphs and $G$ be the coalescence of them with respect to any vertex $u$ of $G_1$ and $w$ of $G_2$.

\begin{enumerate}
\item If $G_1$ is a core graph and $u,w$ are core vertices, then in $G$ the coalesced vertex and its neighbours corresponding to $G_1$ are core vertices.
\item If $G_1$ is a core graph, $u$ is a core vertex and $w$ is a non-core vertex of null spread $-1$, then in $G$ the coalesced vertex is a non-core vertex of null spread $-1$ and its neighbours corresponding to $G_1$ are core vertices.
\item If $G_1$ is a core graph, $u$ is a core vertex and $w$ is a non-core vertex of null spread zero, then in $G$ the coalesced vertex is a non-core vertex of null spread zero and its neighbours corresponding to $G_1$ are core vertices.
\end{enumerate}

Theorem 1.17. [7] Let $G_1$ and $G_2$ be two non-singular graphs and $G$ be the coalescence of them with respect to any vertex $u$ of $G_1$ and $w$ of $G_2$. If $G_1 - u$ and $G_2 - w$ are singular, then in $G$ the coalesced vertex will be a non-core vertex of null spread $-1$. 

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Definition 1.4. [13] Let \((K, u)\) and \((H, w)\) are two rooted graphs. The graph obtained by joining \(u\) and \(w\) by an edge is denoted by \(KH + uw\) (See Figure 3).

Figure 3: Joining \((K, u)\) and \((H, w)\) by an edge uw.

The characteristic polynomial of \(KH + uw\) [11] is
\[
\phi(KH + uw) = \phi(K)\phi(H) - \phi(K - u)\phi(H - w) \tag{1.2}
\]

We have the following results:

Theorem 1.18. [13] Let the components of the graph obtained by deleting the edge uw from \(KH + uw\) be \((K, u)\) and \((H, w)\). If one of the following conditions is satisfied, then \(KH + uw\) is singular.
1. One component and its root-deleted subgraph are singular.
2. One component and the root-deleted subgraph of the other component are singular.

Theorem 1.19. [8] Let \((K, u)\) and \((H, w)\) be the components of the graph obtained by deleting an edge uw from \(KH + uw\).
1. Let \(K\) and \(H\) be singular graphs with nullity \(\eta_1\) and \(\eta_2\) respectively. If \(u\) and \(w\) are core vertices of \(K\) and \(H\) respectively, then nullity of \(KH + uw\) is \(\eta_1 + \eta_2 - 2\).
2. Let \(K\) and \(H\) be singular graphs with nullity \(\eta_1\) and \(\eta_2\) respectively. If \(u\) and \(w\) are noncore vertices (of null spread 0 or \(-1\)) of \(K\) and \(H\) respectively, then the nullity of \(KH + uw\) is \(\eta_1 + \eta_2\).
3. Let \(K\) and \(H\) be singular graphs with nullity \(\eta_1\) and \(\eta_2\) respectively. If \(u\) is a core vertex of \(K\) and \(w\) is a noncore vertex of null spread \(-1\) or vice versa, then the nullity of \(KH + uw\) is \(\eta_1 + \eta_2\).
4. Let \(K\) and \(H\) be singular graphs with nullity \(\eta_1\) and \(\eta_2\) respectively. If \(u\) is a core vertex of \(K\) and \(w\) is a noncore vertex of \(H\) of null spread 0 or vice versa, then the nullity of \(KH + uw\) is \(\eta_1 + \eta_2 - 1\).
5. Let \(K\) and \(H\) be singular graphs with nullity \(\eta_1\) and \(\eta_2\) respectively. If \(u\) is a noncore vertex of \(K\) of null spread 0 and \(w\) is a noncore vertex of \(H\) of null spread \(-1\) or vice versa, then the nullity of \(KH + uw\) is \(\eta_1 + \eta_2\).
6. Let \(K\) be singular with nullity \(\eta_1 \eta > 1\) and \(H\) be nonsingular. If \(u\) is a core vertex and \(H - w\) is nonsingular, then nullity of \(KH + uw\) is \(\eta - 1\).
7. Let \(K\) be singular with nullity \(\eta_1 \eta > 1\) and \(H\) be nonsingular. If \(u\) is a core vertex and \(H - w\) is singular, then nullity of \(KH + uw\) is \(\eta\).
8. Let \(K\) be singular with nullity \(\eta\) and \(H\) be nonsingular. If \(u\) is a noncore vertex (of null spread 0 or \(-1\)), then nullity of \(KH + uw\) is \(\eta\).

The spectrum of \(c_n\) and \(p_n\) are respectively given by
2cos (2πj/n), j = 0, ..., n − 1 and 2cos (nj

The following theorem gives a useful basic property of characteristic polynomial of graphs.

**Theorem 1.20.** [2] Let uv be an edge of G. Then
\[ \emptyset(G) = \emptyset(G - uv) - \emptyset(G - u - v) - 2 \sum_{C \subseteq \Phi(uv)} \emptyset(G - C) \]
where \( C(uv) \) is the set of cycles containing uv. In particular, if uv is a pendant edge with pendant vertex v, then \( \emptyset(G) = x\emptyset(G - v) - \emptyset(G - u - v) \).

Gutman in 1978 gave the following definition for energy of a graph

**Definition 1.5.** [20] If G is a graph on n vertices and \( \lambda_1, \lambda_2, \ldots, \lambda_n \) are its eigenvalues, then the energy of G is
\[ E = E(G) = \sum_{j=1}^{n} \lambda_j \]
A graph with energy, \( E(G) < n \), is said to be hypoenergetic and graph for which \( E(G) \geq n \) are called nonhypoenergetic. If \( E(G) < n - 1 \) and G is connected, G is called strongly hypoenergetic [20].

We have the following basic theorems about energy of graphs.

**Theorem 1.21.** [20] If the graph G is non-singular, then G is nonhypoenergetic.

**Theorem 1.22.** [20] Let G and H be two graphs with disjoint vertex sets and G o H be the coalescence of G and H at u \( \in \) H and v \( \in \) G. Then \( E(G \circ H) \leq E(G) + E(H) \). Equality is attained if and only if either u is an isolated vertex of G or v is an isolated vertex of H or both.

2. Null spread of edges of graphs
In this section first we discuss the null spread of edges of \( P_n \) and \( C_n \). First of all we have the obvious result.

**Theorem 2.23.** A path \( P_n \) of n vertices is singular if n is odd and non-singular if n is even.

**Theorem 2.24.** Let \( P_n \) be a path of n vertices and e be an edge of \( P_n \). Then
(i) \( \eta_e(P_n) = 0 \), if n is odd.
(ii) \( \eta_e(P_n) = 1 \), if n is odd and e is a pendant edge.
(iii) \( \eta_e(P_n) = -1 \), if n is even and e is a pendant edge.
(iv) \( \eta_e(P_n) = -2 \), if n is even and \( P_n - e \) has two components having odd number of vertices.
(v) \( \eta_e(P_n) = 0 \), if n is even and \( P_n - e \) has two components having even number of vertices.

**Proof:** We prove part (ii). If n is odd, by theorem 2.23, we have \( (P_n) = 1 \). Since e is a pendant edge, its removal result in the removal of pendant vertex. So \( P_n - e \) is a path of even number of vertices. Thus \( \eta_e(P_n) = (P_n) - (P_{n-1}) = 1 - 0 = 1 \).

Similarly, using theorem 2.23 we can prove the other parts too.
Theorem 2.25. A cycle $C_n$ of $n$ vertices is non-singular if and only if $n$ is not divisible by 4.

Theorem 2.26. Let $C_n$ be a cycle of $n$ vertices and $e$ be an edge of $C_n$. Then

(i) $\eta_e(C_n) = -1$, if and only if $n$ is odd.
(ii) $\eta_e(C_n) = 0$, if and only if $n$ is even and not divisible by 4.
(iii) $\eta_e(C_n) = 2$, if and only if $n$ is divisible by 4.

Proof:

(i) If $n$ is odd, $C_n - e$ is a path of odd number of vertices and so $\eta(C_n - e) = 1$. Since $\eta(C_n) = 0$ for odd $n$, we have $\eta_e(C_n) = 0 - 1 = -1$. Conversely, if $\eta_e(C_n) = -1$, we have $\eta(C_{n-1}) = \eta(C_n) + (-1) = 1$. This is true only if $n$ is odd.

(ii) If $n$ is even and not divisible by 4, then $C_n$ is non-singular. Also $C_n - e$ is a path of even number of vertices and is non-singular. So $\eta_e(C_n) = 0 - 0 = 0$. Conversely $\eta_e(C_n) = 0$ implies that $\eta(C_{n-1}) = \eta(C_n)$. But for a cycle this is true only if $\eta(C_{n-1}) = \eta(C_n) = 0$. So $n$ must be even and not divisible by four.

(iii) If $n$ is even and is divisible by 4, then $\eta(C_n) = 2$. Also $C_n - e$ is a path of even number of vertices and is non-singular. Thus $\eta_e(C_n) = 2 - 0 = 2$. Conversely $\eta_e(C_n) = 2$ implies that $\eta(C_{n-1}) = \eta(C_n) = 2$. This is true only if $n$ is even and is divisible by 4.

Next we will discuss the null spread of the pendant edge of a singular graph.

Theorem 2.27. Let $G$ be a singular graph of nullity $\eta$ and order $n$. Suppose that $e = uv$ be a pendant edge of $G$ such that $v$ is a pendant vertex.

(i) If $v$ is a core vertex of $G$, then $\eta_e(G) = 1$.
(ii) If $v$ is a noncore vertex of null spread zero of $G$, then $\eta_e(G) = 0$.
(iii) If $v$ is a noncore vertex of null spread $-1$ of $G$, then $\eta_e(G) = -1$.

Proof:

(i) Since core vertex has null spread one, we have $\eta(G - e) = \eta(G - v) = \eta - 1$. So $\eta_e(G) = \eta - (\eta - 1) = 1$.

(ii) Since $v$ is a noncore vertex of zero null spread, we have $\eta(G - e) = \eta(G - v) = \eta(G) = \eta$. So $\eta_e(G) = \eta - \eta = 0$.

(iii) Since $v$ is a noncore vertex of null spread $-1$, we have $\eta(G - e) = \eta(G - v) = \eta + 1$. So $\eta_e(G) = \eta - (\eta + 1) = -1$.

Our next theorem gives the null spread of the cut edge of a graph $G$.

Theorem 2.28. Let $G$ be a graph with a cut edge $e = uv$ and $K,H$ are singular components of $G - e$ having nullity $\eta_1$ and $\eta_2$ respectively. Then

(i) $\eta_e(G) = -2$ if and only if $u$ and $w$ are core vertices of $K$ and $H$ respectively.
(ii) $\eta_e(G) = -1$ if and only if $u$ is a core vertex of $K$ and $w$ is a noncore vertex of $H$ with null spread zero or vice versa.
(iii) If $u$ and $w$ are noncore vertices (of null spread zero or $-1$) of $K$ and $H$ respectively, then $\eta_e(G) = 0$.
(iv) If $u$ is a core vertex of $K$ and $w$ is a noncore vertex of null spread $-1$ or vice versa, then $\eta_e(G) = 0$.
(v) If $u$ is a noncore vertex of $K$ of null spread zero and $w$ is a noncore vertex of $H$ with null spread $-1$ or vice versa, then $\eta_e(G) = 0$.

Proof:

(i) By theorem 1.19, $\eta(G) = \eta(HK + uw) = \eta_1 + \eta_2 - 2$. Since the nullity of $H$ and $K$ are respectively $\eta_1$ and $\eta_2$, it follows that $\eta(G - e) = \eta_1 + \eta_2$. Thus $\eta_e(G) = -2$. 561
Conversely suppose that $\eta_e(G) = -2$. This means that nullity increases by two when we remove the edge $e$. It follows now from the construction of the graph $HK + uw$ that $u$ and $w$ are core vertices.

The proof of other parts follows similarly.

**Remark 2.29.** The part (iii), (iv) and (v) of the above theorem exhibit three situations in which $\eta_e(G) = 0$. So when $\eta_e(G) = 0$, it is impossible to find the type of end vertices of $e = uv$ in these cases uniquely. Thus if $\eta_e(G) = 0$, then either the conditions in the hypothesis of part (iii) or (iv) or (v) holds.

**Theorem 2.30.** Let $G$ be a graph with a cut edge $e = uw$ and $K$, $H$ be the components of $G - e$.

(i) Let $K$ be singular with nullity $\eta$ and $H$ nonsingular. Then $\eta_e(G) = -1$ if and only if $u$ is a core vertex and $H - w$ is nonsingular.

(ii) Let $K$ be singular with nullity $\eta$ and $H$ nonsingular. If $u$ is a core vertex and $H - w$ is singular, then $\eta_e(G) = 0$.

(iii) Let $K$ be singular with nullity $\eta$ and $H$ nonsingular. If $u$ is a noncore vertex (of null spread $0$ or $-1$), then $\eta_e(G) = 0$.

**Proof:** (i) Part 6 of theorem 1.19 shows that if $u$ is a core vertex and $H - w$ is nonsingular, then $\eta_e(G) = -1$. Conversely, when $\eta_e(G) = -1$, the nullity of the graph increases on deleting the edge $e$. It now follows from the construction of the graph $KH + uw$ that $u$ is a core vertex and $H - w$ is nonsingular.

The proof of part (ii) and (iii) follows from part 7 and 8 of theorem 1.19.

**Remark 2.31.** In part (ii) and (iii), there are two different situations which leads to $\eta_e(G) = 0$. Here also it is impossible to find the type of end vertices $u$ and $w$ of the edge $e = uw$ uniquely when $\eta_e(G) = 0$. So if $\eta_e(G) = 0$, then either the conditions in the hypothesis of part (ii) or part (iii) holds.

3. **Concatenation of two graphs**

**Definition 3.6.** Let $G_1$ and $G_2$ be two graphs of orders $n_1$ and $n_2$ respectively. Then the graph having $e(G_1) + e(G_2) - 1$ edges and $n_1 + n_2 - 2$ vertices obtained by identifying an edge from $G_1$ and another from $G_2$ is called the concatenation or edge gluing of $G_1$ and $G_2$.

**Figure 4:** Concatenation of $G_1$ and $G_2$. 

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3.1. Concatenation of paths and cycles
We have the following simple results about the concatenation of paths:

Theorem 3.32. The concatenation of two paths of odd number of vertices concatenated at pendant edges is non-singular.
Proof: The concatenation of two paths of odd number of vertices concatenated at pendant edges is a path of even number of vertices. As paths of even number of vertices are non-singular, the theorem follows.

Theorem 3.33. The concatenation of two paths of even number of vertices concatenated at pendant edges is non-singular.
Proof: The concatenation of two paths of even number of vertices concatenated at pendant edges is a path of even number of vertices. As paths of even number of vertices are non-singular, the theorem follows.

Theorem 3.34. The concatenation of a path of odd number of vertices and a path of even number of vertices concatenated at pendant edges is singular.
Proof: The concatenation of a path of odd number of vertices and a path of even number of vertices concatenated at pendant edges is a path of odd number of vertices. As paths of odd number of vertices are singular, the theorem follows.

Theorem 3.35. The concatenation of two paths of odd number of vertices concatenated at nonpendant edges is either singular with nullity two or non-singular.
Proof: The concatenation of two paths concatenated at nonpendant edges can be regarded as the graph obtained by joining two paths by an edge at nonpendant vertices (see figure 5). Let $G$ be the graph obtained by the concatenation of two paths of odd number of vertices at nonpendant edges. Then $G$ is of the form $P_n P_m + uv$, where both $n$ and $m$ are either even or odd. If both $n$ and $m$ are even, then by equation (1.2) we see that $G$ is nonsingular. If $n$ and $m$ are odd, then both $P_n$ and $P_m$ are singular graphs of nullity one. There arise three cases. First of all if both $u$ and $w$ are core vertices, then by part 1 of theorem 1.19 we get $G$ is non-singular. If both $u$ and $w$ are noncore vertices of null spread $-1$, then by part 2 of theorem 1.19 we get $G$ is singular of nullity two. Finally if $u$ is a core vertex and $w$ is a noncore vertex of null spread $-1$, then by part 3 of theorem 1.19 we see that $G$ is singular with nullity two.

Theorem 3.36. The concatenation of two paths of even number of vertices concatenated at nonpendant edges is either singular with nullity two or non-singular.
Proof is similar to the proof of theorem 3.35.

Theorem 3.37. The concatenation of two paths of even and odd number of vertices concatenated at nonpendant edges is singular with nullity one.
Proof: As in the proof of theorem 3.35, the concatenation of two paths concatenated at nonpendant edges can be regarded as the graph obtained by joining two paths by an edge at nonpendant vertices. Let $G$ be the graph obtained by the concatenation of two paths of odd and even number of vertices at nonpendant edges. Then $G$ is of the form $P_n P_m + uv$, where either $n$ is odd and $m$ is even or $n$ is even and $m$ is odd. Let us fix $m$ as odd and $n$
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as even. So $p_n, p_m$ are paths of even and odd number of vertices respectively. Since $p_n$ is non-singular such that $p_n - u$ is singular and $p_m$ is singular of nullity one, it follows by part 7 and 8 of theorem 1.19 that $G$ is singular with nullity one.

Next we will discuss the concatenation of two cycles.

**Theorem 3.38.** Let $G$ be the concatenation of two cycles $C_k$ and $C_l$, where $k + l = n + 2$. Then the product of the eigenvalues of $G$ is given by

$\prod_{\nu=0}^{n-1} 2 \cos \left(\frac{2\pi \nu}{n}\right) - \prod_{\nu=1}^{k-2} 2 \cos \left(\frac{\nu}{k-1}\right) \prod_{\nu=1}^{l-2} 2 \cos \left(\frac{\nu}{l-1}\right) - 2 \left[ \prod_{\nu=1}^{k-2} 2 \cos \left(\frac{\nu}{k-1}\right) + \prod_{\nu=1}^{l-2} 2 \cos \left(\frac{\nu}{l-1}\right) \right]$.

**Proof:** By theorem 1.20, the characteristic polynomial of $G$ is given by

$\phi(G) = \phi(G - uv) - \phi(G - u - v) - 2 \sum_{e \in E(G)} \phi(G - e)$

$= \phi(C_n) - \phi(P_{k-2}) \phi(P_{l-2}) - 2 \phi(P_{k-2}) + \phi(P_{l-2})$.

Product of the eigenvalues of $G$

$= \text{The coefficient of } x^0 \text{ in } \phi(G)$

$= \prod_{\nu=0}^{n-1} 2 \cos \left(\frac{2\pi \nu}{n}\right) - \prod_{\nu=1}^{k-2} 2 \cos \left(\frac{\nu}{k-1}\right) \prod_{\nu=1}^{l-2} 2 \cos \left(\frac{\nu}{l-1}\right) - 2 \left[ \prod_{\nu=1}^{k-2} 2 \cos \left(\frac{\nu}{k-1}\right) + \prod_{\nu=1}^{l-2} 2 \cos \left(\frac{\nu}{l-1}\right) \right]$.

**Example 3.39.** The product of eigenvalues of the graph in figure 6 is

$\prod_{\nu=0}^{7} 2 \cos \left(\frac{2\pi \nu}{7}\right) - \prod_{\nu=1}^{7} 2 \cos \left(\frac{\nu}{4}\right) \prod_{\nu=1}^{7} 2 \cos \left(\frac{\nu}{3}\right) - 2 \left[ \prod_{\nu=1}^{7} 2 \cos \left(\frac{\nu}{4}\right) + \prod_{\nu=1}^{7} 2 \cos \left(\frac{\nu}{3}\right) \right]$.

$= 2 \cos (0) 2 \cos \left(\frac{2\pi}{3}\right) 2 \cos \left(\frac{4\pi}{7}\right) 2 \cos \left(\frac{6\pi}{7}\right) 2 \cos \left(\frac{8\pi}{7}\right) 2 \cos \left(\frac{10\pi}{7}\right) 2 \cos \left(\frac{12\pi}{7}\right) - 2 \cos \left(\frac{\pi}{4}\right) 2 \cos \left(\frac{2\pi}{4}\right) 2 \cos \left(\frac{3\pi}{4}\right) 2 \cos \left(\frac{4\pi}{3}\right)$.

Figure 5: Concatenation of two odd paths at nonpendant edges

Figure 6: Concatenation of two odd paths at nonpendant edges
Lemma 3.40. If \( n_1 \) is odd and \( n_2 \) is a multiple of 4, then
\[
\prod_{j=0}^{n_1+n_2-3} 2\cos \left( \frac{2\pi j}{n_1+n_2-2} \right) = 2\prod_{j=1}^{n_2-2} 2\cos \left( \frac{\pi j}{n_2-1} \right)
\]

Theorem 3.41. Let \( C_{n_1} \) and \( C_{n_2} \) be cycles with \( n_1 \) and \( n_2 \) vertices respectively. \( G \) be the concatenation of \( C_{n_1} \) and \( C_{n_2} \) with respect to an edge \( e \) of \( C_{n_1} \) and \( e_2 \) of \( C_{n_2} \) and \( e \) be the concatenated edge.

(i) If \( \eta_{e_1}(C_{n_1}) = \eta_{e_2}(C_{n_2}) = 2 \), then \( \eta_e(G) = 0 \).
(ii) If \( \eta_{e_1}(C_{n_1}) = 0 \) and \( \eta_{e_2}(C_{n_2}) = 2 \), then \( \eta_e(G) = -2 \).
(iii) If \( \eta_{e_1}(C_{n_1}) = -1 \) and \( \eta_{e_2}(C_{n_2}) = -1 \), then \( \eta_e(G) = 1 \) or 0 according as \( n_1+n_2-2 \) is divisible by 4 or divisible by only 2.
(iv) If \( \eta_{e_1}(C_{n_1}) = -1 \) and \( \eta_{e_2}(C_{n_2}) = 0 \) (or vice versa), then \( \eta_e(G) = 0 \).
(v) If \( \eta_{e_1}(C_{n_1}) = -1 \) and \( \eta_{e_2}(C_{n_2}) = 2 \) (or vice versa), then \( \eta_e(G) = 1 \).
(vi) If \( \eta_{e_1}(C_{n_1}) = 0 \) and \( \eta_{e_2}(C_{n_2}) = 0 \), then \( \eta_e(G) = 0 \).

Proof: (i) If \( \eta_{e_1}(C_{n_1}) = \eta_{e_2}(C_{n_2}) = 2 \), then \( C_{n_1} \) and \( C_{n_2} \) are singular graphs of nullity 2. So \( n_1 \) and \( n_2 \) are divisible by 4. The concatenated graph has \( n_1+n_2-2 \) vertices and \( G-e \) is a cycle of \( n_1+n_2-2 \) vertices. As \( n_1+n_2-2 \) is not divisible by 4, \( G-e \) is non-singular i.e. \( \eta(G-e) = 0 \). By theorem 1.17, the product of the eigenvalues of
\[
G = \prod_{j=0}^{n_1+n_2-3} 2\cos \left( \frac{2\pi j}{n_1+n_2-2} \right) - \prod_{j=1}^{n_2-2} 2\cos \left( \frac{\pi j}{n_2-1} \right) - 2 \left[ \prod_{j=1}^{n_2-2} 2\cos \left( \frac{\pi j}{n_2-1} \right) \right] = 0,
\]
As \( n_1+n_2-2 \) is not a multiple of 4 and both \( n_1-1 \) and \( n_2-1 \) are odd numbers. So \( \eta(G) = 0 \). Hence \( \eta_e = 0 \).
So if $G = e$ is a cycle of $n_1 + n_2 = 2$ vertices, we have $\eta(G - e) = 2$. The product of the eigenvalues of

$$G = \prod_{j=0}^{n_1+n_2-3} 2 \cos \left( \frac{2\pi j}{n_1+n_2} \right) - \prod_{j=1}^{n_1-2} 2 \cos \left( \frac{\pi j}{n_1-1} \right) \prod_{j=1}^{n_2-2} 2 \cos \left( \frac{\pi j}{n_2-1} \right) - \left[ \prod_{i=1}^{n_1-2} 2 \cos \left( \frac{\pi j}{n_1-1} \right) + \prod_{i=1}^{n_2-2} 2 \cos \left( \frac{\pi j}{n_2-1} \right) \right] \neq 0,$$

as $n_1 - 1$ and $n_2 - 1$ are odd numbers. So $\eta = 0 - 2 = -2$.

(iii) If $\eta_1(C_{n_1}) = -1$ and $\eta_2(C_{n_2}) = -1$, then $n_1$ and $n_2$ are even numbers. So $n_1 + n_2 = 2$ is divisible by 2 or 4. If $n_1 + n_2 = 2$ is divisible by 4, then $G - e$ is singular with nullity 2 i.e. $\eta(G - e) = 2$. If $n_1 + n_2 = 2$ is divisible by 2, then $G - e$ is non-singular and so $\eta(G - e) = 0$. If $n_1 + n_2 = 2$ is divisible by 4, then product of the eigenvalues of

$$G = \prod_{j=0}^{n_1+n_2-3} 2 \cos \left( \frac{2\pi j}{n_1+n_2} \right) - \prod_{j=1}^{n_1-2} 2 \cos \left( \frac{\pi j}{n_1-1} \right) \prod_{j=1}^{n_2-2} 2 \cos \left( \frac{\pi j}{n_2-1} \right) - \left[ \prod_{i=1}^{n_1-2} 2 \cos \left( \frac{\pi j}{n_1-1} \right) + \prod_{i=1}^{n_2-2} 2 \cos \left( \frac{\pi j}{n_2-1} \right) \right] = 0,$$

as $n_1 + n_2 = 2$ is divisible by 4 and $n_1 - 1$, $n_2 - 1$ are divisible by 4. Also the coefficient of $x$ in the characteristic polynomial of $G$ is nonzero. On the other hand if $n_1 + n_2 = 2$ is divisible by 2 only, then the product of the eigenvalues of $G$ is nonzero as $\prod_{j=0}^{n_1+n_2-3} 2 \cos \left( \frac{2\pi j}{n_1+n_2} \right) \neq 0$ and the other products vanish. So $\eta(G) = 1$, if $n_1 + n_2 = 2$ is divisible by 4 and $\eta(G) = 0$, if $n_1 + n_2 = 2$ is divisible by 2 only. Hence $\eta(G) = 1 = 0 = 1$, if $n_1 + n_2 = 2$ is divisible by 2 only.

(iv) If $\eta_1(C_{n_1}) = -1$ and $\eta_2(C_{n_2}) = 0$, then $n_1$ is an odd number and $n_2$ is an even number not divisible by 4. So $n_1 + n_2 = 2$ is an odd number. So $\eta(G - e) = 0$. The product of the eigenvalues of

$$G = \prod_{j=0}^{n_1+n_2-3} 2 \cos \left( \frac{2\pi j}{n_1+n_2} \right) - \prod_{j=1}^{n_1-2} 2 \cos \left( \frac{\pi j}{n_1-1} \right) \prod_{j=1}^{n_2-2} 2 \cos \left( \frac{\pi j}{n_2-1} \right) - \left[ \prod_{i=1}^{n_1-2} 2 \cos \left( \frac{\pi j}{n_1-1} \right) + \prod_{i=1}^{n_2-2} 2 \cos \left( \frac{\pi j}{n_2-1} \right) \right] = 0.$$

Hence $\eta_1(G) = 0$. So $\eta(G) = 0$.

(v) If $\eta_1(C_{n_1}) = -1$ and $\eta_2(C_{n_2}) = 2$, then $n_1$ is an odd number and $n_2$ is an even number divisible by 4. So $n_1 + n_2 = 2$ is an odd number. So $\eta(G - e) = 0$. The product of the eigenvalues of

$$G = \prod_{j=0}^{n_1+n_2-3} 2 \cos \left( \frac{2\pi j}{n_1+n_2} \right) - \prod_{j=1}^{n_1-2} 2 \cos \left( \frac{\pi j}{n_1-1} \right) \prod_{j=1}^{n_2-2} 2 \cos \left( \frac{\pi j}{n_2-1} \right) - \left[ \prod_{i=1}^{n_1-2} 2 \cos \left( \frac{\pi j}{n_1-1} \right) + \prod_{i=1}^{n_2-2} 2 \cos \left( \frac{\pi j}{n_2-1} \right) \right] = 0,$$

since $\prod_{j=0}^{n_1+n_2-3} 2 \cos \left( \frac{2\pi j}{n_1+n_2} \right) = 2 \prod_{j=1}^{n_2-2} 2 \cos \left( \frac{\pi j}{n_2-1} \right)$, by lemma 3.37. Also the coefficient of $x$ in the characteristic polynomial of $G$ is nonzero. So $\eta(G) = 1$. Thus $\eta_1(G) = 1 - 0 = 1$.

(vi) If $\eta_1(C_{n_1}) = 0$ and $\eta_2(C_{n_2}) = 0$, then both $n_1$ and $n_2$ are even numbers not divisible by 4. So $n_1 + n_2$ is an even number divisible by 4. Then $n_1 + n_2 = 2$ is an even number divisible by 2 only. Thus $\eta(G - e) = 0$. The product of the eigenvalues of $G = \prod_{j=0}^{n_1+n_2-3} 2 \cos \left( \frac{2\pi j}{n_1+n_2} \right) - \prod_{j=1}^{n_1-2} 2 \cos \left( \frac{\pi j}{n_1-1} \right) \prod_{j=1}^{n_2-2} 2 \cos \left( \frac{\pi j}{n_2-1} \right) - \left[ \prod_{i=1}^{n_1-2} 2 \cos \left( \frac{\pi j}{n_1-1} \right) + \prod_{i=1}^{n_2-2} 2 \cos \left( \frac{\pi j}{n_2-1} \right) \right] = 0$. 

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\[ 2 \left( \prod_{j=1}^{n_1-2} 2 \cos \left( \frac{n_j}{n_1-1} \right) + \prod_{j=1}^{n_2-2} 2 \cos \left( \frac{n_j}{n_2-1} \right) \right) \neq 0, \text{as } n_1 + n_2 - 2 \text{ is divisible by 2 only and } n_1 - 1, n_2 - 1 \text{ are odd numbers. So } \eta(G) = 0. \text{ Thus } \eta_e(G) = 0 - 0 = 0. \]

3.2. Concatenation of two graphs at pendant edges

Concatenation of two graphs with respect to pendant edges is same as joining two graphs by an edge. Figure 7 illustrate this.

\[ \text{Figure 7: Concatenation of two graphs at pendant edges} \]

The following two theorems can be proved using theorem 1.19.

**Theorem 3.42.** Let \( G_1 \) and \( G_2 \) be two singular graphs with nullity \( \eta_1 \) and \( \eta_2 \) respectively and \( G \) be the concatenation of \( G_1 \) and \( G_2 \) with respect to the pendant edges \( e_1 = uw \) of \( G_1 \) and \( e_2 = u'w' \) of \( G_2 \), where \( w \) and \( w' \) are pendant vertices of \( G_1 \) and \( G_2 \) respectively. Then,

(i) If \( u \) and \( u' \) are core vertices of \( G_1 \) and \( G_2 \) respectively, then nullity of \( G \) is \( \eta_1 + \eta_2 - 2 \).

(ii) If \( u \) and \( u' \) are noncore vertices (of null spread 0 or -1) of \( G_1 \) and \( G_2 \) respectively, then the nullity of \( G \) is \( \eta_1 + \eta_2 \).

(iii) If \( u \) is a core vertex of \( G_1 \) and \( u' \) is a noncore vertex of null spread -1 of \( G_2 \) or vice versa, then the nullity of \( G \) is \( \eta_1 + \eta_2 - 1 \).

(iv) If \( u \) is core vertex of \( G_1 \) and \( u' \) is a noncore vertex of \( G_2 \) of null spread 0 or vice versa, then the nullity of \( G \) is \( \eta_1 + \eta_2 - 1 \).

(v) If \( u \) is a noncore vertex of \( G_1 \) of null spread 0 and \( u' \) is a noncore vertex of \( G_2 \) of null spread -1 or vice versa, then the nullity of \( G \) is \( \eta_1 + \eta_2 \).

**Proof:** Since \( G = G_1G_2 + uu' \), the theorem follows from part 1 to 5 of theorem 1.19

**Corollary 3.43.** Let \( G_1, G_2 \) and \( G \) be as in theorem 3.42 and \( e \) be the concatenated edge. Then

(i) If \( u \) and \( u' \) are core vertices of \( G_1 \) and \( G_2 \) respectively, then \( \eta_e(G) = -2 \).

(ii) If \( u \) and \( u' \) are noncore vertices (of null spread 0 or -1) of \( G_1 \) and \( G_2 \) respectively, then \( \eta_e(G) = 0 \).

(iii) If \( u \) is a core vertex of \( G_1 \) and \( u' \) is a noncore vertex of null spread -1 of \( G_2 \) or vice versa, then \( \eta_e(G) = 0 \).

(iv) If \( u \) is core vertex of \( G_1 \) and \( u' \) is a noncore vertex of \( G_2 \) of null spread 0 or vice versa, then \( \eta_e(G) = -1 \).
Theorem 3.44. Let \( G_1 \) be a singular graph of nullity \( \eta \), \( G_2 \) be non-singular and \( G \) be the concatenation of \( G_1 \) and \( G_2 \) with respect to the pendant edge \( e = uw \) of \( G_1 \) and \( e' = u'w' \) of \( G_2 \), where \( w \) and \( w' \) are pendant vertices of \( G_1 \) and \( G_2 \) respectively. Then,

(i) If \( u \) is a core vertex of \( G_1 \) and \( G_2 - u' \) is nonsingular, then nullity of \( G \) is \( \eta - 1 \).

(ii) If \( u \) is a core vertex of \( G_1 \) and \( G_2 - u' \) is singular, then nullity of \( G \) is \( \eta \).

(iii) If \( u \) is a noncore vertex (of null spread 0 or \(-1\)) of \( G_1 \), then nullity of \( G \) is \( \eta \).

Proof: Since \( G = G_1 G_2 + uu' \), the theorem follows from part 6 to 8 of theorem 1.19.

Corollary 3.45. Let \( G_1, G_2 \) and \( G \) be as in theorem 3.44 and \( e \) be the concatenated edge. Then

(i) If \( u \) is a core vertex of \( G_1 \) and \( G_2 - u' \) is nonsingular, then \( \eta_e(G) = -1 \).

(ii) If \( u \) is a core vertex of \( G_1 \) and \( G_2 - u' \) is singular, then \( \eta_e(G) = 0 \).

(iii) If \( u \) is a noncore vertex (of null spread 0 or \(-1\)) of \( G_1 \), then \( \eta_e(G) = 0 \).

Example 3.46. In Figure 8, the graph \( G_1 \) is a singular graph with nullity one and \( G_2 \) is non-singular. The concatenated graph \( G \) is singular of nullity one. Note that \( u \) is a noncore vertex of null spread \(-1\).

Figure 8: Concatenation of a singular and non-singular graph at pendant edges.

3.3. Concatenation of two graphs at cut edges

Let \( G_1 \) be a graph with cut edge \( e_1 = uv \), so that \( G_1 - e_1 \) has components \( H \) and \( K \). Similarly, \( G_2 \) has cut edge \( e_2 = u'w' \) with \( H' \) and \( K' \) as the components of \( G_2 - e_2 \). The concatenation, \( G \) of \( G_1 \) and \( G_2 \) with respect to \( e_1 \) and \( e_2 \) is same as \((H \cup H')(K \cup K') + vv'\), where \( H \cup H' \) and \( K \cup K' \) are the coalescence of \( H, H' \) and \( K, K' \) respectively.

Figure 9: Concatenation of \( G_1 \) and \( G_2 \) at cut edges.
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**Theorem 3.47.** Let $G_1$ and $G_2$ be singular graphs with nullity $\eta_1$ and $\eta_2$ respectively and $G$ be the concatenation of them with respect to their cut edges $e_1 = uw$ and $e_2 = u'w'$. Assume that the components of $G_1 - e_1$ and $G_2 - e_2$ are singular. Then

(i) If $\eta_{e_1}(G_1) = \eta_{e_2}(G_2) = -2$, then $G$ is singular with nullity $\eta_1 + \eta_2$.

(ii) If $\eta_{e_1}(G_1) = -1$ and $\eta_{e_2}(G_2) = -2$, then $G$ is singular with nullity $\eta_1 + \eta_2$.

(iii) If $\eta_{e_1}(G_1) = \eta_{e_2}(G_2) = -1$, then $G$ is singular of nullity $\eta_1 + \eta_2$.

(iv) If $\eta_{e_1}(G_1) = 0$ and $\eta_{e_2}(G_2) = -2$, then $G$ is singular with nullity $\eta_1 + \eta_2$.

(v) If $\eta_{e_1}(G_1) = \eta_{e_2}(G_2) = -1$, then $G$ is singular with nullity $\eta_1 + \eta_2$ provided $u, u'$ are core vertices, $w$ is a noncore vertex of null spread zero and $w'$ is a noncore vertex of null spread $-1$.

(vi) If $\eta_{e_1}(G_1) = 0$ and $\eta_{e_2}(G_2) = -1$, then $G$ is singular with nullity $\eta_1 + \eta_2 - 1$, provided $u, w'$ are core vertices $u'$ is a noncore vertex of null spread zero and $w$ is a noncore vertex of null spread $-1$.

**Proof:** (i) Since $\eta_{e_1}(G_1) = \eta_{e_2}(G_2) = -2$, by theorem 2.28 we have $u, w, u', w'$ are core vertices. Let $K, H$ are the components of $G_1 - e_1$ and $K', H'$ are the components of $G_2 - e_2$. Assume that $\eta_K, \eta_H, \eta_{K'}$ and $\eta_{H'}$ are the nullities of $K, H, K'$ and $H'$ respectively. By definition, $G$ can be regarded as $(K\circ K') + (H\circ H')$, where $v = u = u'$ and $v' = w = w'$ are the coalesced vertices. Then by theorem 1.2, $K\circ K'$ is singular of nullity $\eta_K + \eta_{K'} - 1$ and $H\circ H'$ is singular of nullity $\eta_H + \eta_{H'} - 1$. Also $v$ and $v'$ are core vertices. So by part 1 of theorem 2.19, we have nullity of $G$ is $\eta_K + \eta_{K'} - 1 + \eta_H + \eta_{H'} - 1 - 2 = \eta_K + \eta_H - 2 + \eta_{K'} + \eta_{H'} - 2 = \eta_1 + \eta_2$, as $\eta_K + \eta_{K'} - 1$ and $\eta_H + \eta_{H'} - 1$.

(ii) Since $\eta_{e_1}(G_1) = -1$, by theorem 2.28, we have $u$ is core vertex of $K$ and $w$ is a noncore vertex of $H$ of null spread zero or vice versa. Let us fix $u$ as core vertex of $K$ and $v$ as noncore vertex of $H$ of null spread zero. Also by theorem 2.28, we have $u'$ and $w'$ are core vertices of $K'$ and $H'$ respectively as $\eta_{e_2}(G_2) = -2$. By definition, $G$ can be regarded as $(K\circ K') + (H\circ H') + vv'$, where $v = u = u'$ and $v' = w = w'$ are the coalesced vertices. Then we get by theorem 1.2 and 1.6 that $K \circ K'$ is a singular graph of nullity $\eta_K + \eta_{K'} - 1$ and $H \circ H'$ is a singular graph of nullity $\eta_H + \eta_{H'} - 1$. Note that the vertex $v$ is a core vertex and $v'$ is a noncore vertex of null spread zero. So by part 3 of theorem 2.19, we see that nullity of $G$ is $\eta_K + \eta_{K'} - 1 + \eta_H + \eta_{H'} - 1 - 1 = \eta_1 + \eta_2$ as $\eta_1 = \eta_K + \eta_{K'} - 1$ and $\eta_2 = \eta_K + \eta_{K'} + \eta_{H'} - 2$.

(iii) Since $\eta_{e_1}(G_1) = -1$ and $\eta_{e_2}(G_2) = -1$, we have by theorem 2.28 that $u$ is a core vertex of $K$ and $w$ is a noncore vertex of $H$ of null spread zero or vice versa. Also $u'$ is a core vertex of $K'$ and $w'$ is a noncore vertex of $H'$ of null spread zero or vice versa. Let us fix $u, u'$ are core vertices and $w, w'$ are noncore vertices of null spread zero. By definition, $G$ can be regarded as $(K\circ K') + (H\circ H') + vv'$, where $v = u = u'$ and $v' = w = w'$ are the coalesced vertices. Then we get by theorem 1.2 and 1.5 that $K \circ K'$ is a singular graph of nullity $\eta_K + \eta_{K'} - 1$ and $H \circ H'$ is a singular graph of nullity $\eta_H + \eta_{H'}$. Note that $v$ is a core vertex and $v'$ is a noncore vertex of null spread zero. Then by part 2 of theorem 2.19, we get nullity of $G$ as $\eta_K + \eta_{K'} - 1 + \eta_H + \eta_{H'} - 1 = \eta_1 + \eta_2$. If we take $u, w'$ as core vertices and $u', w$ as noncore vertices of null spread zero we see that nullity of $G$ is again $\eta_1 + \eta_2$.

Proof of part (iv) and (v) follows similarly.
Corollary 3.48. In the above theorem

(i) If \( \eta_{e_1}(G_1) = \eta_{e_2}(G_2) = -2 \), then the concatenated edge has null spread \(-2\).

(ii) If \( \eta_{e_1}(G_1) = -1 \) and \( \eta_{e_2}(G_2) = -2 \), then the concatenated edge has null spread \(-1\).

(iii) If \( \eta_{e_1}(G_1) = \eta_{e_2}(G_2) = -1 \), then the concatenated edge has null spread \(-1\) or zero according as the pendant vertices of \( e_1 \) and \( e_2 \) at each end are of same type or not.

(iv) If \( \eta_{e_1}(G_1) = 0 \) and \( \eta_{e_2}(G_2) = -2 \), then the concatenated edge has null spread zero.

(v) If \( \eta_{e_1}(G_1) = 0 \) and \( \eta_{e_2}(G_2) = -1 \), then the concatenated edge has null spread zero.

The case in which \( \eta_{e_1}(G_1) = \eta_{e_2}(G_2) = 0 \) is rather complicated. This is because \( \eta_e(G) = 0 \) for any cut edge \( e = uw \) do not uniquely determine the type of vertices \( u \) and \( w \). So this case is specially treated in the next theorem.

Theorem 3.49. Let \( G_1 \) and \( G_2 \) be singular graphs with nullity \( \eta_1 \) and \( \eta_2 \) respectively and \( G \) be the concatenation of them with respect to their cut edges \( e_1 = uw \) and \( e_2 = u'w' \). Suppose that \( \eta_{e_1}(G_1) = \eta_{e_2}(G_2) = 0 \) and the components of \( G_1 - e_1 \) and \( G_2 - e_2 \) are singular.

(i) If \( u, w, u', w' \) are noncore vertices of null spread zero, then \( G \) is singular with nullity \( \eta_1 + \eta_2 \).

(ii) If \( u, w, u', w' \) are noncore vertices of null spread \(-1 \), then \( G \) is singular with nullity \( \eta_1 + \eta_2 + 2 \).

(iii) If \( u, w \) are noncore vertices with null spread zero and \( u', w' \) are noncore vertices having null spread \(-1 \) or vice versa, then nullity of \( G \) is \( \eta_1 + \eta_2 \).

(iv) If \( u, u' \) are core vertices and \( w, w' \) are noncore vertices with null spread \(-1 \) or vice versa, then nullity of \( G \) is \( \eta_1 + \eta_2 \).

(v) If \( u, w' \) are core vertices and \( u', w \) are noncore vertices of null spread \(-1 \) or vice versa, then nullity of \( G \) is \( \eta_1 + \eta_2 - 2 \).

(vi) If \( u, u' \) are noncore vertices of null spread zero and \( w, w' \) are noncore vertices of null spread \(-1 \) or vice versa, then nullity of \( G \) is \( \eta_1 + \eta_2 - 1 \).

(vii) If \( u, w' \) are noncore vertices of null spread zero and \( w, u' \) are noncore vertices of null spread \(-1 \) or vice versa, then nullity of \( G \) is \( \eta_1 + \eta_2 \).

Proof: We prove part (vii). The proofs of other parts follows similarly. Let \( K, H \) be the components of \( G_1 - e_1 \) and \( K', H' \) are the components of \( G_2 - e_2 \). Assume that \( \eta_K, \eta_H, \eta_K' \) and \( \eta_H' \) are the nullities of \( K, H, K' \) and \( H' \) respectively. By definition, \( G \) can be regarded as \( (KoK') \) (HoH') + v\', where \( v = u = u' \) and \( v' = w = w' \) are the coalesced vertices. Since \( u, w' \) are noncore vertices of null spread zero and \( w, u' \) are noncore vertices of null spread \(-1 \), we have \( K o K' \) is singular with nullity \( \eta_K + \eta_K' \) and \( H o H' \) is singular with nullity \( \eta_H + \eta_H' \). Also \( w \) and \( w' \) are noncore vertices of null spread \(-1 \). So by part 5 of theorem 1.19, we have nullity of \( G \) is \( \eta_K + \eta_K' + \eta_H + \eta_H' = \eta_K + \eta_H + \eta_K' + \eta_H' = \eta_1 + \eta_2 \) as \( \eta_1 = \eta_K + \eta_H \) and \( \eta_2 = \eta_K' + \eta_H' \).

Corollary 3.50. In the above theorem, the concatenated edge has null spread zero.

Example 3.51. The graphs \( G_1 \) and \( G_2 \) in figure 10 has nullity 3 and 4 respectively. The vertices \( u, w, u', w' \) are noncore vertices of null spread \(-1 \). Also \( \eta_{e_1}(G_1) = \eta_{e_2}(G_2) = 0 \). The concatenated graph has nullity 9 and the concatenated edge has null spread zero.
Theorem 3.52. Let $G_1$ and $G_2$ be singular graphs having cut edges with nullities $\eta_1$ and $\eta_2$, respectively, and $G$ be the concatenation of them with respect to their cut edges $e_1 = uw$ and $e_2 = u'w'$. Suppose that one of the components $K, H$ of $G_1 - e_1$ and $K', H'$ of $G_2 - e_2$ is singular and the other is non-singular.

(i) If $\eta_{e_1}(G_1) = \eta_{e_2}(G_2) = -1$ and the singular components of both $G_1 - e_1$ and $G_2 - e_2$ are on the same side of the concatenated edge of $G$, then $G$ is singular with nullity $\eta_1 + \eta_2$.

(ii) If $\eta_{e_1}(G_1) = \eta_{e_2}(G_2) = -1$ and the singular components of both $G_1 - e_1$ and $G_2 - e_2$ are on either side of the concatenated edge of $G$, then $G$ is singular with nullity $\eta_1 + \eta_2$.

(iii) If $\eta_{e_1}(G_1) = -1, \eta_{e_2}(G_2) = 0$, the singular components $K$ of $G_1 - e_1$ and $K'$ of $G_2 - e_2$ are on the same side of the concatenated edge of $G$, $u'$ is a core vertex and $H' - w'$ is singular for non-singular component $K'$ of $G_2 - e_2$, then $G$ is singular with nullity $\eta_1 + \eta_2 - 1$.

(iv) If $\eta_{e_1}(G_1) = -1, \eta_{e_2}(G_2) = 0$, the singular components $K$ of $G_1 - e_1$ and $K'$ of $G_2 - e_2$ are on either side of the concatenated edge of $G$, $w'$ is a core vertex and $K' - u'$ is singular for non-singular component $K'$ of $G_2 - e_2$, then $G$ is singular with nullity $\eta_1 + \eta_2$.

(v) If $\eta_{e_1}(G_1) = -1, \eta_{e_2}(G_2) = 0$, the singular components $K$ of $G_1 - e_1$ and $K'$ of $G_2 - e_2$ are on the same side of the concatenated edge of $G$ and $u'$ is a noncore vertex of null spread zero or $-1$, then $G$ is singular with nullity $\eta_1 + \eta_2$.

(vi) If $\eta_{e_1}(G_1) = -1, \eta_{e_2}(G_2) = 0$, the singular components $K$ of $G_1 - e_1$ and $K'$ of $G_2 - e_2$ are on either side of the concatenated edge of $G$ and $w'$ is a noncore vertex of null spread $0$ or $-1$, then $G$ is singular with nullity $\eta_1 + \eta_2$.

Proof: (i) Let $K, H$ be the components of $G_1 - e_1$ and $K', H'$ be the components of $G_2 - e_2$. Suppose that $K, K'$ are singular and $H, H'$ are non-singular. Since $\eta_{e_1}(G_1) = \eta_{e_2}(G_2) = -1$, by theorem 2.30, we have $u, u'$ are core vertices and $H - w, H' - w'$ are non-singular.
graphs. Assume that $\eta_K, \eta_{K'}$ are the nullities of $K, K'$ respectively. By definition, $G$ can be regarded as $(KoK')(HoH') + vv'$, where $v = u = u'$ and $v' = w = w'$ are the coalesced vertices. Theorem 1.2 shows that $KoK'$ is singular with nullity $\eta_K + \eta_{K'} - 1$ and $H o H'$ is non-singular. Also $v$ is a core vertex and $H o H' - v'$ is non-singular. So by part 6 of theorem 1.19, we have nullity of $G$ is $\eta_K + \eta_{K'} - 2 = \eta_K - 1 + \eta_{K'} - 1 = \eta_1 + \eta_2$ as $\eta_1 = \eta_K - 1$ and $\eta_2 = \eta_{K'} - 1$ (Theorem 1.19).

(ii) Let $K, H$ are the components of $G - e_1$ and $K', H'$ are the components of $G - e_2$. Suppose that $K, H'$ are singular and $K, K'$ are nonsingular. Since $\eta_{e_1}(G_1) = \eta_{e_2}(G_2) = -1$, by theorem 2.30, we have $u, w'$ are core vertices and $H - w, K' - u'$ are non-singular graphs. Assume that $\eta_K, \eta_{K'}$ are the nullities of $K, H'$ respectively. By definition, $G$ can be regarded as $(KoK')(HoH') + vv'$, where $v = u = u'$ and $v' = w = w'$ are the coalesced vertices. Theorem 1.3 shows that $KoK'$ is singular with nullity $\eta_K - 1$ and $H o H'$ is singular with nullity $\eta_{H'} - 1$. Note that $v$ and $v'$ are noncore vertices of null spread zero. So by part 2 of theorem 1.19, we have nullity of $G$ is $\eta_K - 1 + \eta_{H'} - 1 = \eta_1 + \eta_2$ as $\eta_1 = \eta_K - 1$ and $\eta_2 = \eta_{K'} - 1$ (Theorem 1.19).

Next we prove (vi). The proofs of other parts follow similarly.

(vi) Let $K, H$ are the components of $G - e_1$ and $K', H'$ are the components of $G - e_2$. Suppose that $K, H'$ are singular and $K, K'$ are nonsingular. Since $\eta_{e_1}(G_1) = -1$ and the singular components of both $G_1 - e_1$ and $G_2 - e_2$ are on either side of $G$, by theorem 2.30 we have $u$ is a core vertex and $H - w$ is nonsingular. Assume that $\eta_K, \eta_{H'}$ are the nullities of $K, H'$ respectively. By definition, $G$ can be regarded as $(KoK')(HoH') + vv'$, where $v = u = u'$ and $v' = w = w'$ are the coalesced vertices. Theorem 1.3, 1.8 and 1.9 shows that $KoK'$ is singular of nullity $\eta_K - 1$ and $H o H'$ is singular of nullity $\eta_{H'} - 1$. Then $v'$ is a noncore vertex of null spread zero or $-1$ according as $w'$ is a noncore vertex of null spread zero or $-1$ and $v$ is a noncore vertex of null spread zero. So part 2 of theorem 1.19 shows that nullity of $G$ is $\eta_K + \eta_{H'} - 1 = \eta_1 + \eta_2$ as $\eta_1 = \eta_K - 1$ and $\eta_2 = \eta_{H'} - 1$ (Theorem 1.19).

**Corollary 3.53.** In the above theorem,

(i) If $\eta_{e_1}(G_1) = \eta_{e_2}(G_2) = -1$ and the singular components of both $G_1 - e_1$ and $G_2 - e_2$ are on one side of the concatenated edge of $G$, then the concatenated edge has null spread $-1$.

(ii) If $\eta_{e_1}(G_1) = \eta_{e_2}(G_2) = -1$ and the singular components of both $G_1 - e_1$ and $G_2 - e_2$ are on either side of the concatenated edge of $G$, then the concatenated edge has null spread zero.

(iii) If $\eta_{e_1}(G_1) = -1, \eta_{e_2}(G_2) = 0$, then the concatenated edge has null spread zero. The case of $\eta_{e_1}(G_1) = \eta_{e_2}(G_2) = 0$ is separately treated in the next theorem.

**Theorem 3.54.** Let $G_1$ and $G_2$ be singular graphs with nullities $\eta_1$ and $\eta_2$ respectively and $G$ be the concatenation of them with respect to their cut edges $e_1 = uv$ and $e_2 = u'w'$. Suppose that one of the components $K, H$ of $G_1 - e_1$ and $K', H'$ of $G_2 - e_2$ is singular and the other is non-singular.
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(i) If $\eta_{e_1}(G_1) = \eta_{e_2}(G_2) = 0$, the singular components of both $G_1 - e_1$ and $G_2 - e_2$ are on the same side of the concatenated edge of $G$ and both $e_1$, $e_2$ are cut edges which satisfies the hypothesis in part 2 of theorem 2.30, then $G$ is singular with nullity $\eta_1 + \eta_2$.

(ii) If $\eta_{e_1}(G_1) = \eta_{e_2}(G_2) = 0$, the singular components of both $G_1 - e_1$ and $G_2 - e_2$ are on either side of the concatenated edge of $G$ and both $e_1$, $e_2$ are cut edges which satisfies the hypothesis in part 2 of theorem 2.30, then $G$ is singular with nullity $\eta_1 + \eta_2 - 2$.

(iii) If $\eta_{e_1}(G_1) = \eta_{e_2}(G_2) = 0$, the singular components of both $G_1 - e_1$ and $G_2 - e_2$ are on the same side of the concatenated edge of $G$ and both $e_1$, $e_2$ are cut edges which satisfies the hypothesis in part 3 of theorem 2.30, then $G$ is singular with nullity $\eta_1 + \eta_2 + 1$.

(iv) If $\eta_{e_1}(G_1) = \eta_{e_2}(G_2) = 0$, the singular components of both $G_1 - e_1$ and $G_2 - e_2$ are on either side of the concatenated edge of $G$ and both $e_1$, $e_2$ are cut edges which satisfies the hypothesis in part 3 of theorem 2.30, then $G$ is singular with nullity $\eta_1 + \eta_2$.

Proof: (i) Since the singular components of both $G_1 - e_1$ and $G_2 - e_2$ are on one side of the concatenated edge of $G$ and both $e_1$, $e_2$ are cut edges which satisfies the hypothesis in part 2 of theorem 2.30, assume that $K$, $K'$ are singular and $H$, $H'$ are nonsingular. Then $u$, $u'$ are core vertices and $H - w$, $H' - w'$ are singular. Let $\eta_K$, $\eta_{K'}$ be the nullities of $K$, $K'$ respectively. By definition, $G$ can be regarded as $(KoK') (H HoH') + vv'$, where $v = u = u'$ and $v' = w = w'$ are the coalesced vertices. Theorem 1.3 and 1.10 shows that $KoK'$ is singular of nullity $\eta_K + \eta_{K'} - 1$ and $H o H'$ is singular of nullity one. The coalesced vertex $w$ is a core vertex and $w'$ is a noncore vertex of null spread $-1$. So part 3 of theorem 1.19 shows that nullity of $G$ is $\eta_K + \eta_{K'} - 1 + 1 = \eta_1 + \eta_2$ as $\eta_K = \eta_1$ and $\eta_{K'} = \eta_2$.

Next we prove (iv). The proof of other parts follow similarly.

(iv) Since the singular components of both $G_1 - e_1$ and $G_2 - e_2$ are on either side of the concatenated edge of $G$ and both $e_1$, $e_2$ are cut edges which satisfies the hypothesis in part 3 of theorem 2.30, assume that $K$, $H'$ are singular and $H$, $K'$ are non-singular. Then $u$, $u'$ are noncore vertices of null spread $-1$ or zero and $H - w$, $K' - u'$ are nonsingular. Let $\eta_K$, $\eta_{H'}$ be the nullities of $K$, $H'$ respectively. By definition, $G$ can be regarded as $(KoK') (H HoH') + vv'$, where $v = u = u'$ and $v' = u = u'$ are the coalesced vertices. Theorem 1.8 and 1.9 shows that $KoK'$ is singular with nullity $\eta_K$ and $H o H'$ is singular with nullity $\eta_{H'}$. Note that the coalesced vertices $v$ and $v'$ are noncore vertices of null spread zero or $-1$ according as $u$ and $u'$ are noncore vertices of null spread zero or $-1$. So part 3 of theorem 1.19 shows that nullity of $G$ is $\eta_K + \eta_{H'} = \eta_1 + \eta_2$ as $\eta_K = \eta_1$ and $\eta_{H'} = \eta_2$.

Corollary 3.55. In the above theorem, if $\eta_{e_1}(G_1) = \eta_{e_2}(G_2) = 0$, then the concatenated edge has null spread zero.

Example 3.56. The graph $G$ in figure 11 is the concatenation of $G_1$ and $G_2$ with respect to their cut edges $e_1$ and $e_2$ respectively. Here $\eta_{e_1}(G_1) = -1$ and $\eta_{e_2}(G_2) = 0$. The nullities of $G_1$ and $G_2$ are two. We have concatenated $G_1$ and $G_2$ with respect to their cut edges $e_1$ and $e_2$ in such a way that the singular components of both $G_1$ and $G_2$ are either sides of the concatenated edge of $G$. Here $u$ and $u'$ core vertices. $H - w$ is non-singular and $H' - w'$ is singular. Note that the nullity of $G$ is $2 + 2 - 1 = 3$. This is what we have said in part (iv) of theorem 3.49.
3.4. Concatenation of a cycle and a graph having cut edge

**Theorem 3.57.** Let $G_1$ be a cycle with $\eta_{e_1}(G_1) = -1$, $G_2$ be a singular graph with nullity $\eta$ having a cut edge $e_2 = uv'$ and the components of $G_2 - e_2$ are singular. Let $G$ be the concatenation of $G_1$ and $G_2$ with respect to $e_1$ and $e_2$.

(i) If $\eta_{e_2}(G_2) = -2$, then $G$ is singular with nullity $\eta + 1$.

(ii) If $\eta_{e_2}(G_2) = -1$, then $G$ is singular with nullity $\eta$.

(iii) If $\eta_{e_2}(G_2) = 0$ and $u'$, $w'$ are noncore vertices of null spread zero, then $G$ is singular with nullity $\eta$.

(iv) If $\eta_{e_2}(G_2) = 0$ and $u'$, $w'$ are noncore vertices of null spread $-1$, then $G$ is singular with nullity $\eta + 1$.

(v) If $\eta_{e_2}(G_2) = 0$, $u'$ is a core vertex and $w'$ is a noncore vertex of null spread $-1$, then $G$ is singular with nullity $\eta - 1$.

(vi) If $\eta_{e_2}(G_2) = 0$, $u'$ is a noncore vertex of null spread zero and $w'$ is a noncore vertex of null spread $-1$, then $G$ is singular with nullity $\eta$.

**Proof:** We prove part (v). The proof of other parts follow similarly. Since $G_1$ is a cycle with $\eta_{e_1}(G_1) = -1$, we see that $G_1$ is a cycle of odd number of vertices. So $G_1$ is nonsingular. Let $K$ and $H$ be singular components of $G_2 - e_2$ having nullities $\eta_K$ and $\eta_H$ respectively. Given that $u'$ is a core vertex and $w'$ is a noncore vertex of null spread $-1$. Let $e_1 = uv$. The concatenation of $G_1$ and $G_2$ with respect to $e_1 = uw$ and $e_2 = uv'$ is same as taking coalescence of $G_1$ with $K$ and $H$ with respect to the end vertices of $e_1$ and $e_2$ (as in figure 12). Suppose that $u'$ is the root of $K$ and $w'$ is the root of $H$. First coalesce $G_1$ and $K$ with respect to $u$ and $u'$. Since $u'$ is a core vertex, $G_1 \circ K$ is a singular graph with nullity $\eta_K - 1$ by theorem 1.3. After coalescence the vertex $w$ of $G_1$ becomes a noncore vertex of null spread zero (theorem 1.12). Next take coalescence of $G_1 \circ K$ and $H$ with respect to $w$ and $w'$. As $w$ is a noncore vertex of null spread zero and $w'$ is a noncore vertex of null spread $-1$, we see by theorem 1.7 that $(G_1 \circ K) \circ H$ is a singular graph with nullity $\eta_K - 1 + \eta_H = \eta - 1$, where $\eta_K + \eta_H = \eta$. 
Corollary 3.58. In the above theorem the concatenated edge has null spread zero.

Theorem 3.59. Let \( G_1 \) be a cycle with \( \eta_{e_1}(G_1) = -1 \) and \( G_2 \) be a singular graph with nullity \( \eta \) with a cut edge \( e_2 = u'w' \). Assume that one component \( K \) of \( G_2 - e_2 \) is singular with nullity \( \eta \) and other component \( H \) is non-singular.

(i) If \( \eta_{e_2}(G_2) = -1 \), then \( G \) is singular with nullity \( \eta - 1 \).
(ii) If \( \eta_{e_2}(G_2) = 0, u' \) is a core vertex and \( H - w' \) is singular, then \( G \) is singular with nullity \( \eta - 1 \).
(iii) If \( \eta_{e_2}(G_2) = 0 \) and \( u' \) is a noncore vertex (of null spread 0 or \(-1\)), then \( G \) is singular with nullity \( \eta \).

Proof: Similar to theorem 3.57.

Corollary 3.60. In the above theorem the concatenated edge has null spread zero.

Example 3.61. The graph \( G \) in figure 12 is the concatenation of \( G_1 \) and \( G_2 \) concatenated with respect to the edges \( e_1 \) of \( G_1 \) and \( e_2 \) of \( G_2 \). The nullity of the graph \( G_2 \) is two. Also \( \eta_{e_1}(G_1) = -1 \) and \( \eta_{e_2}(G_2) = 0 \) with \( u' \) is a noncore vertex of null spread zero and \( w' \) is a noncore vertex of null spread \(-1\). The nullity of the concatenated graph \( G \) is two.

![Figure 12: Concatenation of an odd cycle and a graph with a cut edge.](image)

Theorem 3.62. Let \( G_1 \) be a cycle with \( \eta_{e_1}(G_1) = 0 \), \( G_2 \) be a singular graph with nullity \( \eta \) having a cut edge \( e_2 = u'w' \) and the components of \( G_2 - e_2 \) are singular. Let \( G \) be the concatenation of \( G_1 \) and \( G_2 \) with respect to \( e_1 \) and \( e_2 \).

(i) If \( \eta_{e_2}(G_2) = -2 \), then \( G \) is singular with nullity \( \eta \).
(ii) If \( \eta_{e_2}(G_2) = -1 \), then \( G \) is singular with nullity \( \eta \).
1.8. After coalescence the vertex \( w \) of \( G \) by theorem 1.7 that (concatenation of having a cut edge with respect to (theorem 1.15). Next take coalescence of (ii)

Corollary 3.63. In the above theorem the concatenated edge has null spread zero.

**Theorem 3.64.** Let \( G_1 \) be a cycle with \( \eta_{e_1}(G_1) = 0 \) and \( G_2 \) be a singular graph with nullity \( \eta \) having a cut edge \( e_2 = u'w' \). Assume that one component \( K \) of \( G_2 - e_2 \) is singular with nullity \( \eta_K \) and \( \eta_H \) respectively. Given that \( u' \) is a non-core vertex of null spread zero and \( w' \) is a non-core vertex of null spread \(-1\). Let \( e_1 = uw \). The concatenation of \( G_1 \) and \( G_2 \) with respect to \( e_1 = uw \) and \( e_2 = u'w' \) is same as taking coalescence of \( G_1 \) with \( K \) and \( H \) with respect to the end vertices of \( e_1 \) and \( e_2 \) (as in figure 12). Suppose that \( u' \) is the root of \( K \) and \( w' \) is the root of \( H \). First coalesce \( G_1 \) and \( K \) with respect to \( u \) and \( u' \). Since \( u' \) is a non-core vertex of null spread zero, \( G_0 \) is a singular graph with nullity \( \eta_K \), by theorem 1.8. After coalescence the vertex \( w \) of \( G_1 \) becomes a non-core vertex of null spread zero (theorem 1.15). Next take coalescence of \( G_1 \) of \( K \) and \( H \) with respect to \( w \) and \( w' \). As \( w \) is a non-core vertex of null spread zero and \( w' \) is a non-core vertex of null spread \(-1\), we see by theorem 1.7 that \((G_1 \cup K) \cup H \) is a singular graph with nullity \( \eta_K + \eta_H = \eta \).

**Proof:** We prove only part (vi). The proof of other parts follow similarly. Since \( G_1 \) is a cycle with \( \eta_{e_1}(G_1) = 0 \), we see that \( |G_1| = n \), where \( n \) is an even number not divisible by four. So \( G_1 \) is nonsingular. Let \( K \) and \( H \) be singular components of \( G_2 - e_2 \) having nullities \( \eta_K \) and \( \eta_H \) respectively. Given that \( u' \) is a non-core vertex of null spread zero and \( w' \) is a non-core vertex of null spread \(-1\). Let \( e_1 = uw \). The concatenation of \( G_1 \) and \( G_2 \) with respect to \( e_1 = uw \) and \( e_2 = u'w' \) is same as taking coalescence of \( G_1 \) with \( K \) and \( H \) with respect to the end vertices of \( e_1 \) and \( e_2 \) (as in figure 12). Suppose that \( u' \) is the root of \( K \) and \( w' \) is the root of \( H \). First coalesce \( G_1 \) and \( K \) with respect to \( u \) and \( u' \). Since \( u' \) is a non-core vertex of null spread zero, \( G_0 \) is a singular graph with nullity \( \eta_K \), by theorem 1.8. After coalescence the vertex \( w \) of \( G_1 \) becomes a non-core vertex of null spread zero (theorem 1.15). Next take coalescence of \( G_1 \cup K \) and \( H \) with respect to \( w \) and \( w' \). As \( w \) is a non-core vertex of null spread zero and \( w' \) is a non-core vertex of null spread \(-1\), we see by theorem 1.7 that \((G_1 \cup K) \cup H \) is a singular graph with nullity \( \eta_K + \eta_H = \eta \).

**Corollary 3.65.** In the above theorem the concatenated edge has null spread zero.

**Theorem 3.66.** Let \( G_1 \) be a cycle with \( \eta_{e_1}(G_1) = 2 \) and \( G_2 \) be a singular graph with nullity \( \eta \) having a cut edge \( e_2 = u'w' \) and the components of \( G_2 - e_2 \) are singular. Let \( G \) be the concatenation of \( G_1 \) and \( G_2 \) with respect to \( e_1 \) and \( e_2 \).

(i) If \( \eta_{e_2}(G_2) = -2 \), then \( G \) is singular with nullity \( \eta + 2 \).

(ii) If \( \eta_{e_2}(G_2) = -1 \), then \( G \) is singular with nullity \( \eta + 1 \).

(iii) If \( \eta_{e_2}(G_2) = 0 \) and \( u' \), \( w' \) are non-core vertices of null spread zero, then \( G \) is singular with nullity \( \eta \).

**Proof:** Similar to the proof of theorem 3.57.

**Corollary 3.65.** In the above theorem the concatenated edge has null spread zero.
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(iv) If \( \eta_{e_2}(G_2) = 0 \) and \( u', w' \) are noncore vertices of null spread \(-1\), then \( G \) is singular with nullity \( \eta \).
(v) If \( \eta_{e_2}(G_2) = 0 \), \( u' \) is a core vertex and \( w' \) is a noncore vertex of null spread \(-1\), then \( G \) is singular with nullity \( \eta \).
(vi) If \( \eta_{e_2}(G_2) = 0 \), \( u' \) is a noncore vertex of null spread zero and \( w' \) is a noncore vertex of null spread \(-1\), then \( G \) is singular with nullity \( \eta \).

**Corollary 3.67.** In the above theorem, if \( \eta_{e_2}(G_2) = -1 \), then the concatenated edge has null spread one and in all other cases the concatenated edge has null spread zero.

**Theorem 3.68.** Let \( G_1 \) be a cycle with \( \eta_{e_1}(G_1) = 2 \) and \( G_2 \) be a singular graph with nullity \( \eta \) with a cut edge \( e_2 = u'w' \). Assume that one component \( K \) of \( G_2 - e_2 \) is singular with nullity \( \eta \) and the other component \( H \) is non-singular.
(i) If \( \eta_{e_2}(G_2) = -1 \), then \( G \) is singular with nullity \( \eta \).
(ii) If \( \eta_{e_2}(G_2) = 0 \), \( u' \) is a core vertex and \( H - w' \) is singular, then \( G \) is singular with nullity \( \eta \).
(iii) If \( \eta_{e_2}(G_2) = 0 \) and \( u' \) is a noncore vertex (of null spread 0 or \(-1\)), then \( G \) is singular with nullity \( \eta \).

**Proof:** Similar to the proof of theorem 3.57.

**Corollary 3.69.** In the above theorem, if \( \eta_{e_2}(G_2) = -1 \), then the concatenated edge has null spread one and in all other cases the concatenated edge has null spread zero.

We conclude this section with the following two results about the energy of graphs.

**Theorem 3.70.** Let \( G_1 \) be a singular graph having a cycle and \( G_2 \) be a singular graph with a cut edge \( e_2 = u'w' \) and the components of \( G_2 - e_2 \) are singular. Let \( G \) be the concatenation of \( G_1 \) and \( G_2 \) with respect to an edge \( e_1 \) of the cycle of \( G_1 \) and \( e_2 \) of \( G_2 \). If \( G_1 \) is hypoenergetic and the components of \( G_2 - e_2 \) are strongly hypoenergetic, then \( G \) is hypoenergetic.

**Proof:** Let \( |G_1| = n_1 \) and \( |G_2| = n_2 \). Let \( K, H \) be the components of \( G_1 - e_2 \). The concatenation of \( G_1 \) and \( G_2 \) with respect to \( e_1 \) and \( e_2 \) is same as taking coalescence of \( G_1 \) with \( K \) and \( H \) with respect to the end vertices of \( e_1 \) and \( e_2 \). So by theorem 1.22, we have \( E(G) \leq E(G_1) + E(K) + E(H) < |G_1| + |K| - 1 + |H| - 1 = n_1 + n_2 - 2 \).

**Theorem 3.71.** Let \( G_1 \) and \( G_2 \) be singular graphs with nullity \( \eta_1 \) and \( \eta_2 \) respectively and \( G \) be the concatenation of \( G_1 \) and \( G_2 \) with respect to their cut edges \( e_1 \) and \( e_2 \) as \( u = uw \) and \( e_2 = u'w' \). If the components of \( G_1 - e_1 \) and \( G_2 - e_2 \) are strongly hypoenergetic, then \( G \) is hypoenergetic.

**Proof:** Let \( |G_1| = n_1 \) and \( |G_2| = n_2 \). Let \( K, H \) be the components of \( G_1 - e_1 \) and \( K', H' \) be the components of \( G_2 - e_2 \). The concatenation, \( G \) of \( G_1 \) and \( G_2 \) with respect to \( e_1 \) and \( e_2 \) is same as \( (H_0H')(K_0K') + vv' \), where \( v, v' \) are the coalesced vertices and \( H_0H', K_0K' \) are
the coalescence of $H, H'$ and $K, K'$ respectively. So by theorem 1.22, $E(G) \leq E(K) + E(H) + E(K') + E(H') + E(K_2) < |K| - 1 + |H| - 1 + |K'| - 1 + |H'| - 1 + 2 = n_1 + n_2 - 2$.

4. Conclusion
Theory of large graphs are widely applicable not only in mathematics but also in computer science, statistical physics, biology, engineering, and many other fields. Concatenation or edge gluing is a technique used in the construction of larger graphs. In this paper we made a humble attempt to construct a theoretical basis for the study of concatenation of graphs. Some of the basic results are stated and proved using the techniques we have developed in our earlier research. There remains several areas to be explored in the study of spectral properties of concatenated graphs both theoretical and applied.

Acknowledgement. The second author wishes to thank the University Grants Commission, India for awarding fellowship to pursue research under faculty development programme during 12th plan.

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