Results on Relatively Prime Dominating Sets in Graphs

C. Jayasekaran\(^1\) and A. Jancy Vini\(^2\)

\(^1\)Department of Mathematics, Pioneer Kumaraswamy College
Nagercoil-629003, Tamilnadu, India.
E-mail: jaya\_pkc@yahoo.com

\(^2\)Department of Mathematics, Holy Cross College (Autonomous)
Nagercoil-629004, Tamilnadu, India.
E-mail: jancyvini@gmail.com

\(^1\)Corresponding author

Received 24 July 2017; accepted 19 August 2017

Abstract. In this paper we introduce relatively prime dominating set of a graph \(G\). Let \(G\) be a non–trivial graph. A set \(S \subseteq V\) is said to be relatively prime dominating set if it is a dominating set with at least two elements and for every pair of vertices \(u\) and \(v\) in \(S\) such that \((\deg u, \deg v) = 1\). The minimum cardinality of a relatively prime dominating set is called relatively prime domination number and it is denoted by \(\gamma_{rpd}(G)\). If there is no such pair exist then \(\gamma_{rpd}(G) = 0\). We characterize connected unicyclic graphs with \(\gamma_{rpd}(G) = 2\) and also we prove that \(\gamma_{rpd}(K_{m,n}) = 2\) iff \((m, n) = 1\) and \(\gamma_{rpd}(P_n) = 2\) for \(n \geq 4\).

Keywords: Dominating set, relatively prime dominating set, unicyclic graph.

AMS Mathematics Subject Classification (2010): 05C69

1. Introduction

By a graph \(G = (V, E)\) we mean a finite undirected graph without loops or multiple edges. The order and size of \(G\) are denoted by \(p\) and \(q\) respectively. For graph theoretical terms, we refer to Harary [2] and for terms related to domination we refer to Haynes [3]. A subset \(S\) of \(V\) is said to be a dominating set in \(G\) if every vertex in \(V - S\) is adjacent to at least one vertex in \(S\). The domination number \(\gamma(G)\) is the minimum cardinality of a dominating set in \(G\).

Berge and Ore [1,6] formulated the concept of domination in graphs. It was further extended to define many other domination related parameters in graphs.

A graph which contains exactly one cycle is called a unicyclic graph. A branch at \(v\) in \(G\) is a maximal connected subgraph \(B\) of \(G\) such that the intersection of \(B\) with the vertex \(v\) is \(v\) and \(B - v\) is connected [8]. The distance \(d(u, v)\) between two vertices \(u\) and \(v\) in a connected graph \(G\) is the length of a shortest \(u\)-\(v\) path in \(G\). The diameter of a connected graph \(G\) is the maximum distance between two vertices of \(G\) and is denoted by \(diam(G)\). Many other domination parameters in domination theory were studied in [5, 7]. In this paper we define relatively prime dominating set \(\gamma_{rpd}(G)\) and initiate a study of...
C. Jayasekaran and A. Jancy Vini

this parameter. We obtain $\gamma_{rd}(G)$ for various classes of graphs. Now consider the following results, which are required in the subsequent section.

**Theorem 1.1.** [1] For any $G$, 
\[ \frac{p}{1+\Delta(G)} \leq \gamma(G) \leq p - \Delta(G). \]

**Theorem 1.2.** [6] If a graph $G$ has no isolated vertices then $\gamma(G) \leq \frac{p}{2}$.

**Notations.** Consider a cycle $C_r = (v_1, v_2, \ldots, v_r)$ (clock-wise). For our convenience we denote it by $C_{r(v)}$. Identifying an end vertex of paths $P_m$ at $v_i$ and $P_s$ at $v_j$, then $C_{r(v)}$ is denoted by $C_{r(v)}(0, \ldots, P_m, 0, \ldots, P_s, 0, \ldots, 0)$. Identifying an end vertex of paths $P_m$ and $P_s$ at the vertex $v_j$, then $C_{r(v)}$ is denoted by $C_{r(v)}(0, \ldots, P_m \cup P_s, 0, \ldots, 0)$.

The graphs $C_{4(v)}(0, 0, P_2, P_3)$, $C_{4(v)}(0, 2P_2 \cup P_3, 0, 0)$, $C_{4(v)}(0, 2P_2 \cup P_3, P_2, P_3)$ and $C_{3(v)}(4P_2, 2P_2, 3P_2)$ are given in figure 1.

![Diagram](image)

**Figure 1:**

**Result 1.3.** For $n \geq 2$, $\gamma(P_n) = 2$.

2. Definition and example

**Definition 2.1.** A set $S \subseteq V$ is said to be relatively prime dominating set if it is a dominating set with at least two elements and for every pair of vertices $u$ and $v$ in $S$ such
that \((\deg u, \deg v) = 1\). The minimum cardinality of a relatively prime dominating set is called relatively prime domination number and it is denoted by \(\gamma_{rpd}(G)\).

**Example 2.2.** Consider the connected graph \(G\) given in figure 2. Clearly \(\{v_1, v_4, v_6\}\) is a minimal dominating set and \((d(v_1), d(v_4)) = (1, 3) = 1, (d(v_1), d(v_6)) = (1, 5) = 1\) and \((d(v_4), d(v_6)) = (3, 5) = 1\). By definition, \(\{v_1, v_4, v_6\}\) is a relatively prime dominating set and hence \(\gamma_{rpd}(G) = 3\). Also \(\gamma(G) = \gamma_{rpd}(G) = 3\).

![Figure 2: The graph G](image)

**Example 2.3.** Consider the disconnected graph \(G\) given in figure 3. Clearly \(\{v_1, v_3\}\) is a dominating set and \((d(v_1), d(v_3)) = (1, 2) = 1\). By definition, \(\{v_1, v_3\}\) is a relatively prime dominating set and hence \(\gamma_{rpd}(G) = 2\).

![Figure 3:](image)

**Example 2.4.** For \(n = 2\), \(\gamma_{rpd}(K_n) = 2\) and for \(n > 2\), \(\gamma_{rpd}(K_n) = 0\).

**Observations 2.5.**

1. For any graph \(G\) of order at least 2, \(\gamma_{rpd}(G) \geq 0\) and \(\gamma_{rpd}(G) \neq 1\).
2. If \(\gamma_{rpd}(G) \neq 0\) then \(\gamma(G) \leq \gamma_{rpd}(G)\).
3. For any \(k\)-regular graph \((k > 1) G\), \(\gamma_{rpd}(G) = 0\).

**Example 2.6.** Consider the connected graph \(G\) given in figure 4. Clearly \(\{v_2, v_3\}\) is a minimal dominating set and hence \(\gamma(G) = 2\). Also \(\{v_1, v_3, v_5\}\) is a minimal relatively prime dominating set and hence \(\gamma_{rpd}(G) = 3 > \gamma(G)\).

![Figure 4:](image)
C. Jayasekaran and A. Jancy Vini

Figure 4:

Definition 2.7. [6] A branch at \( v \) in \( G \) is a maximal connected subgraph \( B \) of \( G \) such that the intersection of \( B \) with the vertex \( v \) is \( v \) and \( B-v \) is connected.

Notation 2.8. [6] Let \( v \) be a cut vertex of a connected graph \( G \). Let \( B_1, B_2, \ldots, B_k \) be the branches with \( n_1, n_2, \ldots, n_k \) number of copies at \( v \) in \( G \), respectively. In this case we denote the graph \( G \) by \( G(v; n_1B_1, n_2B_2, \ldots, n_kB_k) \).

Example 2.9. Consider the graph \( G \) given in figure 5.0. There are four distinct branches \( B_1, B_2, B_3 \) and \( B_4 \) at \( v \) in \( G \) and they are given in figures 5.1, 5.2, 5.3 and 5.4, respectively. Therefore, \( G = G(v; 2B_1, B_2, B_3, B_4) \).

3. Main results

Theorem 3.1. For a connected unicyclic graph \( G \), \( \gamma_{rd}(G) = 2 \) iff \( G \) is one of the graphs

1. \( C_{3(n)}(mP_2, nP_2, 0) \) where \((m+2, n+2) = 1 \) and \( m, n \geq 1 \)
2. \( C_{3(n)}(K_{1, m}, nP_2, 0) \) where \((m, n+2) = 1 \) and \( m \geq 2, n \geq 1 \)
3. \( C_{3(n)}(mP_2, 0, 0) \) and \( m \geq 1 \)
4. \( C_{3(n)}(K_{1, m}, 0, 0) \) where either \( m \) is odd or even but not a multiple of 3
5. \( C_{3(n)}(mP_2 \cup K_{1, m}, 0, 0) \) where \((m+3, n) = 1 \) and \( m \geq 1, n \geq 2 \)
Results on Relatively Prime Dominating Sets in Graphs

(6) \( C_{4(v_1)} (mP_2, nP_2, 0, 0) \) where \((m+2, n+2) = 1 \) and \( m, n \geq 1 \)

(7) \( C_{4(v_1)} (mP_2, 0, nP_2, 0) \) where \((m+2, n+2) = 1 \) and \( m, n \geq 1 \)

(8) \( C_{4(v_1)} (mP_2, 0, K_{1,n}, 0) \) where \((m+2, n+1) = 1 \) \( m \geq 1 \) and \( n \geq 2 \)

(9) \( C_{4(v_1)} (mP_2, 0, 0, 0) \) where \( m \) is odd

(10) \( C_{4(v_1)} (K_{1,m}, 0, 0, 0) \) where \( m \) is odd

(11) \( C_{3(v_1)} (mP_2, 0, nP_2, 0, 0) \) where \((m+2, n+2) = 1 \) and \( m, n \geq 1 \)

(12) \( C_{3(v_1)} (mP_2, 0, 0, 0, 0) \) where \( m \) is odd

(13) \( C_{6(v_1)} (mP_2, 0, 0, nP_2, 0, 0) \) where \((m+2, n+2) = 1 \) and \( m, n \geq 1 \)

(14) \( C_{6(v_1)} (mP_2, 0, 0, 0, 0, 0) \) where \((m+2, 2) = 1 \) and \( m \) is odd

**Proof:** Let \( G \) be a connected unicyclic graph with cycle \( C_n \). Let \( \gamma_{rpd}(G) = 2 \). If \( n \geq 7 \) then the cardinality of every dominating set is \( \geq 3 \) and hence \( n \leq 6 \).

**Case 1.** \( n = 3 \)

Let \( v_1v_2v_3 \) be the cycle \( C_3 \). If \( d(v_i) \geq 3, 1 \leq i \leq 3 \), then \( \gamma_{rpd}(G) \geq 3 \). Hence at most two vertices, say \( v_i \) and \( v_j \), can have degree \( \geq 3 \).

**Subcase 1(a).** \( d(v_1) \geq 3 \) and \( d(v_2) \geq 3 \)

In this case, \( d(v_3) = 2 \). Since \( G \) is unicyclic graph, the branches at \( v_1 \) and \( v_2 \) are trees other than the branch which contains the cycle \( C_3 \). If all branches are \( P_2 \) then \( G = C_{3(v_1)} (mP_2, nP_2, 0) \) where \( m \) and \( n \) are the number of \( P_2 \)'s at \( v_1 \) and \( v_2 \), respectively. If \((m+2, n+2) = 1 \) then \( \{v_1, v_2\} \) is a relatively prime dominating set of \( G \). Thus \( G = C_{3(v_1)} (mP_2, nP_2, 0) \) where \((m+2, n+2) = 1 \) and \( m, n \geq 1 \).

Clearly at most one branch can be different from \( P_2 \), otherwise \( \gamma_{rpd}(G) \geq 3 \). Let \( B \neq P_2 \) be the tree branch at \( v_1 \). Let \( u \) be the vertex adjacent to \( v_1 \) in \( B \). Since \( d(v_2) \geq 3 \), any relatively prime dominating set must contain \( v_2 \). If \( d(v_1) \geq 4 \) then \( \gamma_{rpd}(G) \geq 3 \). Hence \( d(v_1) = 3 \). Since \( \gamma_{rpd}(G) = 2 \) and \( d(u) \geq 2 \), any relatively prime dominating set must contain \( u \). Clearly all branches at \( u \) are \( P_2 \)'s other than the branch which contains the cycle \( C_3 \). In this case \( G = C_{3(v_1)} (K_{1,m}, nP_2, 0) \). If \( (m, n+2) = 1 \) then \( \{u, v_2\} \) is a relatively prime dominating set of \( G \). Thus \( G = C_{3(v_1)} (K_{1,m}, nP_2, 0) \) where \((m, n+2) = 1 \) and \( m \geq 2, n \geq 1 \).

**Subcase 1(b).** \( d(v_1) \geq 3 \)

In this case, \( d(v_2) = d(v_3) = 2 \). If all branches at \( v_1 \) other than the cyclic branch are \( P_2 \), then \( \{u, v_1\} \) is a relatively prime dominating set of \( G \) where \( u \) is a vertex adjacent to \( v_1 \) in \( P_2 \). In this case \( G = C_{3(v_1)} (mP_2, 0, 0) \) where \( m \geq 1 \).
C. Jayasekaran and A. Jancy Vini

Suppose at least two branches at \( v_1 \) other than the cyclic branches are not \( P_2 \) then \( \gamma_{rpm}(G) \geq 3 \). Hence exactly one tree branch, say \( B \) at \( v_1 \) is not \( P_2 \). Let \( u \) be the vertex adjacent to \( v_1 \) in \( B \). If \( d(u) \geq 2 \) then each branch at \( u \) is \( P_2 \) other than the cyclic branch which contains \( C_3 \) otherwise \( \gamma_{rpm}(G) \geq 3 \). In this case \( G = C_{3(v_1)}(K_{1,m}, 0, 0) \). Clearly \( \{u, v_1\} \) and \( \{u, v_2\} \) and \( \{u, v_3\} \) are minimal dominating sets. If \( m \) is odd then \( (m, 2) = 1 \) and if \( m \) is even and \( m \) is not a multiple of 3 then \( (m, 3) = 1 \). Thus \( G = C_{3(v_1)}(K_{1,m}, 0, 0) \) when \( m \) is odd or \( m \) is even but not a multiple of 3.

If \( G \) has more than one tree branch at \( v_1 \) in \( G \) then \( G = C_{3(v_1)}(mP_2 \cup K_{1,n}, 0, 0) \). If \( (m+3, n) = 1 \) then \( \{v_1, u\} \) is a relatively prime dominating set of \( G \). Thus \( G = C_{3(v_1)}(mP_2 \cup K_{1,n}, 0, 0) \) where \( m \geq 1, n \geq 2 \).

**Case 2.** \( n = 4 \)

Let \( v_1 v_2 v_3 v_4 v_1 \) be the cycle \( C_4 \). If \( d(v_i) \geq 3 \) for any three vertices \( v_i \) then \( \gamma_{rpm}(G) \geq 3 \) and hence at most two vertices can have degree \( \geq 3 \).

**Subcase 2.1.** Two vertices have degree \( \geq 3 \)

Here we consider the two sub cases either they are adjacent or non-adjacent.

**Subcase 2.1(a) Adjacent**

Let the vertices be \( v_1 \) and \( v_2 \). Then \( d(v_1) \geq 3 \) and \( d(v_2) \geq 3 \). In this case \( d(v_3) = d(v_4) = 2 \). Since \( G \) is unicycle graph, the branches at \( v_1 \) and \( v_2 \) are trees other than the branch which contains the cycle \( C_4 \). Suppose \( B \neq P_2 \) be a branch at \( v_1 \) in \( G \). Then any dominating set has at least one vertex from \( B \) other than \( v_1 \), the vertex \( v_2 \), since \( d(v_2) \geq 3 \) and \( v_3 \). This implies that \( \gamma_{rpm}(G) \geq 3 \) and hence each branch at \( v_1 \) is \( P_2 \). Similarly, each branch of \( v_2 \) is \( P_2 \).

Hence \( G = C_{4(v_1)}(mP_2, nP_2, 0, 0) \) where \( m \) and \( n \) are the number of \( P_2 \)'s at \( v_1 \) and \( v_2 \), respectively. Clearly \( \{v_1, v_2\} \) is a minimal dominating set. If \( (m+2, n+2) = 1 \) then \( \{v_1, v_2\} \) is a relatively prime dominating set. Thus \( G = C_{4(v_1)}(mP_2, nP_2, 0, 0) \) where \( (m+2, n+2) = 1 \) and \( m, n \geq 1 \).

**Subcase 2.1(b) Non-adjacent**

Let the vertices be \( v_1 \) and \( v_3 \). Then \( d(v_1) \geq 3 \) and \( d(v_3) \geq 3 \). In this case \( d(v_2) = d(v_4) = 2 \). Since \( G \) is unicycle graph, the branches at \( v_1 \) and \( v_3 \) are trees other than the branch which contains the cycle \( C_4 \). If all the branches at \( v_1 \) and \( v_3 \) are \( P_2 \)'s other than the branch which contains the cycle \( C_4 \) then \( G = C_{4(v_1)}(mP_2, 0, nP_2, 0) \). If \( (m+2, n+2) = 1 \) then \( \{v_1, v_3\} \) is a relatively prime dominating set. Thus \( G = C_{4(v_1)}(mP_2, 0, nP_2, 0) \) where \( (m+2, n+2) = 1 \) and \( m, n \geq 1 \).

Suppose there are two tree branches \( B_1 \) and \( B_2 \) which are not \( P_2 \) either at \( v_1 \) or \( v_2 \) or both. Then any dominating set has at least one vertex from each of the branches \( B_1 \) and \( B_2 \) other than \( v_1 \) and \( v_3 \) and at least two vertices from \( \{v_1, v_2, v_3, v_4\} \) and hence
\( \gamma_{\text{rpd}}(G) \geq 4 \) which is a contradiction. Therefore, at most one tree branch say \( B \neq P_2 \) at \( v_3 \) in \( G \). If there is a branch \( P_2 \) other than \( B \) at \( v_3 \) in \( G \) then any dominating set has the vertices \( v_1, v_3 \) and at least one vertex from \( V(B) - v_3 \) and hence \( \gamma_{\text{rpd}}(G) \geq 3 \) which is a contradiction. Therefore, exactly one tree branch namely \( B \neq P_2 \) at \( v_3 \) in \( G \). Let \( u \) be the vertex adjacent to \( v_3 \) in \( G \). Since \( d(v_i) \geq 3 \) and all branches at \( v_1 \) are \( P_2 \), any minimal dominating set contains the vertex \( v_1 \). Since \( \gamma_{\text{rpd}}(G) = 2 \) and \( d(u) \geq 2 \) the minimal dominating set is \( \{v_1, u\} \) and all the vertices adjacent to \( u \) other than \( v_3 \) are end vertices and hence \( B = K_{1,n}, n \geq 2 \). This implies that \( G = C_{4} \) \((mP_2, 0, K_{1,n}, 0)\). If \( (m+2, n+1) = 1 \) then \( \{u, v_1\} \) is a relatively prime dominating set. Thus \( G = C_{4} \) \((mP_2, 0, K_{1,n}, 0)\) where \( (m+2, n+1) = 1 \) \( m \geq 1 \) and \( n \geq 2 \).

**Subcase 2.2.** One vertex has degree \( \geq 3 \)

Let the vertex be \( v_1 \) with \( d(v_1) \geq 3 \). Then \( d(v_2) = d(v_3) = d(v_4) = 2 \). If \( \text{diam}(G) \geq 6 \) then cardinality of every dominating set is \( \geq 3 \). Therefore, \( \text{diam}(G) \leq 5 \). Since \( d(v_1) \geq 3 \), \( \text{diam}(G) \geq 4 \). Therefore, \( \text{diam}(G) = 4 \) or \( 5 \). If \( \text{diam}(G) = 4 \) then any branch \( B \) at \( v_1 \) other than the cyclic branch is \( P_2 \). Let \( m \) be the number of \( P_2 \)'s at \( v_1 \). Then \( G = C_{4} \) \((mP_2, 0, 0, 0)\). If \( (m+2, 2) = 1 \) then \( m \) is odd and hence \( \{v_1, v_3\} \) is a relatively prime dominating set. Thus \( G = C_{4} \) \((mP_2, 0, 0, 0)\) where \( m \) is odd. Suppose \( \text{diam}(G) = 5 \) and \( d(v_1) > 3 \). Since \( G \) is unicyclic, each branch at \( v_1 \) other than \( C_4 \) is a tree. Since \( \text{diam}(G) = 5 \), one tree branch at \( v_1 \) must be \( K_{1, m} \) \((m \geq 2)\) with centre \( u \neq v_1 \). If \( d(v_1) > 3 \) then any minimal dominating set has \( u, v_1 \) and one of \( v_2, v_3 \) and \( v_4 \) and hence \( \gamma_{\text{rpd}}(G) > 2 \). This implies that \( d(v_1) = 3 \) and \( G = C_{4} \) \((K_{1, m}, 0, 0, 0)\), \( m \geq 2 \). Clearly, \( \{u, v_3\} \) is a minimal dominating set with \( d(u) = m \) and \( d(v_3) = 2 \). If \( m \) is odd then \( (m, 2) = 1 \) and hence \( \{u, v_3\} \) is a relatively prime dominating set and \( \gamma_{\text{rpd}}(G) = 2 \).

**Case 3.** \( n = 5 \)

Let \( v_1v_2v_3v_4v_5v_1 \) be the cycle \( C_5 \). If \( d(v_i) \geq 3 \) for at least 3 vertices \( v_i \) \((i = 1 \to 5)\) then \( \gamma_{\text{rpd}}(G) \geq 3 \) and hence at most two \( v_i \)'s can have degree \( \geq 3 \).

**Subcase 3.1.** Two vertices have degree \( \geq 3 \)

Let \( v_i \) and \( v_j \) \((i \neq j)\) be the vertices such that \( d(v_i) \geq 3 \) and \( d(v_j) \geq 3 \), \( 1 \leq i, j \leq 5 \). Since \( G \) is unicyclic, each branch at \( v_i \) and \( v_j \) is a tree. Suppose \( B \neq P_2 \) is a branch at \( v_i \). Then any minimal dominating set has two vertices from \( \{v_1, v_2, v_3, v_4, v_5\} \) and one vertex from \( V(B) - v_i \) and hence \( \gamma_{\text{rpd}}(G) \geq 3 \) which is a contradiction. Therefore, each branch at \( v_i \) is \( P_2 \).

Similarly each branch at \( v_j \) is \( P_2 \). If \( d(v_i, v_j) = 1 \) then \( \gamma_{\text{rpd}}(G) \geq 3 \) which is a contradiction. Hence \( d(v_i, v_j) \) is either 2 or 3. Without loss of generality we may assume that \( d(v_i, v_j) = 2 \) and let \( v_i = v_1 \) and \( v_j = v_5 \). This implies that \( G = C_{5} \) \((mP_2, 0, nP_2, 0, 0)\). If \( (m+2, n+2) = 1 \) then \( \{v_1, v_5\} \) is a relatively prime dominating set of \( G \). Thus \( G = C_{5} \) \((mP_2, 0, nP_2, 0, 0)\) where \( (m+2, n+2) = 1 \) and \( m, n \geq 1 \).
Subcase 3.2. One vertex has degree $\geq 3$
Without loss of generality, let $d(v_i) \geq 3$. Then each branch at $v_i$ other than the branch which contains $C_4$ is $P_2$. Then $G = C_{S(v_i)}(mP_2, 0, 0, 0, 0)$. If $m$ is odd then $\{v_1, v_3\}$ and $\{v_1, v_4\}$ are relatively prime dominating sets of $G$. Thus $G = C_{S(v_i)}(mP_2, 0, 0, 0, 0)$ and $m$ is odd.

Case 4. $n = 6$
Let $v_1v_2v_3v_4v_5v_6v_1$ be the cycle $C_6$. If $d(v_i) \geq 3$ for at least 3 vertices $v_i$ $(i = 1$ to $6)$ then $r_{pd}(G) \geq 3$ and hence at most two $v_i$’s can have degree $\geq 3$.

Subcase 4.1. Two vertices have degree $\geq 3$
Let $v_i$ and $v_j$ $(i \neq j)$ be the vertices such that $d(v_i) \geq 3$ and $d(v_j) \geq 3$, $1 \leq i \neq j \leq 6$. Clearly any minimal dominating set must contain $v_i$ and $v_j$. If $d(v_i, v_j) = 1$ or $2$, then $r_{pd}(G) \geq 3$ which is a contradiction. Hence $d(v_i, v_j) = 3$ and let $v_i = v_1$ and $v_j = v_4$. Clearly all branches at $v_1$ and $v_4$ are $P_2$’s other than the cyclic branch otherwise $r_{pd}(G) \geq 3$ and the only dominating set is $\{v_1, v_4\}$. This implies that $G = C_{6(v_i)}(mP_2, 0, 0, nP_2, 0, 0)$. If $(m+2, n+2) = 1$ then $\{v_1, v_4\}$ is a relatively prime dominating set of $G$. Thus $G = C_{6(v_i)}(mP_2, 0, 0, nP_2, 0, 0)$ where $(m+2, n+2) = 1$ and $m, n \geq 1$.

Subcase 4.2. One vertex has degree $\geq 3$
Without loss of generality, let $d(v_1) \geq 3$. Clearly each branch at $v_1$ other than the branch which contains $C_6$ is $P_2$. This implies that $G = C_{6(v_i)}(mP_2, 0, 0, 0, 0, 0)$. If $m$ is odd then $\{v_1, v_4\}$ is a relatively prime dominating set of $G$. Thus $G = C_{6(v_i)}(mP_2, 0, 0, 0, 0, 0)$ and $m$ is odd.

Conversely, let $G$ be the graph given in the statement. Then for each graph $G$, $r_{pd}(G) = 2$.

Theorem 3.2. For a complete bipartite graph $K_{m,n}$, $r_{pd}(K_{m,n}) = 2$ iff $(m, n) = 1$.
Proof: Let $V_1, V_2$ be the bipartition of the vertex set of $K_{m,n}$ with $|V_1| = m$ and $|V_2| = n$. Clearly, $d(u) = n$ and $d(v) = m$ for $u \in V_1$ and $v \in V_2$. Any minimum dominating set of $K_{m,n}$ has one vertex in $V_1$ and another vertex in $V_2$. Hence a minimum dominating set of $K_{m,n}$ becomes a relatively prime dominating set iff $(m, n) = 1$. Therefore, $r_{pd}(K_{m,n}) = 2$ iff $(m, n) = 1$.

Example 3.3. For $K_{3,4}$, $\{v_1, u_1\}$ is a relatively prime dominating set.
Results on Relatively Prime Dominating Sets in Graphs

\[ u_1 \quad u_2 \quad u_3 \quad u_4 \]

\begin{figure}[h]
\centering
\includegraphics[scale=0.5]{K3_4}
\caption{K\textsubscript{3,4}}
\end{figure}

\textbf{Theorem 3.4.} \( \gamma_{rpd}(P_n) = \begin{cases} 2 & \text{if } 2 \leq n \leq 5 \\ 3 & \text{if } n = 6, 7 \\ 0 & \text{otherwise} \end{cases} \)

\textbf{Proof:} Let \( v_1 v_2 \ldots v_n \) be a path \( P_n \).

\textbf{Case 1.} \( 2 \leq n \leq 5 \)
If \( n = 2 \) then \( \{v_1, v_2\} \) is the required minimal relatively prime dominating set and hence \( \gamma_{rpd}(P_n) = 2 \). Let \( n > 2 \). In this case \( \{v_1, v_{n-1}\} \) is a dominating set. Also \( (d(v_1), d(v_{n-1})) = (1, 2) = 1 \). Therefore, \( \{v_1, v_{n-1}\} \) is a relatively prime dominating set and hence \( \gamma_{rpd}(P_n) = 2 \).

\textbf{Case 2.} \( n = 6, 7 \)
In this case \( \{v_1, v_4, v_n\} \) is a dominating set. Also \( (d(v_1), d(v_4)) = (1, 2) = 1 \), \( (d(v_1), d(v_n)) = (1, 1) = 1 \) and \( (d(v_4), d(v_n)) = (2, 1) = 1 \). Therefore, \( \{v_1, v_4, v_n\} \) is a relatively prime dominating set and hence \( \gamma_{rpd}(P_n) = 3 \).

\textbf{Case 3.} \( n \geq 8 \)
Clearly any dominating set contains at least two internal vertices \( v_i, v_j \), \( 2 \leq i \neq j \leq n-1 \) and \( (d(v_i), d(v_j)) = 2 \) which implies that \( \gamma_{rpd}(P_n) = 0 \). The theorem follows from above three cases.

\textbf{Theorem 3.5.} \( \gamma_{rpd}(\overline{P_n}) = \begin{cases} 2 & \text{if } n \geq 3 \\ 0 & \text{otherwise} \end{cases} \)

\textbf{Proof:} If \( n = 2 \) then \( \overline{P_2} = K_2 \) which is a regular graph of degree 0 and hence \( \gamma_{rpd}(\overline{P_2}) = 0 \). If \( n = 3 \) then \( \overline{P_3} = K_2 \cup K_1 \) and hence \( \gamma_{rpd}(\overline{P_3}) = 2 \). Let \( n \geq 4 \). Let \( v_1 v_2 \ldots v_n \) be a path \( P_n \). In \( \overline{P_n} \), \( v_i \) is adjacent to all vertices except \( v_2 \). Clearly \( \{v_1, v_2\} \) is a dominating set of \( \overline{P_n} \). In \( \overline{P_n} \), \( v_1 \) has degree \( n-2 \) and \( v_2 \) has degree \( n-3 \). Since \( (n-2, n-3) = 1 \), \( \{v_1, v_2\} \) is a relatively prime dominating set for \( \overline{P_n} \) and hence \( \gamma_{rpd}(\overline{P_n}) = 2 \) for \( n \geq 4 \). Thus, the theorem is proved.

\textbf{Theorem 3.6.} If \( G_1 \cong G_2 \) then \( \gamma_{rpd}(G_1) = \gamma_{rpd}(G_2) \).

\textbf{Proof:} Let \( G_1 \cong G_2 \). Let \( f \) be an isomorphism between graphs \( G_1 \) and \( G_2 \). Let \( V(G_1) = \{v_1, v_2, \ldots, v_n\} \). Since \( f : V(G_1) \to V(G_2) \) is a bijection, let \( V(G_2) = \{f(v_1), f(v_2), \ldots, f(v_n)\} \). Let \( \{v_1, v_2, \ldots, v_m\} \) be a relatively prime dominating set of \( G_1 \). Since \( f \) is an isomorphism,
{f(v_1), f(v_2), ..., f(v_m)} is a dominating set of G_2. Since isomorphism preserves degrees of the vertices, \((d(f(v_i)), d(f(v_j))) = (d(v_i), d(v_j)) = 1\) for \(i \neq j, 1 \leq i \leq j \leq m\). Therefore, \{f(v_1), f(v_2), ..., f(v_m)\} is a relatively prime dominating set of G_2 and hence \(\gamma_{rpd}(G) = \gamma_{rpd}(G_2)\).

**Note 3.7.** Converse of theorem 3.7 is not true. For example, consider the graphs G_1 and G_2 given in figure 7 and figure 8, respectively. Here \(\gamma_{rpd}(G_1) = \gamma_{rpd}(G_2) = 2\), but the two graphs are not isomorphic.

**4. Conclusion**

In this paper, we surveyed selected results on relatively prime dominating sets in graphs. These results establish key relationships between the relatively prime numbers and the dominating sets in graphs. Further we characterize connected unicyclic graphs with \(\gamma_{rpd}(G) = 2\). In Theorem 3.2, we prove that for a complete bipartite graph \(K_{m,n}\), the relatively prime domination number is 2 iff \((m, n) = 1\). We also extend the results for \(P_n\) and \(\overline{P}_n\). Finally, we have proved that if two graphs are isomorphic then their relatively prime domination numbers must be same.

**Acknowledgement.** The authors are thankful to the reviewer for their comments and suggestions for improving the quality of this paper.

**REFERENCES**

Results on Relatively Prime Dominating Sets in Graphs

