A sum divisor cordial labeling of a graph $G$ with vertex set $V(G)$ is a bijection $f$ from $V(G)$ to $\{1, 2, \ldots, |V(G)|\}$ such that each edge $uv$ assigned the label 1 if $2$ divides $f(u) + f(v)$ and 0 otherwise. Further, the number of edges labeled with 0 and the number of edges labeled with 1 differ by at most 1. A graph with a sum divisor cordial labeling is called a sum divisor cordial graph. In this paper, we prove that Herschel graph $H_s$, fusion of any two adjacent vertices of degree 3 in a Herschel graph $H_s$, duplication of any vertex of degree 3 in a Herschel graph $H_s$, switching of a central vertex in the Herschel graph $H_s$, path union of two copies of $H_s$ are sum divisor cordial graphs.

**Keywords:** Divisor cordial labeling, sum divisor cordial labeling, fusion, duplication, switching, path union.

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**1. Introduction**

By a graph, we mean a finite undirected graph without loops or multiple edges. For standard terminology and notations related to graph theory we refer to Harary [3]. A labeling of graph is a map that carries the graph elements to the set of numbers, usually to the set of non-negative or positive integers. If the domain is the set of edges, then we speak about edge labeling. If the labels are assigned to both vertices and edges, then the labeling is called total labeling. Cordial labeling is extended to divisor cordial labeling, prime cordial labeling, total cordial labeling, Fibonacci cordial labeling etc.

For dynamic survey of various graph labeling, we refer to Gallian [1]. Lourdusamy and Patrick [5] introduced the concept of sum divisor cordial labeling. Sugumaran and Rajesh [7] proved that Swastik graph $Sw_n$, path union of finite copies of Swastik graph $Sw_n$, cycle of $k$ copies of Swastik graph $Sw_n$ ($k$ is odd), Jelly fish $J(n,n)$ and Petersen graph are sum divisor cordial graphs. Sugumaran and Rajesh [8] proved that the Theta graph and some graph operations in Theta graph are sum divisor cordial graphs. Ganesan and Balamurugan [2] have discussed the prime labeling of Herschel graph. Our primary objective of this paper is to prove the Herschel graph and some graph operations in Herschel graph namely fusion, duplication, switching of a central vertex, path union of two copies of Herschel graphs are sum divisor cordial graphs.

**Definition 1.1.** [10] Let $G = (V(G), E(G))$ be a simple graph and let $f: V(G) \to \{1, 2, \ldots, |V(G)|\}$ be a bijection. For each edge $uv$, assign the label $1$ if either $f(u)|f(v)$ or $f(v)|f(u)$ and the label $0$ otherwise. The function $f$ is called a divisor cordial labeling if $|e_f(0) - e_f(1)| \leq 1$. A graph which admits a divisor cordial labeling is called a divisor cordial graph.

**Definition 1.2.** [5] Let $G = (V(G), E(G))$ be a simple graph and let $f: V(G) \to \{1, 2, \ldots, |V(G)|\}$ be a bijection. For each edge $e = uv$, assign the label $1$ if either $2|f(u) + f(v))$ and assign the label $0$ otherwise. The function $f$ is called a sum divisor cordial labeling if $|e_f(0) - e_f(1)| \leq 1$. A graph which admits a sum divisor cordial labeling is called a sum divisor cordial graph.

**Definition 1.3.** A *Herschel graph* $H_s$ is a bipartite undirected graph with 11 vertices and 18 edges.

![Herschel graph](image)

**Figure 1:** Herschel graph $H_s$

In this paper, we always fix the position of vertices $v, u_1, u_2, \ldots, u_{10}$ of $H_s$ as indicated in the above Figure 1, unless or otherwise specified.

**Definition 1.4.** Let $u$ and $v$ be two distinct vertices of a graph $G$. A new graph $G_1$ is constructed by fusing (identifying) two vertices $u$ and $v$ by a single vertex $x$ in $G_1$. 466
such that every edge which was incident with either \( u \) (or) \( v \) in \( G \) now incident with \( x \) in \( G_1 \).

**Definition 1.5.** Duplication of a vertex \( v_k \) of a graph \( G \) produces a new graph \( G_1 \) by adding a vertex \( v_k' \) with \( N(v_k) = N(v_k') \). In other words, a vertex \( v_k' \) is said to be a duplication of \( v_k \) if all the vertices which are adjacent to \( v_k \) are now adjacent to \( v_k' \).

**Definition 1.6.** A vertex switching \( G_v \) of a graph \( G \) is obtained by taking a vertex \( v \) of \( G \), removing the entire edges incident with \( v \) and adding edges joining \( v \) to every vertex which are non-adjacent to \( v \) in \( G \).

**Definition 1.7.** [4] Let \( G \) be a graph and let \( G_1 = G_2 = \cdots = G_n = G \), where \( n \geq 2 \). Then the graph obtained by adding an edge from each \( G_i \) to \( G_{i+1} \) (\( 1 \leq i \leq n - 1 \)) is called the path union of \( G \).

2. Main results

**Theorem 2.1.** The Herschel graph \( H_s \) is a sum divisor cordial graph.

**Proof:** Let \( G = H_s \) be a Herschel graph and let \( v \) be the central vertex and \( u_i (1 \leq i \leq 10) \) be the remaining vertices of the Herschel graph. Then \( |V(G)| = 11 \) and \( |E(G)| = 18 \). We define the vertex labeling \( f: V(G) \to \{1, 2, \ldots , |V(G)|\} \) as follows.
\[
\begin{align*}
f(v) &= 11, \\
f(u_1) &= 1, \\
f(u_2) &= 4, \\
f(u_i) &= i; \quad i = 3, 9, 10, \\
f(u_i) &= i + 1; \quad i = 1, 4, 5, 6, 7.
\end{align*}
\]
From the above labeling pattern, we have \( e_f(0) = e_f(1) = 9 \).

Hence \( |e_f(0) - e_f(1)| \leq 1 \).

Thus \( G \) is a sum divisor cordial graph.

**Example 2.1.** The sum divisor cordial labeling of Herschel graph \( H_s \) is shown in Figure 2.

![Figure 2: Sum divisor cordial labeling of Herschel graph \( H_s \).](image-url)
Theorem 2.2. The fusion of any two adjacent vertices of degree 3 in the Herschel graph is a sum divisor cordial graph.

Proof: Let \( H_s \) be the Herschel graph with \(|V(H_s)| = 11\) and \(|E(H_s)| = 18\). Let \( v \) be the central vertex of the Herschel graph and it has 3 vertices of degree 4 and 8 vertices of degree 3. Let \( G \) be the graph obtained by fusion of two adjacent vertices of degree 3 in the Herschel graph of \( H_s \). Then \(|V(G)| = 10\) and \(|E(G)| = 17\).

We define the vertex labeling \( f : V(G) \to \{1, 2, ..., |V(G)|\} \) as follows.

Case 1. Fusion of \( u_6 \) and \( u_{10} \).
Suppose that \( u_6 \) and \( u_{10} \) are fused together as a single vertex \( u \).
\[
\begin{align*}
  f(v) &= 1, \\
  f(u_6) &= 7, \\
  f(u) &= 4, \\
  f(u_i) &= i + 1; \ i = 1, 2, 4, 5, 7, 8, 9.
\end{align*}
\]

Case 2. Fusion of \( u_6 \) and \( u_2 \).
Suppose that \( u_6 \) and \( u_2 \) are fused together as a single vertex \( u \).
\[
\begin{align*}
  f(v) &= 1, \quad f(u_4) = 7, \quad f(u_{10}) = 4, \\
  f(u) &= 2, \\
  f(u_i) &= i + 1; \ i = 5, 7, 8, 9. \\
  f(u_i) &= i + 2; \ i = 1, 3.
\end{align*}
\]

Case 3. Fusion of \( u_6 \) and \( u_3 \).
Suppose that \( u_6 \) and \( u_3 \) are fused together as a single vertex \( u \).
\[
\begin{align*}
  f(v) &= 1, \quad f(u_9) = 7, \quad f(u_{10}) = 4, \\
  f(u) &= 2, \\
  f(u_i) &= i; \ i = 7, 8. \\
  f(u_i) &= i + 1; \ i = 2, 4, 5, 9.
\end{align*}
\]

Case 4. Fusion of \( u_5 \) and \( u_9 \).
Suppose that \( u_5 \) and \( u_9 \) are fused together as a single vertex \( u \).
\[
\begin{align*}
  f(v) &= 1, \quad f(u_4) = 7, \\
  f(u) &= 4, \\
  f(u_i) &= i; \ i = 6, 10. \\
  f(u_i) &= i + 1; \ i = 1, 2, 4, 7, 8.
\end{align*}
\]

Case 5. Fusion of \( u_5 \) and \( u_4 \).
Suppose that \( u_5 \) and \( u_4 \) are fused together as a single vertex \( u \).
\[
\begin{align*}
  f(v) &= 1, \quad f(u_9) = 7, \quad f(u_5) = 5.
\end{align*}
\]
Sum Divisor Cordial Labeling of Herschel Graph

\[ f(u) = 4, \]
\[ f(u_i) = i; \ i = 6,10. \]
\[ f(u_i) = i+1; \ i = 1,2,7,8. \]

**Case 6.** Fusion of \( u_5 \) and \( u_i \).

Suppose that \( u_5 \) and \( u_i \) are fused together as a single vertex \( u \).

\[ f(v) = 1, \ f(u_i) = 7, \ f(u_6) = 4, \]
\[ f(u) = 2, \]
\[ f(u_i) = i; \ i = 6,10. \]
\[ f(u_i) = i+1; \ i = 2,4,7,8. \]

From all the above cases, we have \( |e_f(0) - e_f(1)| \leq 1 \).

Thus \( G \) is a sum divisor cordial graph.

**Example 2.2.** The sum divisor cordial labeling of fusion of \( u_6 \) and \( u_{10} \) in \( H_s \) is shown in Figure 3.

![Figure 3: Sum divisor cordial labeling of fusion of \( u_6 \) and \( u_{10} \) in \( H_s \).](image)

**Theorem 2.3.** The duplication of any vertex of degree 3 in a Herschel graph is a sum divisor cordial graph.

**Proof:** Let \( H_s \) be the Herschel graph with \( |V(H_s)| = 11 \) and \( |E(H_s)| = 18 \). Let \( v \) be the central vertex and \( u_k' \) be the duplication of the vertex \( u_k \) in the Herschel graph \( H_s \).

Let \( G \) be the graph obtained by duplicating the vertex \( u_k \) of degree 3 in \( H_s \). Then \( |V(G)| = 12 \) and \( |E(G)| = 21 \).

**Case 1.** Duplication of vertex \( u_k \), where \( k = 1,2,3,4,5,6,10 \).

We define the vertex labeling \( f : V(G) \to \{1,2,\ldots,|V(G)|\} \) as follows.

\[ f(v) = 1, \ f(u_i) = 11, \ f(u_6) = 4, \ f(u_k') = 12, \]
\[ f(u_i) = i; \ i = 9,10. \]
\[ f(u_i) = i+1; \ i = 1,2,4,5,6,7. \]

**Case 2.** Duplication of vertex \( u_6 \).

\[ f(v) = 9, \ f(u_6) = 7, \ f(u_6) = 4, \ f(u_6') = 12, \]
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\[ f(u_i) = i; \quad i = 1, 2, 3, 5, 6. \]
\[ f(u_i) = i + 1; \quad i = 7, 9, 10. \]

From the above two cases, we have \( |e_f(0) - e_f(1)| \leq 1. \)
Thus \( G \) is a sum divisor cordial graph.

**Example 2.3.** The sum divisor cordial labeling of the duplication of the vertex \( u_{10} \) in \( H_s \) is shown in Figure 4.

**Figure 4:** Sum divisor cordial labeling of the duplication of the vertex \( u_{10} \)

**Theorem 2.4.** The switching of a central vertex \( v \) in the Herschel graph \( H_s \) is a sum divisor cordial graph.

**Proof:** Let \( H_s \) be the Herschel graph with \( |V(H_s)| = 11 \) and \( |E(H_s)| = 18. \) Let \( v \) be the central vertex and \( G \) be the new graph obtained by switching the central vertex \( v \).

We define the vertex labeling \( f : V(G) \to \{1, 2, \ldots, |V(G)|\} \) as follows.

\[ f(v) = 1, \]
\[ f(u_i) = 11, \]
\[ f(u_i) = 6, \]
\[ f(u_i) = i; \quad i = 4, 5, 9, 10. \]
\[ f(u_i) = i + 1; \quad i = 1, 2, 6, 7. \]

From the above labeling pattern, we observe that
\[ e_f(0) = e_f(1) = 10. \]
Hence \( |e_f(0) - e_f(1)| \leq 1. \)
Thus \( G \) is a sum divisor cordial graph.

**Example 2.4.** The sum divisor cordial labeling of switching of a central vertex \( v \) in \( H_s \) is shown in Figure 5.
Theorem 2.5. The graph obtained by path union of two copies of Herschel graphs $H_s$ is a sum divisor cordial graph.

Proof: Consider two copies of Herschel graphs $H^1_s$ and $H^2_s$ respectively. Let $V(H^1_s) = \{u_i : 1 \leq i \leq 10\}$ and let $V(H^2_s) = \{v_i : 1 \leq i \leq 10\}$. Then $|V(H^1_s)| = 11$ and $|E(H^1_s)| = 18$ and $|V(H^2_s)| = 11$ and $|E(H^2_s)| = 18$. Let $G$ be the graph obtained by the path union of two copies of Herschel graphs $H^1_s$ and $H^2_s$. Then $V(G) = V(H^1_s) \cup V(H^2_s)$ and $E(G) = E(H^1_s) \cup E(H^2_s) \cup \{u_i v_i\}$. Note that $G$ has 22 vertices and 37 edges.

We define the vertex labeling $f : V(G) \rightarrow \{1, 2, \ldots, |V(G)|\}$ as follows.

Labeling of $H^1_s$:

- $f(u) = 1$,
- $f(u_9) = 9$,
- $f(u_3) = 11$,
- $f(u_8) = 6$,
- $f(u_4) = 4$,
- $f(u_{10}) = 10$,
- $f(u_i) = i + 1; \ i = 1, 2, 4, 6, 7$.

Labeling of $H^2_s$:

- $f(v) = 12$,
- $f(v_3) = 19$,
- $f(v_2) = 15$,
- $f(v_6) = 20$,
- $f(v_8) = 17$,
- $f(v_9) = 22$,
- $f(v_{10}) = 21$,
- $f(v_i) = i + 12; \ i = 1, 2, 4, 6$.

From the above labeling pattern, we observe that $e_f(0) = 18$ and $e_f(1) = 19$.

Hence $|e_f(0) - e_f(1)| \leq 1$.

Thus $G$ is a sum divisor cordial graph.

Example 2.5. The sum divisor cordial graph of the path union of $H^1_s$ and $H^2_s$ is shown in Figure 6.
3. Conclusion
In this paper, we have investigated the sum divisor cordiality on special graph namely Herschel graph and proved that the Herschel graph $H_s$, fusion of any two vertices of degree 3 in $H_s$, duplication of any vertex of degree 3 in $H_s$, switching of central vertex in $H_s$ and path union of two copies of $H_s$ are sum divisor cordial graphs.

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