All the Solutions of the Diophantine Equation

\[ p^4 + q^2 = z^2 \] when \( p \) is Prime

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Abstract. In this paper, we consider the title equation in the particular case when \( p \) is prime. It is established that the equation has exactly two distinct solutions. One solution for each and every prime \( p \geq 3 \), the other solution for each and every prime \( p \geq 2 \). The solutions are demonstrated for each prime \( p \) in the form of identities. Furthermore, the connection between the equation and the Pythagorean triples is also discussed when the prime \( p \) is replaced by any odd value \( A \geq 3 \).

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1. Introduction

The field of Diophantine equations is ancient, vast, and no general method exists to decide whether a given Diophantine equation has any solutions, or how many solutions. In most cases, we are reduced to study individual equations, rather than classes of equations.

The literature contains a very large number of articles on non-linear such individual equations involving primes and powers of all kinds. Among them are for example [1, 4, 5, 6, 7, 8]. The title equation stems from the equation \( p^4 + q^2 = z^2 \). Whereas in most articles, the values \( x, y \) are investigated for the solutions of the equation, in this paper these values are fixed positive integers. In the equation \( p^4 + q^2 = z^2 \) we consider all primes \( p \geq 2 \) and \( q > 1 \). We are mainly interested in how many solutions exist for any given prime \( p \). This is established in Section 2. In Section 3, we discuss the connection between the equation and the Pythagorean triple, i.e., \( a^2 + b^2 = c^2 \).

2. All the solutions of \( p^4 + q^2 = z^2 \) when \( p \) is prime

In this section we consider the equation \( p^4 + q^2 = z^2 \) for all primes \( p \geq 2 \). For \( p = 2 \) it will be shown that the equation has exactly one solution. For each and every \( p \geq 3 \), we shall establish the existence of two distinct solutions demonstrated in the form of two identities. This is done in the following Theorem 2.1.

Theorem 2.1. Suppose that \( p \geq 2 \) is prime, and

\[ p^4 + q^2 = z^2. \]
The values \( q, z \) in (a) and in (b)

(a) \[ q = \frac{p^4 - 1}{2}, \quad z = \frac{p^4 + 1}{2}, \quad p \geq 3, \]

(b) \[ q = \frac{(p-1)p(p+1)}{2}, \quad z = \frac{p(p^2 + 1)}{2}, \quad p \geq 2, \]

form two distinct solutions of equation (1).

**Proof:** The equation \( p^4 + q^2 = z^2 \) yields

\[ p^4 = z^2 - q^2 = (z - q)(z + q). \] (2)

Since \( p \) is prime, it follows that the values \( z - q \) and \( z + q \) in (2) satisfy five possibilities, three of which are a priori impossible. Hence, we have

(i) \[ z - q = 1 \quad \text{and} \quad z + q = p^4, \]

(ii) \[ z - q = p \quad \text{and} \quad z + q = p^3. \]

(i) Suppose \( z - q = 1 \) and \( z + q = p^4 \). The value \( z - q = 1 \) yields \( z = q + 1 \) implying \( 2q + 1 = p^4 \) and \( q = \frac{p^4 - 1}{2} \). The sum of \( z - q = 1 \) and \( z + q = p^4 \) is equal to \( 2z = p^4 + 1 \) or \( z = \frac{p^4 + 1}{2} \). Hence, the values \( q = \frac{p^4 - 1}{2} \) and \( z = \frac{p^4 + 1}{2} \) satisfy equation (1) for all primes \( p \geq 3 \) and (a) is established.

(ii) Suppose \( z - q = p \) and \( z + q = p^3 \). The sum of \( z - q = p \) and \( z + q = p^3 \) implies that \( 2z = p(p^2 + 1) \) or \( z = \frac{p(p^2 + 1)}{2} \). The difference of \( z + q = p^3 \) and \( z - q = p \) yields \( 2q = p(p^2 - 1) \) or \( q = \frac{p(p^2 - 1)}{2} \). Thus, the values \( q = \frac{(p-1)p(p+1)}{2} \) and \( z = \frac{p(p^2 + 1)}{2} \) satisfy equation (1) for each and every prime \( p \geq 2 \) and (b) has been established.

The two distinct solutions (a) and (b) are identities valid for each and every designated value \( p \). Equation (1) has therefore infinitely many solutions.

This completes the proof of Theorem 2.1. \( \square \)

**3. The equation** \( p^4 + q^2 = z^2 \) **and the Pythagorean triples**

In Section 2 we have considered equation (1) for all primes \( p \). In this section, we omit the condition that \( p \geq 3 \) is prime, and use instead every odd value \( A \geq 3 \). We show that for every odd composite equal to \( A \), equation (1) has a solution.

A set of positive integers \( a, b, c \) is called a "pythagorean triple" (abbreviated triple) denoted \((a, b, c)\) if \( a^2 + b^2 = c^2 \). Let \( a^2 + b^2 = c^2 \) be a triple. For every integer \( M > 1 \), \( Ma^2 + Mb^2 = Mc^2 \) is also a triple. For example: The triple \((7, 24, 25)\) yields the
All the Solutions of the Diophantine Equation $p^4 + q^2 = z^2$ when $p$ is Prime
triples $(2 \cdot 7, 2 \cdot 24, 2 \cdot 25), (3 \cdot 7, 3 \cdot 24, 3 \cdot 25)$ where respectively $M = 2, 3,$
and so on for $M > 3$. Suppose that $A^2$ is any odd value.

If $A^2 + b^2 = c^2$ is a triple, set $b = \frac{A^2 - 1}{2}, \ c = \frac{A^2 + 1}{2},$ and the triple is

$$A^2 + \frac{(A^2 - 1)^2}{4} = \frac{(A^2 + 1)^2}{4}.$$  
The above triple with $M = A^2$ yields the triple

$$A^4 + \frac{(A^2 - 1)^2}{4} A^2 = \frac{(A^2 + 1)^2}{4} A^2.$$  

Substituting the values $p^2 = A^2, \ q = \frac{(A^2 - 1)A}{2}, \ z = \frac{(A^2 + 1)A}{2}$ in

$p^4 + q^2 = z^2$ results in $A^4 + \frac{(A^2 - 1)^2}{4} A^2 = \frac{((A^2 - 1)A)^2}{4}$. Thus, equation (1)
has a solution for each and every odd composite equal to $A^2$. The equation has therefore
infinitely many solutions.

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