Sum Divisor Cordial Labeling of Theta Graph

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Abstract. A sum divisor cordial labeling of a graph $G$ with vertex set $V$ is a bijection $f$ from $V$ to $\{1, 2, \ldots, |V|\}$ such that each edge $uv$ assigned the label 1 if $2 \mid f(u) + f(v)$ and 0 otherwise. Further, the number of edges labeled with 0 and the number of edges labeled with 1 differ by at most 1. A graph with a sum divisor cordial labeling is called a sum divisor cordial graph. In this paper, we prove that Theta graph $T_a$, fusion of any two vertices in the cycle of $T_a$, duplication of any vertex $v_i$ in the cycle of $T_a$, switching of a central vertex in $T_a$, path union of two copies of $T_a$, star of Theta graph are sum divisor cordial graphs.

Keywords: Divisor cordial labeling, sum divisor cordial labeling, fusion, duplication, switching, path union.

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1. Introduction

By a graph, we mean a finite undirected graph without loops or multiple edges. For standard terminology and notations related to graph theory we refer to Harary [3]. A labeling of graph is a map that carries the graph elements to the set of numbers, usually to the set of non-negative or positive integers. If the domain is the set of edges, then we speak about edge labeling. If the labels are assigned to both vertices and edges, then the labeling is called total labeling. Cordial labeling is extended to divisor cordial labeling, prime cordial labeling, total cordial labeling, Fibonacci cordial labeling etc.

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For dynamic survey of various graph labeling, we refer to Gallian [1]. Lourdusamy and Patrick [6] introduced the concept of sum divisor cordial labeling. Sugumaran and Rajesh [8] proved that Swastik graph $Sw_n$, path union of finite copies of Swastik graph $Sw_n$, cycle of $k$ copies of Swastik graph $Sw_n$ ($k$ is odd), Jelly fish $J(n,n)$ and Petersen graph are sum divisor cordial graphs. Ganesan and Balamurugan [2] have discussed the prime labeling of Theta graph. Our primary objective of this paper is to prove that the Theta graph and some graph operations in Theta graph namely fusion, duplication, switching of a central vertex, path union of two copies and the star of Theta graphs are sum divisor cordial graphs.

**Definition 1.1.** [11] Let $G = (V(G), E(G))$ be a simple graph and let $f : V(G) \to \{1, 2, ..., |V(G)|\}$ be a bijection. For each edge $uv$, assign the label 1 if either $f(u) | f(v)$ or $f(v) | f(u)$ and the label 0 otherwise. The function $f$ is called a divisor cordial labeling if $|e_f(0) - e_f(1)| \leq 1$. A graph which admits a divisor cordial labeling is called a divisor cordial graph.

**Definition 1.2.** [6] Let $G = (V(G), E(G))$ be a simple graph and let $f : V(G) \to \{1, 2, ..., |V(G)|\}$ be a bijection. For each edge $e = uv$, assign the label 1 if either $2 | (f(u) + f(v))$ and assign the label 0 otherwise. The function $f$ is called a sum divisor cordial labeling if $|e_f(0) - e_f(1)| \leq 1$. A graph which admits a sum divisor cordial labeling is called a sum divisor cordial graph.

**Definition 1.3.** A Theta graph $T_a$ is a block with two non adjacent vertices of degree 3 and all other vertices of degree 2.

![Theta graph](image)

**Figure 1:** Theta graph

In this paper, we always fix the position of the vertices $v_0, v_1, ..., v_6$ of $T_a$ as indicated in the above figure 1, unless or otherwise specified.

**Definition 1.4.** Let $u$ and $v$ be two distinct vertices of a graph $G$. A new graph $G_1$ is constructed by fusing (identifying) two vertices $u$ and $v$ by a single vertex $x$ in $G_i$ such that every edge which was incident with either $u$ (or) $v$ in $G$ now incident with $x$ in $G_1$. 

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**Definition 1.5.** Duplication of a vertex $v_k$ of a graph $G$ produces a new graph $G'$ by adding a vertex $v'_k$ with $N(v_k) = N(v'_k)$. In other words, a vertex $v'_k$ is said to be a duplication of $v_k$ if all the vertices which are adjacent to $v_k$ are now adjacent to $v'_k$.

**Definition 1.6.** A vertex switching $G'$ of a graph $G$ is obtained by taking a vertex $v$ of $G$, removing the entire edges incident with $v$ and adding edges joining $v$ to every vertex which are not adjacent to $v$ in $G$.

**Definition 1.7.** [4] Let $G$ be a graph and let $G_1 = G_2 = \cdots = G_n = G$, where $n \geq 2$. Then the graph obtained by adding an edge from each $G_i$ to $G_{i+1}$ $(1 \leq i \leq n-1)$ is called the path union of $G$.

**Definition 1.8.** [5] Let $G$ be a graph with $n$ vertices. The graph obtained by replacing each vertex of the star $K_{1,n}$ by a copy of $G$ is called a star graph of graph $G$ and it is denoted by $G_n^*$.

2. Main results

**Theorem 2.1.** The Theta graph $T_n$ is a sum divisor cordial graph.

**Proof:** Let $G = T_n$ be a Theta graph with centre $v_0$. Then

$V(G) = \{v_0, v_1, v_2, v_3, v_4, v_5, v_6\}$ and $E(G) = \{v_iv_{i+1} / 1 \leq i \leq 5\} \cup \{v_0v_1, v_0v_4, v_3v_6\}$.

Note that $|V(G)| = 7$ and $|E(G)| = 8$. We define the vertex labeling $f : V(G) \to \{1, 2, \ldots, |V(G)|\}$ as follows.

- $f(v_0) = 3$,
- $f(v_i) = i; \ 1 \leq i \leq 2$,
- $f(v_i) = i+1; \ 3 \leq i \leq 6$.

From the above labeling pattern, we have $e_f(0) = e_f(1) = 4$.

Hence $|e_f(0) - e_f(1)| \leq 1$.

Thus $G$ is a sum divisor cordial graph.

**Example 2.1.** The sum divisor cordial labeling of Theta graph $T_n$ is shown in Figure 2.

![Figure 2: Sum divisor cordial labeling of Theta graph $T_n$.](image)
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**Theorem 2.2.** The fusion of any two vertices in the cycle of $T_a$ is a sum divisor cordial graph.

**Proof:** Let $T_a$ be a Theta graph with centre $v_0$. Then $V(T_a) = \{v_0, v_1, v_2, v_3, v_4, v_5, v_6\}$ and $E(T_a) = \{v_i v_{i+1} / 1 \leq i \leq 5\} \cup \{v_0 v_1, v_0 v_4, v_1 v_6\}$. Note that $|V(T_a)| = 7$ and $|E(T_a)| = 8$. Let $G$ be a graph obtained by fusion of two vertices $v_2$ and $v_3$ in the cycle of $T_a$ and we call it as vertex labeling $f : V(G) \to \{1, 2, ..., |V(G)|\}$ as follows.

- $f(v_0) = 6$, $f(v_3) = 3$, $f(v_i) = i$; $i = 1, 2, 4, 5$.

From the above labeling pattern, we observe that $e_f(0) = 3$ and $e_f(1) = 4$.

Hence, $|e_f(0) - e_f(1)| \leq 1$.

Thus $G$ is a sum divisor cordial graph.

**Example 2.2.** The sum divisor cordial labeling of fusion of $v_2$ and $v_3$ in $T_a$ is shown in Figure 3.

![Figure 3: Sum divisor cordial labeling of fusion of $v_2$ and $v_3$ in $T_a$](image)

**Theorem 2.3.** The duplication of any vertex $v_i$ in the cycle of $T_a$ is a sum divisor cordial graph.

**Proof:** Let $T_a$ be a Theta graph with centre $v_0$. Then $V(T_a) = \{v_0, v_1, v_2, v_3, v_4, v_5, v_6\}$ and $E(T_a) = \{v_i v_{i+1} / 1 \leq i \leq 5\} \cup \{v_0 v_1, v_0 v_4, v_1 v_6\}$. Note that $|V(T_a)| = 7$ and $|E(T_a)| = 8$. Let $G$ be a graph obtained from $T_a$ after duplication of the vertex $v_i$ in $T_a$.

- Let $v'_i$ be the duplication vertex of $v_i$ in $T_a$. Clearly $|V(G)| = 8$ and $|E(G)| = 11$.

**Case 1.** Duplication of vertex $v_i$, where $i = 1, 2, 3, 4$ and 6.

We define the vertex labeling $f : V(G) \to \{1, 2, ..., |V(G)|\}$ as follows.

- $f(v_0) = 3$, $f(v'_i) = 8$,
- $f(v_i) = i$; $1 \leq i \leq 2$,
- $f(v'_i) = i + 1$; $3 \leq i \leq 6$.

**Case 2.** Duplication of vertex $v_3$.

We define the vertex labeling $f : V(G) \to \{1, 2, ..., |V(G)|\}$ as follows.
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\[ f(v_0) = 3, \quad f(v_1) = 7, \quad f(v_2) = 8, \]
\[ f(v_i) = i; \quad i = 1, 2, 6, \]
\[ f(v_i) = i + 1; \quad i = 3, 4. \]

From the above two cases, we have \(|e_f(0) - e_f(1)| \leq 1\).

Hence \(G\) is a sum divisor cordial graph.

**Example 2.3.** The sum divisor cordial labeling of the duplication of the vertex \(v_1\) in \(T_a\) is shown in Figure 4.

![Figure 4: Sum divisor cordial labeling of the duplication of the vertex \(v_1\) in \(T_a\).](image)

**Theorem 2.4.** The switching of a central vertex in the Theta graph \(T_a\) is a sum divisor cordial graph.

**Proof:** Let \(T_a\) be a Theta graph with centre \(v_0\). Then \(V(T_a) = \{v_0, v_1, v_2, v_3, v_4, v_5, v_b\}\) and \(E(T_a) = \{v_i v_{i+1} \mid 1 \leq i \leq 5\} \cup \{v_0 v_1, v_0 v_4, v_0 v_b\}\). Note that \(T_a\) has 7 vertices and 8 edges. Let \(G\) be the graph obtained from \(T_a\) after switching the central vertex \(v_0\) of \(T_a\).

We define the vertex labeling \(f : V(G) \to \{1, 2, \ldots, |V(G)|\}\) as follows:
\[ f(v_0) = 1, \quad f(v_1) = 6, \quad f(v_b) = 7, \]
\[ f(v_i) = i; \quad 2 \leq i \leq 5. \]

From the above labeling pattern, we observe that \(|e_f(0) - e_f(1)| = 5\).

Hence \(|e_f(0) - e_f(1)| = 1\).

Thus \(G\) is a sum divisor cordial graph.

**Example 2.4.** The sum divisor cordial labeling of switching of a central vertex \(v_0\) in \(T_a\) is shown in Figure 5.

![Figure 5: Sum divisor cordial labeling of switching of a central vertex \(v_0\) in \(T_a\).](image)
Theorem 2.5. The graph obtained by path union of two copies of Theta graphs $T_a$ is a sum divisor cordial graph.

Proof: Consider two copies of Theta graphs $T_a^1$ and $T_a^2$ respectively. Then $V(T_a^1) = \{u_0, u_1, u_2, ..., u_6\}$ and $E(T_a^1) = \{u_iu_{i+1} : 1 \leq i \leq 5\} \cup \{u_0u_1, u_4u_5, u_6u_0\}$. And $V(T_a^2) = \{v_0, v_1, v_2, ..., v_6\}$ and $E(T_a^2) = \{v_iv_{i+1} : 1 \leq i \leq 5\} \cup \{v_0v_1, v_0v_4, v_1v_6\}$.

Let $G$ be the graph obtained by the path union of two copies of Theta graph $T_a^1$ and $T_a^2$. Then $V(G) = V(T_a^1) \cup V(T_a^2)$ and $E(G) = E(T_a^1) \cup E(T_a^2) \cup \{u_3v_2\}$.

Note that $G$ has 14 vertices and 17 edges. We define the vertex labeling $f : V(G) \to \{1, 2, ..., |V(G)|\}$ as follows.

$f(u_0) = 3$, $f(v_0) = 10$,

$f(u_i) = i + 1$; $3 \leq i \leq 6$.

$f(v_i) = i + 7$; $i = 1, 2$,

$f(v_i) = i + 8$; $3 \leq i \leq 6$.

From the above labeling pattern, we observe that $e_f(0) = 9$ and $e_f(1) = 8$.

Hence $|e_f(0) - e_f(1)| = 1$.

Thus $G$ is a sum divisor cordial graph.

Example 2.5. The sum divisor cordial graph of the path union of $T_a^1$ and $T_a^2$ is shown in Figure 6.

![Figure 6: The sum divisor cordial graph of the path union of $T_a^1$ and $T_a^2$](image)

Theorem 2.6. The star graph of Theta graph $T_a$ is a sum divisor cordial graph.

Proof: Let $G = T_a^*$ be a star of Theta graph $T_a$. Then $G$ has 8 copies of Theta graph, namely $T_a^i$, $i = 0, 1, 2, ..., 7$. We denote $V(T_a^i) = \{v_0^i, v_1^i, ..., v_6^i\}$ and $E(T_a^i) = \{v_j^iv_{j+1}^i : 1 \leq j \leq 5\} \cup \{v_0^iv_1^i, v_0^iv_4^i, v_1^iv_6^i\}$ be the vertex and edge sets of $T_a^i$ for each $i = 0, 1, 2, ..., 7$. Note that $G$ has 56 vertices and 71 edges. We assume that the central copy of $G = T_a^*$ is $T_a^0$ and the other copies of
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\( G = T^*_u \) is \( T^*_u \) for \( 1 \leq k \leq 7 \). We define the vertex labeling \( f : V(G) \to \{1, 2, \ldots, 56\} \) as follows.

\[
\begin{align*}
  f(v^0_0) &= 3, \\
  f(v^0_1) &= i; i = 1, 2, \\
  f(v^0_2) &= i + 1; 3 \leq i \leq 6. \\
  f(v^0_3) &= 10, \\
  f(v^0_4) &= i + 7; i = 1, 2, \\
  f(v^0_5) &= i + 8; 3 \leq i \leq 6. \\
  f(v^0_6) &= 17, \\
  f(v^0_7) &= i + 14; i = 1, 2, \\
  f(v^0_8) &= i + 15; 3 \leq i \leq 6. \\
  f(v^0_9) &= 24, \\
  f(v^0_{10}) &= i + 21; i = 1, 2, \\
  f(v^0_{11}) &= i + 22; 3 \leq i \leq 6. \\
  f(v^0_{12}) &= 31, \\
  f(v^0_{13}) &= i + 28; i = 1, 2, \\
  f(v^0_{14}) &= i + 29; 3 \leq i \leq 6. \\
  f(v^0_{15}) &= 38, \\
  f(v^0_{16}) &= i + 35; i = 1, 2, \\
  f(v^0_{17}) &= i + 36; 3 \leq i \leq 6. \\
  f(v^0_{18}) &= 45, \\
  f(v^0_{19}) &= i + 42; i = 1, 2, \\
  f(v^0_{20}) &= i + 43; 3 \leq i \leq 6. \\
  f(v^0_{21}) &= 52, \\
  f(v^0_{22}) &= i + 49; i = 1, 2, \\
  f(v^0_{23}) &= i + 50; 3 \leq i \leq 6.
\end{align*}
\]

From the above labeling pattern, we observe that \( e_f(0) = 36 \) and \( e_f(1) = 35 \).

Hence \( |e_f(0) - e_f(1)| \leq 1 \).

Thus \( G \) is a sum divisor cordial graph.

**Example 2.6.** The sum divisor cordial labeling of star of Theta graph is shown in Figure 7.
3. Conclusion
In this paper, we investigated the sum divisor cordial graph on a special graph namely Theta graph and proved that Theta graph $T_a$, fusion of any two vertices in the cycle of $T_a$, duplication of any vertex $v_i$ in the cycle of $T_a$, switching of central vertex in $T_a$, path union of two copies of $T_a$, star of Theta graphs are sum divisor cordial graphs.

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