Annals of Generalized Minimal Closed Sets in Bitopological Spaces

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Abstract. In this paper, we introduce and characterize generalized minimal closed sets in bitopological spaces and study some of their properties. A subset A of X is said to be \((\tau_i, \tau_j)\)-generalized minimal closed (briefly \((\tau_i, \tau_j)\)-g-m closed) set in a bitopological space if \(\tau_j\)-cl \((A)\) \(\subseteq\) U whenever A \(\subseteq\) U and U is \(\tau_i\)- minimal open set in \((X; \tau_i, \tau_j)\).

Keywords: \(\tau_i\) minimal open set, \(\tau_i\)-maximal closed set, \((\tau_i, \tau_j)\)-g-closed set, \((\tau_i, \tau_j)\)-\(\omega\)-closed set, \((\tau_i, \tau_j)\)-\(g\)-open set

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1. Introduction and preliminaries

The triple \((X; \tau_i, \tau_j)\) where X is a set and \(\tau_i\) and \(\tau_j\) are two topologies on X is a bitopological space. Kelly [5] initiated the systematic study of such spaces. After the work of Kelly [5] various authors [2,3,7,8] turned their attention to generalization of various concepts of topology by considering bitopological spaces. The concept of generalized closed sets in bitopological spaces was introduced and investigated by T [7].

Throughout this chapter \((X; \tau_i, \tau_j)\) denote non empty bitopological spaces on which no separation axioms are assumed unless otherwise mentioned and the fixed integers i, j \(\in\) {1,2}.

We recall the following definitions, which are useful in the sequel.

Definition 1.1. Let i, j \(\in\) {1,2} be fixed integers. In a bitopological space \((X; \tau_i, \tau_j)\), a subset A of X is said to be

(i) \((\tau_i, \tau_j)\)-g-closed set [7] if \(\tau_j\)-cl \((A)\) \(\subseteq\) U whenever A \(\subseteq\) U and U is \(\tau_i\)-open set.

(ii) \((\tau_i, \tau_j)\)-\(g\)-open set iff A \(c^c\) is \((\tau_i, \tau_j)\)-g-closed set.

(iii) \((\tau_i, \tau_j)\)-\(\omega\)-closed set [6] if \(\tau_j\)-cl \((A)\) \(\subseteq\) U whenever A \(\subseteq\) U and U is \(\tau_i\)-semi open set in \((X, \tau_i)\).

(iv) \((\tau_i, \tau_j)\)-\(\omega\)-open set [6] iff A \(c^c\) is \((\tau_i, \tau_j)\)-\(\omega\)-closed set.
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**Definition 1.2.** Let \( i, j \in \{1, 2\} \) be fixed integers. In a bitopological space \((X; \tau_i, \tau_j)\), a proper nonempty \((\tau_i, \tau_j)\)-g-open set \( A \) of \((X; \tau_i, \tau_j)\) is said to be

(i) \((\tau_i, \tau_j)\)-minimal g-open (resp. \((\tau_i, \tau_j)\)-minimal g-closed) set if any \((\tau_i, \tau_j)\)-g-open (respectively \((\tau_i, \tau_j)\)-g-closed) subset of \((X; \tau_i, \tau_j)\) which is contained in \( A \) is either \( A \) or \( \emptyset \).

(ii) \((\tau_i, \tau_j)\)-maximal g-open (resp. \((\tau_i, \tau_j)\)-maximal g-closed) set if any \((\tau_i, \tau_j)\)-g-open (respectively \((\tau_i, \tau_j)\)-g-closed) subset of \((X; \tau_i, \tau_j)\) which contains \( A \) is either \( A \) or \( X \).

**2. Generalized minimal closed sets in bitopological spaces**

In this section, we introduce and investigate generalized minimal closed sets in bitopological spaces.

**Definition 2.1.** Let \( i, j \in \{1, 2\} \) be fixed integers. In a bitopological space \((X; \tau_i, \tau_j)\), a subset \( A \) of \( X \) is said to be \((\tau_i, \tau_j)\)-generalized minimal closed (briefly \((\tau_i, \tau_j)\)-g-m-closed) set if \( \tau_i\text{-cl}(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is \( \tau_i\)-minimal open set in \((X; \tau_i, \tau_j)\).

**Remark 2.2.** By setting \( \tau_i = \tau_j \) in the Definition 2.1, a \((\tau_i, \tau_i)\)-g-m-closed set is a g-closed set in a topological space.

**Theorem 2.3.** Let \( i, j \in \{1, 2\} \) be fixed integers. Every \((\tau_i, \tau_j)\)-g-m-closed set in a bitopological space \((X; \tau_i, \tau_j)\) is a \((\tau_i, \tau_j)\)-g-closed set.

**Proof:** Let \( A \subseteq X \) be any \((\tau_i, \tau_j)\)-g-m-closed set in \((X; \tau_i, \tau_j)\). By Definition 2.1 \( \tau_i\text{-cl}(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is \( \tau_i\)-minimal open set. But every minimal open set is an open set. Therefore \( \tau_i\text{-cl}(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is a \( \tau_i\)-open set. Hence \( A \) is a \((\tau_i, \tau_j)\)-g-closed set in \((X; \tau_i, \tau_j)\).

**Remark 2.4.** Converse of the Theorem 2.3 need not be true.

**Example 2.5.** Let \( X = \{a, b, c, d\} \) with \( \tau_1 = \{\emptyset, \{a\}, \{a, b\}, \{c, d\}, \{a, c, d\}, \} \) and \( \tau_2 = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, X\} \).

\((\tau_1, \tau_2)\)-g-m-closed sets = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, X\}.

\((\tau_2, \tau_2)\)-g-closed sets = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, X\}.

**Theorem 2.6.** Let \( i, j \in \{1, 2\} \) be fixed integers. Every \((\tau_i, \tau_j)\)-g-m-closed set in a bitopological space \((X; \tau_i, \tau_j)\) is a \((\tau_i, \tau_j)\)-\(\omega\)-closed set.

**Proof:** Let \( A \subseteq X \) be any \((\tau_i, \tau_j)\)-g-m-closed set in \((X; \tau_i, \tau_j)\). By Definition 2.1 \( \tau_i\text{-cl}(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is a \( \tau_i\)-minimal open set. But every minimal open set is an open set and hence is a semi-open set. Therefore \( \tau_i\text{-cl}(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is a \( \tau_i\)-semi-open set. Hence \( A \) is a \((\tau_i, \tau_j)\)-\(\omega\)-closed set in \((X; \tau_i, \tau_j)\).

**Remark 2.7.** Converse of the above Theorem 2.6 need not be true.
Example 2.8. Let $X = \{a, b, c, d\}$ with $\tau_1 = \{\phi, \{b\}, \{a, b\}, \{c, d\}, \{b, c, d\}, X\}$ and $\tau_2 = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}, \{a, b, c\}, \{a, b, d\}, \{b, c, d\}, X\}$. ($\tau_1$, $\tau_2$)-g-$m$ closed sets: $\{\phi, \{c\}, \{d\}, \{c, d\}\}$.

($\tau_2$, $\tau_1$)-g-$m$ closed sets: $\{\phi, \{a\}\}$.

($\tau_1$, $\tau_2$)-$\omega$-closed sets $= \{\phi, \{a\}, \{c\}, \{d\}, \{a, c, d\}, X\}$.

($\tau_2$, $\tau_1$)-$\omega$-closed sets $= \{\phi, \{a\}, \{a, b\}, \{c, d\}, \{a, c, d\}, X\}$.

Proposition 2.9. Let $i, j \in \{1, 2\}$ be fixed integers. If $A$ is a $\tau_i$-minimal closed subset of a bitopological space $(X; \tau_i, \tau_j)$, then $A$ is a $(\tau_i, \tau_j)$-g-$m$ closed set in $(X; \tau_i, \tau_j)$.

Proof: Let $A \subseteq U$, such that $U$ is a $\tau_i$-minimal open set. By hypothesis $A$ is a $\tau_i$-minimal closed subset of $(X; \tau_i, \tau_j)$, then $A$ is a $\tau_i$-closed subset of $(X; \tau_i, \tau_j)$, so that $\tau_i$-cl$(A) = A$. Therefore, $\tau_i$-cl$(A) \subseteq A$, whenever $A \subseteq U$ and $U$ is a $\tau_i$-minimal open set in $(X; \tau_i, \tau_j)$. Hence $A$ is a $(\tau_i, \tau_j)$-g-$m$ closed set in $(X; \tau_i, \tau_j)$.

Remark 2.10. If $\tau_i \subseteq \tau_j$ in $(X; \tau_i, \tau_j)$ then, $(\tau_2, \tau_1)$-g-$m$ closed sets $\not\subseteq (\tau_1, \tau_2)$-g-$m$ closed sets.

Example 2.11. Let $X = \{a, b, c, d\}$ with $\tau_1 = \{\phi, \{a\}, \{a, b\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}, X\}$ and $\tau_2 = \{\phi, \{a\}, \{b\}, \{a, b\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}, \{a, b, c\}, \{a, b, d\}, \{b, c, d\}, X\}$.($\tau_1$, $\tau_2$)-g-$m$ closed sets: $\{\phi, \{a\}, \{c\}, \{d\}, \{c, d\}\}$.

($\tau_2$, $\tau_1$)-g-$m$ closed sets: $\{\phi, \{b\}, \{c\}, \{d\}, \{c, d\}\}$.

Theorem 2.12. Let $i, j \in \{1, 2\}$ be fixed integers. If $A$ is a $(\tau_i, \tau_j)$-g-$mi$ closed set in a bitopological space $(X; \tau_i, \tau_j)$ and $A \subseteq B \subseteq \tau_j$-cl$(A)$ then $B$ is a $(\tau_i, \tau_j)$-g-$m$ closed set in a bitopological space $(X; \tau_i, \tau_j)$.

Proof: Let $B$ be any set such that $B \subseteq U$ and $U$ is a $\tau_j$-minimal open set in $(X; \tau_j, \tau_i)$. Given that $A \subseteq B \subseteq \tau_j$-(cl) (i)

Since $A \subseteq B \subseteq U$, then $A \subseteq U$ where $U$ is a $\tau_j$-minimal open set. But $A$ is a $(\tau_i, \tau_j)$-g-$m$ closed set, by Definition 2.1, $\tau_j$-cl$(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is a $\tau_j$-minimal open set in $(X; \tau_j, \tau_i)$. From (i) $A \subseteq B \subseteq \tau_j$-cl$(A)$ implies $B \subseteq \tau_j$-cl$(A)$ which implies $\tau_i$-cl$(B) \subseteq \tau_j$-cl$(\tau_j$-cl$(A)) = \tau_j$-cl$(A)$. That is $\tau_i$-cl$(B) \subseteq \tau_j$-cl$(A)$. But $\tau_j$-cl$(A) \subseteq U$. Therefore, $\tau_j$-cl$(B) \subseteq U$ whenever $B \subseteq U$ and $U$ is a $\tau_j$-minimal open set in $(X; \tau_j, \tau_i)$. Hence $B$ is a $(\tau_i, \tau_j)$-g-$m$ closed set in $(X; \tau_i, \tau_j)$.

Theorem 2.13. Let $i, j \in \{1, 2\}$ be fixed integers. If $A$ is a $(\tau_i, \tau_j)$-g-$mi$ closed set in a bitopological space $(X; \tau_i, \tau_j)$, then $\tau_j$-cl$(A) \cap A$ contains no nonempty $\tau_j$-maximal closed subset.

Proof: Let $F$ be a $\tau_i$-maximal closed subset of cl$(A) \cap A$. Then $F^c$ is a $\tau_i$-minimal open set. Let $A$ be such that $A \subseteq F^c$ where $F^c$ is a $\tau_i$-minimal open set in $(X; \tau_i, \tau_j)$. Since $A$ is a $(\tau_i, \tau_j)$-g-$m$ closed set, by the Definition 2.1, $\tau_j$-cl$(A) \subseteq F$ whenever $A \subseteq F^c$ and $F^c$ is a $\tau_i$-minimal open set in $(X; \tau_i, \tau_j)$. So $F \subseteq [\tau_j$-cl$(A)]^c$. On the other hand $F \subseteq \tau_j$-cl$(A)$. Therefore $F \subseteq [\tau_j$-cl$(A)] \cap \tau_j$-cl$(A) = \phi$. 271
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Therefore $F = \emptyset$.

**Theorem 2.14.** Let $i, j \in \{1, 2\}$ be fixed integers. If $A$ is a $(\tau_i, \tau_j)$-g-m closed set in a bitopological space $(X; \tau_i, \tau_j)$, then $\tau_{ij} - \text{cl}(A) - A$ contains no nonempty $\tau_i$ - closed subset.

**Proof:** Let $A$ be a $(\tau_i, \tau_j)$-g-m closed set in $(X; \tau_i, \tau_j)$ and $F$ be a nonempty $\tau_i$ - closed set contained in $\tau_{ij} - \text{cl}(A) - A$. So $F \subseteq \tau_{ij} - \text{cl}(A) - A = \tau_{ij} - \text{cl}((A \cap A^c))$. Then $F \subseteq \tau_{ij} - \text{cl}(A)$ and $F \subseteq A^c$. Now $F \subseteq A^c$ means $A \subseteq F^c$ where $F^c$ is an open set. Since every $(\tau_i, \tau_j)$-g-m closed set in a bitopological space $(X; \tau_i, \tau_j)$ is a $(\tau_i, \tau_j)$-g-m closed set, $A$ is a $(\tau_i, \tau_j)$-g-m closed set. Then by the Definition [7], $\tau_{ij} - \text{cl}(A) \subseteq F^c$ whenever $A \subseteq F^c$ and $F^c$ is an open set in $(X; \tau_i, \tau_j)$, so that $F \subseteq [\tau_{ij} - \text{cl}(A)]^c$. On the other hand $F \subseteq \tau_{ij} - \text{cl}(A)$, so that $F \subseteq [\tau_{ij} - \text{cl}(A)]^c \cap \tau_{ij} - \text{cl}(A) = \emptyset$. Therefore $F = \emptyset$.

**Corollary 2.15.** Let $i, j \in \{1, 2\}$ be fixed integers. $A$ $(\tau_i, \tau_j)$-g-m closed set $A$ in a bitopological space $(X; \tau_i, \tau_j)$ is $\tau_{ij}$-closed iff $\tau_{ij} - \text{cl}(A) - A$ is $\tau_i$ - closed.

**Proof:** Let $A$ be any $(\tau_i, \tau_j)$-g-m closed set in a bitopological space $(X; \tau_i, \tau_j)$ which is a $\tau_{ij}$-closed set so that $\tau_{ij} - \text{cl}(A) = A$, then $\tau_{ij} - \text{cl}(A) - A = \emptyset$. Therefore $\tau_{ij} - \text{cl}(A) - A$ is a $\tau_i$ - closed set.

Conversely, let $A$ be any $(\tau_i, \tau_j)$-g-m closed set in a bitopological space $(X; \tau_i, \tau_j)$ such that $\tau_{ij} - \text{cl}(A) - A$ is a $\tau_i$ - closed set. Since $\tau_{ij} - \text{cl}(A)$ - $A$ is a subset of itself and is a $\tau_i$ - closed set, by the Theorem 2.14, $\tau_{ij} - \text{cl}(A) - A = \emptyset$, so that $A = \tau_{ij} - \text{cl}(A)$. Therefore $A$ is a $\tau_{ij}$-closed set.

**Proposition 2.16.** Let $i, j \in \{1, 2\}$ be fixed integers. If $A$ is an $\tau_i$ minimal open set and a $(\tau_i, \tau_j)$-g-m closed set, then $A$ is a $\tau_{ij}$-closed set.

**Proof:** Since $A \subseteq A$ and as $A$ is a $\tau_i$-minimal open and a $(\tau_i, \tau_j)$-g-m closed set, we have $\text{cl}(A) \subseteq A$. Therefore $\tau_{ij} - \text{cl}(A) = A$. Hence $A$ is a $\tau_{ij}$-closed set.

**Theorem 2.17.** Let $i, j \in \{1, 2\}$ be fixed integers. If $A$ is a $(\tau_i, \tau_j)$-g-m closed set in a bitopological space $(X; \tau_i, \tau_j)$, then for each $x \in \tau_{ij} - \text{cl}(A)$, $\tau_{ij} - \text{cl} \{x\} \cap A \neq \emptyset$.

**Proof:** Let $A$ be any $(\tau_i, \tau_j)$-g-m closed set in a bitopological space $(X; \tau_i, \tau_j)$, such that $A$ contains no nonempty $\tau_i$ - closed set. Then $A \subseteq [\tau_{ij} - \text{cl}(A)]^c$. But $A$ is a $(\tau_i, \tau_j)$-g-m closed set. By the Definition 2.1 $\tau_{ij} - \text{cl}(A) \subseteq [\tau_{ij} - \text{cl}(A)]^c$. This is a contradiction to the fact that $x \in \tau_{ij} - \text{cl}(A)$. Therefore $\tau_{ij} - \text{cl}(A) \cap A \neq \emptyset$.

**Lemma 2.18.** If $Y \subseteq X$ is any subspace of a bitopological space $(X; \tau_1, \tau_2)$ and $U$ is any $\tau_i$-minimal open set in $(X; \tau_i, \tau_j)$ then $Y \cap U$ is a $\tau_{i_Y}$ minimal open set.

**Proof:** Let $U$ be a $\tau_i$-minimal open set in a bitopological space $(X; \tau_1, \tau_2)$ such that $Y \cap U$ is not a $\tau_{i_Y}$ minimal open set in $Y$. Then there exists an $\tau_{i_Y}$ open set $G \neq Y$ in $Y$ such that $G \subseteq Y \cap U$ where $G = Y \cap H$ and $H$ is an $\tau_i$-open set in $(X; \tau_1, \tau_2)$. Now $Y \cap H \subseteq Y \cap U$.
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implies $H \subseteq U$. This contradicts the fact that $U$ is a $\tau_i$-minimal open set. Therefore $Y \cap U$ is a $\tau_i$-$\gamma$ minimal open set.

**Theorem 2.21.** Let $i, j \in \{1, 2\}$ be fixed integers. If $B \subseteq A \subseteq X$ such that $B$ is a $(\tau_i, \tau_i)$-g-$m_i$ closed relative to $A$ and that $A$ is an $\tau_i$-open and $(\tau_i, \tau_i)$-g-$m_i$ closed set in $(X; \tau_i, \tau_i)$ then $B$ is a $(\tau_i, \tau_i)$-g-$m_i$ closed set in $(X; \tau_i, \tau_i)$.

**Proof:** Let $B \subseteq U$ such that $U$ is a $\tau_i$-minimal open set in $(X; \tau_i, \tau_i)$. Given $B \subseteq A \subseteq X$, so $B \subseteq A \cap U$ and $A$ is an $\tau_i$-open set in $X$. Then by the Lemma 2.20 $A \cap U$ is a $\tau_i$-minimal open set in $X$. Now $A \cap U \subseteq A \subseteq X$, then $A \cap U$ is a $\tau_i$-minimal open set in $A$ by the Lemma 2.19. Therefore $B \subseteq A \cap U$ and $A \cap U$ is a $\tau_i$-minimal open set in $A$. By hypothesis $A \cap \tau_i\text{-cl}(B) \subseteq A \cap U$ implies $\tau_i\text{-cl}(B) \subseteq U$. Hence $B$ is a $(\tau_i, \tau_i)$-g-$m_i$ closed set in $(X; \tau_i, \tau_i)$.

**Definition 2.22.** Let $i, j \in \{1, 2\}$ be fixed integers. In a bitopological space $(X; \tau_i, \tau_j)$, a subset $A$ of $X$ is said to be a $(\tau_i, \tau_j)$-generalized maximal open (briefly $(\tau_i, \tau_j)$-g-$m_a$ open) set iff $A^Z$ is a $(\tau_i, \tau_j)$-g-$m_a$ generalized minimal closed set.

**Theorem 2.23.** Let $i, j \in \{1, 2\}$ be fixed integers. A subset $A$ of a bitopological space $(X; \tau_i, \tau_j)$ is a $(\tau_i, \tau_j)$-g-$m_a$ open set iff $F \subseteq \tau_i\text{-int} A$ whenever $F \subseteq A$ and $F$ is a $\tau_i$-maximal closed set in $(X; \tau_i, \tau_j)$.
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**Proof:** Let $A$ be any $(\tau_i, \tau_j)$-$g$-$m$ set in $(X; \tau_i, \tau_j)$ such that $F \subseteq A$ and $F$ is a $\tau_i$-maximal closed set in $(X; \tau_i, \tau_j)$. Then, by the Definition 2.22, $A^c$ is a $(\tau_i, \tau_j)$-$g$-$m$ closed set in $(X; \tau_i, \tau_j)$. That is $A^c$ is a $(\tau_i, \tau_j)$-$g$-$m$ closed set whenever $A^c \subseteq F^c$. Therefore by the Definition 2.1, $(\tau_i, \tau_j)$-$g$-$m$ closed set whenever $A^c \subseteq F^c$ and $F$ is a $\tau_i$-minimal open set. Then $(\tau_i$-int $A^c) \subseteq F^c$, which implies $F \subseteq \text{int} A$. Conversely, let $A$ be any subset of $X$ such that $F \subseteq \text{int} A$ whenever $F \subseteq A$ and $F$ is a $\tau_i$-maximal closed set in $(X; \tau_i, \tau_j)$. Then $(\tau_i$-int $A^c) \subseteq F^c$ whenever $A^c \subseteq F$ and $F$ is a $\tau_i$-minimal open set. We have $\tau_i$-$cl (A^c) \subseteq F^c$ whenever $A^c \subseteq F$ and $F$ is a $\tau_i$-minimal open set. Therefore by the Definition 2.1, $A^c$ is a $(\tau_i, \tau_j)$-$g$-$m$ closed set. Thus $A$ is a $(\tau_i, \tau_j)$-$g$-$m$ open set in $(X; \tau_i, \tau_j)$.

**Theorem 2.24.** Let $i, j \in \{1, 2\}$ be fixed integers. Every $(\tau_i, \tau_j)$-$g$-$m$ open set is a $(\tau_i, \tau_j)$-$g$-$open$ set in a bitopological space $(X; \tau_i, \tau_j)$.

**Proof:** Let $A$ be a $(\tau_i, \tau_j)$-$g$-$m$ open set in $(X; \tau_i, \tau_j)$. Then $A^c$ is a $(\tau_i, \tau_j)$-$g$-$m$ closed set and by the Theorem 2.3, $A^c$ is a $(\tau_i, \tau_j)$-$g$-$open$ set in $(X; \tau_i, \tau_j)$.

**Remark 2.25.** Converse of the Theorem 2.24 need not be true.

**Example 2.26.** Let $X = \{a, b, c, d\}$ with $\tau_1 = \{\phi, \{a\}, \{a, b\}, \{a, c\}, \{a, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{a, b, c, d\}, X\}$ and $\tau_2 = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{b, c\}, \{b, d\}, \{a, b, d\}, \{b, c, d\}, X\}$.

$(\tau_1, \tau_2)$-$g$-$m$ open sets: $\{\{a, b\}, \{a, b, c\}, \{a, b, d\}, \{b, c\}, \{b, d\}\}$.

$(\tau_1, \tau_2)$-$g$-$open$ sets: $\{\phi, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}, \{b, d\}\}$.

**Theorem 2.27.** Let $i, j \in \{1, 2\}$ be fixed integers. Every $(\tau_i, \tau_j)$-$g$-$m$ open set is a $(\tau_i, \tau_j)$-$\omega$-$open$ set in a bitopological space $(X; \tau_i, \tau_j)$.

**Proof:** Let $A$ be a $(\tau_i, \tau_j)$-$g$-$m$ open set in $(X; \tau_i, \tau_j)$. Then $A^c$ is a $(\tau_i, \tau_j)$-$g$-$m$ closed set and by the Theorem 2.6, $A^c$ is a $(\tau_i, \tau_j)$-$\omega$-$open$ set in $(X; \tau_i, \tau_j)$.

**Remark 2.28.** Converse of the Theorem 2.27 need not be true.

**Example 2.29.** Let $X = \{a, b, c, d\}$ with $\tau_1 = \{\phi, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}, \{b, d\}, \{a, b, c\}, \{a, b, d\}, \{b, c, d\}, X\}$ and $\tau_2 = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{b, d\}, \{a, b, c\}, \{a, b, d\}, \{b, c, d\}, X\}$.

$(\tau_1, \tau_2)$-$g$-$m$ open sets: $\{\{a, b\}, \{a, b, c\}, \{a, b, d\}\}$.

$(\tau_1, \tau_2)$-$\omega$-$open$ sets: $\{\phi, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}, \{b, d\}, \{a, b, c\}, \{a, b, d\}, \{b, c, d\}, X\}$.

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**Theorem 2.30.** Let i, j ∈ \{1, 2\} be fixed integers. If \( \tau_i \)-int \( A \subseteq B \subseteq A \) and \( A \) is a \((\tau_i, \tau_j)\)-g-m, open set in a bitopological space \((X; \tau_i, \tau_j)\), then \( B \) is a \((\tau_i, \tau_j)\)-g-m, open set in \((X; \tau_i, \tau_j)\).

**Proof:** Given \( \tau_i \)-int \( A \subseteq B \subseteq A \) and \( A \) is a \((\tau_i, \tau_j)\)-g-m, open set in \((X; \tau_i, \tau_j)\). Then \( A^c \subseteq B^c \subseteq (\tau_i \text{-int } A)^c \) and \( A^c \) is a \((\tau_i, \tau_j)\)-g-m, closed set. That is \( A^c \subseteq B^c \subseteq \tau_j \text{-cl } (A^c) \) and \( A^c \) is a \((\tau_i, \tau_j)\)-g-m, closed set. By the Theorem 2.13 \( B^c \) is a \((\tau_i, \tau_j)\)-g-m, closed set. Thus by the Definition 2.22 \( B \) is a \((\tau_i, \tau_j)\)-g-m, open set in \((X; \tau_i, \tau_j)\).

**Theorem 2.31.** Let i, j ∈ \{1, 2\} be fixed integers. If a set \( A \) is any \((\tau_i, \tau_j)\)-g-m, open set in a bitopological space \((X; \tau_i, \tau_j)\), then \( O = X \) whenever \( O \) is an \( \tau_i \)-open set and \( \tau_j \)-int \((A) \cup A^c \subseteq O \).

**Proof:** Let \( A \) be any \((\tau_i, \tau_j)\)-g-m, open set in \((X; \tau_i, \tau_j)\) and \( O \) be an \( \tau_i \)-open set such that \( \tau_j \)-int \((A) \cup A^c \subseteq O \). Then \( A^c \) is a \((\tau_i, \tau_j)\)-g-m, closed set and \( O^c \) is a \( \tau_j \)-closed set such that \( O^c \subseteq [\tau_j \text{-int } (A) \cup A^c] = (\tau_i \text{-int } A^c) \cap (A^c)^c = \tau_j \text{-cl } (A^c) - A^c \). Since \( A^c \) is a \((\tau_i, \tau_j)\)-g-m, closed set and \( O^c \) is a \( \tau_j \)-closed set, by the Theorem 2.13 \( \tau_j \text{-cl } (A^c) - A^c \) contains no nonempty closed subset, which implies \( O^c = \emptyset \). Hence \( O = X \).

**Remark 2.32.** Converse of the Theorem 2.32 need not be true.

**Example 2.33.** In Example 2.26 let \( A = \{a, c\} \). The only \( \tau_i \)-open set containing \( \tau_j \)-int \((A) \cup A^c \) is \( X \), but \( A \) is not a \((\tau_i, \tau_j)\)-g-m, open set.

**Theorem 2.34.** Let i, j ∈ \{1, 2\} be fixed integers. If \( A \subseteq Y \subseteq X \) and \( A \) is a \((\tau_i, \tau_j)\)-g-m, open set in a bitopological space \((X; \tau_i, \tau_j)\), then \( A \) is a \((\tau_i, \tau_j)\)-g-m, open set relative to \( Y \).

**Proof:** Let \( A^c \subseteq Y \cap O \) such that \( Y \cap O \) is \( \tau_i,Y \) minimal open set and \( O \) is \( \tau_i \)-minimal open set in \((X; \tau_i, \tau_j)\). Then \( A^c \subseteq O \). By hypothesis \( A^c \) is a \((\tau_i, \tau_j)\)-g-m, closed set. Therefore \( \tau_j\text{-cl } (A^c) \subseteq O \cap Y \cap \tau_j\text{-cl } (A^c) \subseteq Y \cap O \). Hence \( A^c \) is \((\tau_i, \tau_j)\)-g-m, closed relative to \( Y \) which implies \( A \) is \((\tau_i, \tau_j)\)-g-m, open relative to \( Y \).

**Theorem 2.35.** Let i, j ∈ \{1, 2\} be fixed integers. If \( A \subseteq B \subseteq X \) and \( A \) is \((\tau_i, \tau_j)\)-g-m, open relative to \( B \) and \( B \) is \((\tau_i, \tau_j)\)-g-m, open set in \((X; \tau_i, \tau_j)\), then \( A \) is \((\tau_i, \tau_j)\)-g-m, open relative to \( Y \).

**Proof:** From [1] it is followed that \( A \) is \((\tau_i, \tau_j)\)-g-m, open relative to \( Y \).

**REFERENCES**