All the Solutions of the Diophantine Equation \( p^3 + q^2 = z^3 \)

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Abstract. In this paper, it is established that the title equation has exactly four solutions, all of which are exhibited.

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1. Introduction

In the huge field of Diophantine equations, no general method exists to decide whether a given Diophantine equation has any solutions, or how many solutions. In most cases, we are reduced to study individual equations, rather than classes of equations.

The literature contains a very large number of articles on non-linear such individual equations involving primes and powers of all kinds. Among them are for example [1, 3, 4, 5, 6, 9, 11, 13]. The title equation stems from \( p^3 + q^2 = z^3 \).

In this paper, the values \( x, y \), are fixed positive integers. Our main objective revolves around the existence and the number of solutions of the equation \( p^3 + q^2 = z^3 \).

2. The main result

In this section, we determine all the solutions of the equation \( p^3 + q^2 = z^3 \), where \( p, q, z \), and all other values represent positive integers. This is done in Theorem 2.1.

Theorem 2.1. Suppose that \( p \) is prime and \( q > 1 \). Then the equation

\[
p^3 + q^2 = z^3
\]

has exactly four solutions in all of which \( p = 7 \). In one solution \( q \) is prime, and in all other solutions \( q \) is composite.

Proof: From (1) we obtain

\[
q^2 = z^3 - p^3 = (z - p)(p^2 + pz + z^2).
\]

Denote \( z - p = T \) where \( T \geq 1 \). Substituting \( z = p + T \) into (2) results in

\[
q^2 = T(3p^2 + 3pT + T^2).
\]

We distinguish two cases for which equality (3) may be satisfied, namely: (i) When \( T > 1 \) and \( 3p^2 + 3pT + T^2 \) are squares simultaneously. (ii) When \( T \geq 1 \) and \( 3p^2 + 3pT + T^2 \) are not necessarily squares simultaneously.

We will now show that case (i) is actually impossible.
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(i) Suppose that \( T > 1 \) and \( 3p^2 + 3pT + T^2 \) are squares simultaneously. Denote \( T = U^2 \) and \( 3p^2 + 3pT + T^2 = V^2 \). Then
\[
3p^2 + 3pT + T^2 = 3p^2 + 3pU^2 + (U^2)^2 = V^2
\]
or
\[
3p(p + U^2) = V^2 - (U^2)^2 = (V - U^2)(V + U^2).
\] (4)

It follows from (4) that \( p \) divides at least one of the values \( V - U^2 \), \( V + U^2 \). We will now show that this statement does not hold.

If \( p \mid (V - U^2) \), denote \( pR = V - U^2 \) or \( V = pR + U^2 \). Then from (4) we have
\[
3p(p + U^2) = p^2R^2 + 2pRU^2 + (U^2)^2 - (U^2)^2
\]
or
\[
p^2(R^2 - 3) + pU^2(2R - 3) = 0
\]
which is impossible for all values \( R \). Hence \( p \nmid (V - U^2) \).

If \( p \mid (V + U^2) \), denote \( pS = V + U^2 \) or \( V = pS - U^2 \). From (4) we obtain
\[
p^2(S^2 - 3) - pU^2(2S + 3) = 0
\]
implying
\[
p = U^2, \quad 2S + 3
\]
The divisors of \( p \) are \( 1 \) and \( p \), and since \( T > 1 \) therefore \( T = U^2 > 1 \) or \( U > 1 \).

Then from (5) it follows that: either (a) \( \dfrac{U^2}{S^2 - 3} = 1 \) and \( 2S + 3 = p \), or (b) \( U = p \)
and \( \dfrac{U(2S + 3)}{S^2 - 3} = \dfrac{p(2S + 3)}{S^2 - 3} = 1 \). If (a), then \( \dfrac{U^2}{S^2 - 3} = 1 \) or \( U^2 = S^2 - 3 \). But \( S^2 - U^2 = 3 \) has the only solution \( U = 1 \) and \( S = 2 \) which is impossible. If (b), then
\[
\dfrac{p(2S + 3)}{S^2 - 3} = 1 \quad \text{or} \quad p = \dfrac{S^2 - 3}{2S + 3}
\]
which is impossible since \( \dfrac{S^2 - 3}{2S + 3} \) is never an integer.

Thus \( p \nmid (V + U^2) \), and case (i) is complete.

(ii) Suppose that \( T \geq 1 \) and \( 3p^2 + 3pT + T^2 \) are not necessarily squares simultaneously. In equality (3) set \( 3p^2 + 3pT + T^2 = TA^2 \) as
\[
3p^2 + 3pT + T^2 = TA^2
\] (6)
for some value \( A \) which guarantees that equality (3) is indeed a square \( q^2 = (TA)^2 \).

Then, from (6) it follows that \( T \mid 3p^2 \). The value \( T \) may assume all possible divisors of \( 3p^2 \), namely: \( T = 1 \), \( T = 3 \), \( T = p \), \( T = 3p \), \( T = p^2 \), \( T = 3p^2 \). The six cases are considered separately.

The case \( T = 1 \). Substituting \( T = 1 \) in (3) yields
\[
q^2 = 3p^2 + 3p + 1,
\] (7)
from which
\[
q^2 - 1 = (q - 1)(q + 1) = 3p(p + 1).
\]
Therefore, either \( p \mid (q - 1) \) or \( p \mid (q + 1) \). Note that \( p \neq 2 \).

If \( p \mid (q - 1) \), denote \( Bp = q - 1 \) where \( B \geq 1 \). Substituting \( q = Bp + 1 \) into (7) results in
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\[ B^2 p^2 + 2Bp + 1 = 3p^2 + 3p + 1, \]

and after simplifications implies that  \( p = \frac{3 - 2B}{B^2 - 3}. \) The term \( \frac{3 - 2B}{B^2 - 3} \) is negative for all values  \( B \geq 1, \) and therefore is impossible. Thus,  \( p \not| (q - 1) \) and  \( p \not| (q + 1). \)

If  \( p \not| (q + 1), \) denote  \( Cp = q + 1 \) where  \( C \geq 1. \) Then  \( q = Cp - 1, \) and from (7) it follows that
\[ q^2 = (Cp - 1)^2 = C^2p^2 - 2Cp + 1 = 3p^2 + 3p + 1. \] (8)

After simplifications of (8), one obtains that
\[ p = \frac{2C + 3}{C^2 - 3}. \]

Evidently, the only value that  \( C \) may assume is  \( C = 2. \) Hence,  \( C = 2 \) yields
\[ \frac{2 \cdot 2 + 3}{2^2 - 3} = 7 = p. \] The values  \( p = 7, q = 2p - 1 = 13 \) prime, and  \( z = p + 1 = 8 \) form a solution of equation (1).

The case  \( T = 1 \) is complete.

The case  \( T = 3. \) From (3) we obtain  \( q^2 = 3(3p^2 + 9p + 9) \) or  \( q^2 = 3^2(p^2 + 3p + 3) \)
implying that  \( p^2 + 3p + 3 \) must equal a square, say  \( A^2. \) If  \( p^2 + 3p + 3 = A^2, \) then
\[ A^2 - p^2 = (A - p)(A + p) = 3(p + 1). \] (9)

We now show that  \( 3 \not| (A - p) \) and  \( 3 \not| (A + p) \) implying that  \( T \neq 3. \) If  \( 3 \not| (A - p), \)
denote  \( 3D = A - p \) where  \( D \geq 1. \) Hence, from (9)  \( 3D(A + p) = 3(p + 1) \) or  \( D(A + p) = p + 1. \) Since  \( A > p, \) this equality is impossible and  \( 3 \not| (A - p). \) If  \( 3 \not| (A + p) \) then  \( 3E = A + p. \) We have from (9) that  \( (3E - 2p)3E = 3(p + 1) \) or  \( (3E - 2p)E = p + 1. \)

Thus,  \( p(2E + 1) = 3E^2 - 1, \) and  \( p = \frac{3E^2 - 1}{2E + 1}. \) But, this fraction never equals an integer, and therefore it follows that  \( 3 \not| (A + p). \) Hence  \( T \neq 3. \)

As an immediate consequence, it follows that for every prime  \( p, \)  \( p^2 + 3p + 3 \) is never equal to a square.

The case  \( T = p. \) With  \( T = p \) in (3), we obtain  \( q^2 = p(7p^2) = 7p^3 \) implying that  \( p = 7 \) and  \( q = 7 \). Hence, the values  \( p = 7, q = 7 \) and  \( z = 2p = 14 \) yield a solution of equation (1).

The case  \( T = 3p. \) When  \( T = 3p \) in (3), then  \( q^2 = 3p\cdot21p^2 = 3^2\cdot7p^3. \) Thus,  \( p = 7 \) and  \( q^2 = 3^2p^4 = 3^2\cdot7^2 \) \( \text{The values } p = 7, q = 3\cdot7^2 \) and  \( z = 4p = 28 \) form a solution of equation (1).

The case  \( T = p^2. \) From (3) we have
\[ q^2 = p^4(3p^2 + 3p + 3) = p^4(3 + 3p + p^2). \]

It now follows that the value  \( p^2 + 3p + 3 \) must equal a square say  \( M^2, \) so that  \( q^2 = (p^2M)^2. \) But,  \( p^2 + 3p + 3 \neq M^2 \) as was shown in the case  \( T = 3. \) Thus  \( T \neq p^2. \)

The case  \( T = 3p^2. \) From (3) we obtain
\[ q^2 = 3p^2(3p^2 + 9p + 9p^2) = 9p^4(1 + 3p + 3p^2). \]

Therefore, the value  \( 3p^2 + 3p + 1 \) must be equal to a square, say  \( N^2, \) in order that  \( q^2 = (3p^2N)^2. \) The value  \( 3p^2 + 3p + 1 \) appears in equality (7) of the case  \( T = 1, \) and is indeed equal to a square only when  \( p = 7 \) for which a solution of equation (1) exists.
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Hence, the values \( p = 7, q = 3p^2(3p^2 + 3p + 1)^{1/2} = 3 \cdot 7 \cdot 13 \) and \( z = p + 3p^2 = p(3p+1) = 2 \cdot 7 \cdot 11 \) yield a solution to equation (1).

The four solutions of equation (1) have been established and exhibited.

This concludes the proof of Theorem 2.1. □

As a consequence of Theorem 2.1 we have:

Remark 2.1. The unique solution of the square \( K^2 = 3p^2 + 3pT + T^2 \) consists of the value \( T = 1 \) and the primes \( p = 7 \) and \( K = 13 \).

As a summary, and for the convenience of the readers, we now demonstrate the four solutions in the order of their occurrence.

Solution 1. \( 7^3 + 13^3 = (2 \cdot 3)^3 \).

Solution 2. \( 7^3 + (7)^3 = (2 \cdot 7)^3 \).

Solution 3. \( 7^3 + (3 \cdot 7)^2 = (2 \cdot 7)^3 \).

Solution 4. \( 7^3 + (3 \cdot 7^2 \cdot 13)^2 = (2 \cdot 7 \cdot 11)^3 \).

3. Conclusion

We conclude by giving a glimpse on the equation \( p^3 + q^m = z^3 \) when \( m = 1, 2 \) and 3.

It is easily seen that infinitely many solutions exist for the equation \( p^3 + q^1 = z^3 \) when \( p \) is prime and \( q \) is prime/composite. Few such examples are:

\[
2^3 + 19 = 3^3, \quad 3^3 + 37 = 4^3, \quad 5^3 + 91 = 6^3, \quad 7^3 + 386 = 9^3.
\]

In this paper, the equation \( p^3 + q^2 = z^3 \) yields quite surprisingly only four solutions in all of which \( p = 7 \) and in only one of them \( q \) is prime.

In 1637, Fermat (1601 – 1665) stated that the Diophantine equation \( x^n + y^n = z^n \), with integral \( n > 2 \), has no solutions in positive integers \( x, y, z \). This is known as Fermat's "Last Theorem". In 1995, 358 years later, the validity of the Theorem was established and published by A. Wiles. Thus, the equation \( p^3 + q^3 = z^3 \) has no solutions in positive integers \( p, q, z \).

REFERENCES

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