Characteristic Subgroups of a finite Abelian Group

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Abstract. We consider the following questions: (i) number of characteristic subgroups of a finite abelian p-group \( Z_{p^n} \times Z_{p^n} \) (ii) number of characteristic subgroups of a finite abelian group \( Z_n \times Z_n \) and (iii) characteristic subgroup lattice of \( Z_n \times Z_n \) is isomorphic to subgroup lattice of \( Z_n \).

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1. Introduction

In 1939, Baer [1] considered the following question “When two groups have isomorphic subgroups lattices?” Since this is a very difficult problem. Here authors consider a related question “When two groups have isomorphic lattices of characteristic subgroups?” In general problem considered by Baer [1] or related question consider by authors seems to very difficult. We will consider only the particular case of finite Abelian group of rank two i.e., \( Z_n \times Z_n \).

A subgroup \( N \) of a group \( G \) is called a Characteristic Subgroup if \( \Phi(N) = N \) for all Automorphism \( \Phi \) of \( G \). This term was first used by Frobenius in 1895.

Theorem 1.1. If \( \gcd(|H|, |K|) = 1 \), \( H \times K \) is characteristic subgroup of \( G \) if and only if \( H \) and \( K \) are characteristic subgroup of \( G \).

Proof: Let \( x \in H \times K \)
\[
\therefore \text{x is uniquely expressed as product of } h \in H \text{ and } k \in K \text{ such that } x = hk. 
\]

Then \( f(x) = f(hk) = f(h)f(k) \) \( \forall f \in Aut(G) \)

It is given that \( H \) and \( K \) is characteristic subgroups of \( G \), therefore \( f(h) \in H \) and \( f(k) \in K \).
\[
\therefore f(x) \in HK 
\]

Here \( HK = H \times K \) [ Because \( H \triangleleft G, K \triangleleft G \) and \( H \cap K = \{e\} \)]
\[
\therefore H \times K \text{ is characteristic subgroup of } G. 
\]

Converse \( \therefore \) Let \( h(\neq e) \in H \), then \( h = he \in H \times K \).
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\[ f(h) \in H \times K \quad \forall f \in Aut(G) \quad [\text{Because } H \times K \text{ is characteristic subgroup of } G]. \]

Therefore \( f(h) \) is uniquely expressed as product of elements of \( H \) and \( K \), then \( f(h) = f(h)e. \)

If possible \( f(h) \in K \implies |f(h)|||K| \quad (1) \)

But \( |h||H| \) and \( |f(h)| = |h| \implies |f(h)|||H| \quad (2) \)

From (1) and (2), we have

\[ |f(h)|||H|, |K| \implies |f(h)||1 = f(h) = e \implies h = e. \]

This contradiction shows that \( f(h) \in H. \)

Hence \( H \) is characteristic subgroup of \( G. \)

Similarly, \( K \) is characteristic subgroup of \( G. \)

If we denote \( NC(G) \) the number of characteristic subgroups of the group \( G \), then by use of theorem 1.1 we have,

\[ p \leq 1 \leq \prod_{i=1}^{r} NC(Z_{p^i} \times Z_{p^i}) \]

\[ n = p_1 a_1 p_2 a_2 p_3 a_3 \ldots p_r a_r. \]

Now our problem is reduced to find number of characteristic subgroups of a finite abelian of type \( Z_{p^a} \times Z_{p^a}. \)

2. Partition

Firstly we partition the set \( S \) (non-trivial cyclic subgroups of \( Z_{p^m} \times Z_{p^n} \) \( 1 \leq m \leq n \)) into \((p+1)\) partitions.

Two cyclic subgroups \( H \) and \( K \) in \( S \) are equivalent, denoted by \( H \sim K \), if and only if \( H \cap K \) contains a subgroup of order \( p \) (clearly such subgroup is unique and cyclic)

**Lemma 2.1.** The relation \( \sim \) between elements of the \( S \) is an equivalence relation on \( S. \)

**Proof:** **Reflexive.** Since \( H \) is a non-trivial cyclic subgroup of \( Z_{p^a_1} \times Z_{p^a_2} \), then \( H \) contains a subgroup of order \( p \). Hence \( H \cap H = H \) contains a subgroup of order \( p \), then \( H \sim H. \)

**Symmetric.** If \( H \sim K \), then \( H \cap K \) contains a subgroup of order \( p \), since \( H \cap K = K \cap H \).

We deduce that \( K \cap H \) contains a subgroup of order \( p \) and consequently \( K \sim H. \)

**Transitive.** If \( H \sim K \) and \( K \sim L \), then \( H \cap K \) and \( K \cap L \) contains a subgroup of order \( p \). By using result “every cyclic subgroup of order \( p^\alpha (\alpha \geq 1) \) has unique subgroup of order \( p \)”, hence \( H \) and \( L \) contains same cyclic subgroup of order \( p \) which is contained by \( K. \)

Therefore \( H \cap L \) contains a subgroup of order \( p \) and consequently \( H \sim L. \)

Hence relation \( \sim \) is called equivalence relation.

**Theorem 2.2.** An equivalence relation \( \sim \) on a non-empty set \( S \) partitions the set \( S \) into the disjoint union of distinct equivalence classes.

Here group \( Z_{p^m} \times Z_{p^n} \) has only \( p+1 \) cyclic subgroups of order \( p \), using above theorem we can partition set \( S \) into \( p+1 \) distinct equivalence class and these partition are as follows:

\( a) \quad [(0,p^{n-1})] = \{ H \in S|H \sim ((0,p^{n-1})}\) and denoted by class-0

\( b) \quad [(p^{m-1},ip^{n-1})] = \{ H \in S|H \sim ((p^{m-1},ip^{n-1})\) and denoted by class-i where \( 1 \leq i \leq p. \)

3. Main theorem

**Theorem 3.1.** Prove that there is exactly one characteristic subgroup of order \( p \) in group \( Z_{p^m} \times Z_{p^n} \) where \( m < n \) i.e., \( ((0,p^{n-1})\) which belong to class-0.
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**Proof:** From [2], we know that there are exactly $p+1$ subgroups of order $p$ in group $Z_p^m \times Z_p^n$ and they are given below:

(i) $\langle (0, p^{n-1}) \rangle$ from class-0
(ii) $\langle (p^{m-1}, ip^{n-1}) \rangle$ from class-1 where $1 \leq i \leq p$

Firstly, we prove that $\langle (0, p^{n-1}) \rangle$ is a characteristic subgroup of group $Z_p^m \times Z_p^n$.

In group $Z_p^m \times Z_p^n$, order of element $(0,1)$ is $p^n$ and therefore in any automorphism $(0,1)$ is transferred to element of group $Z_p^m \times Z_p^n$ which has order $p^n$, they are written as $(j, k)$ where $(k,p)=1$.

Let $x$ be any element of subgroup $\langle (0, p^{n-1}) \rangle$, then $x = (0, rp^{n-1})$.

Let $f(x) = f(0, rp^{n-1}) = rp^{n-1}f(0,1) = rp^{n-1}(j,k) = (rjp^{n-1},rkp^{n-1})$

Hence $f(x) = (0, rp^{n-1}) \in \langle (0, p^{n-1}) \rangle$

Therefore, subgroup $\langle (0, p^{n-1}) \rangle$ is a characteristic subgroup of group $Z_p^m \times Z_p^n$.

Secondly, we prove that $\langle (p^{m-1}, ip^{n-1}) \rangle$ is not a characteristic subgroup of group $Z_p^m \times Z_p^n$ for $1 \leq i \leq p$.

In group $Z_p^m \times Z_p^n$, order of element $(1,0)$ is $p^m$ and therefore in any automorphism $(1,0)$ is transferred to element of group $Z_p^m \times Z_p^n$ which has order $p^m$ which belong to class other than-0. Take $(j \neq 0 (modp)$.

Let $f_j$ be an Automorphism of group $Z_p^m \times Z_p^n$ such that $f_j(1,0) = (1,jp^{n-m})$ and $f_j(0,1) = (0,1)$

Then $f_j(kp^{m-1},ikp^{n-1}) = kp^{m-1}f_j(1,0) + ikp^{n-1}f_j(0,1) = kp^{m-1}(1,jp^{n-m}) + ikp^{n-1}(0,1) = (kp^{m-1},k(i+j)p^{n-1}) \notin \langle (p^{m-1}, ip^{n-1}) \rangle \forall k \neq 0 (modp)$

Hence, subgroup $\langle (p^{m-1}, ip^{n-1}) \rangle$ is a not characteristic subgroup of group $Z_p^m \times Z_p^n$.

**Theorem 3.2.** Prove that there is no subgroup of order $p$ which is characteristic subgroup of group $Z_p^n \times Z_p^n$.

**Proof:** From [2], we know that there are exactly $p+1$ subgroups of order $p$ in group $Z_p^n \times Z_p^n$ and they are given below:

(i) $\langle (0, p^{n-1}) \rangle$
(ii) $\langle (p^{n-1}, ip^{n-1}) \rangle$ where $1 \leq i \leq p$.

Firstly, we prove that $\langle (0, p^{n-1}) \rangle$ is not a characteristic subgroup of group $Z_p^n \times Z_p^n$.

Let $f_0$ be an Automorphism of group $Z_p^n \times Z_p^n$ such that $f_0(1,0) = (0,1)$ and $f_0(0,1) = (1,0)$.

Then $f_0(0,kp^{n-1}) = kp^{n-1}f_0(0,1) = kp^{n-1}(1,0) = (kp^{n-1},0) \notin \langle (0, p^{n-1}) \rangle \forall k \neq 0 (modp)$.

Secondly, we prove that $\langle (p^{n-1}, ip^{n-1}) \rangle$ is not a characteristic subgroup of group $Z_p^n \times Z_p^n$ for $1 \leq i \leq p$.

Let $f_i$ be an Automorphism of group $Z_p^n \times Z_p^n$ such that $f_i(1,0) = (p-i,1)$ and $f_i(0,1) = (1,0)$.

Then $f_i(kp^{n-1},ikp^{n-1}) = kp^{n-1}f_i(1,0) + ikp^{n-1}f_i(0,1) = kp^{n-1}(p-i,1) + ikp^{n-1}(1,0) = (kp^{n-1},kp^{n-1}) = (0,kp^{n-1}) \notin \langle (p^{n-1}, ip^{n-1}) \rangle \forall k \neq 0 (modp)$.
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Hence there is no subgroup of order $p$ which is characteristic subgroup of group $Z_p^n \times Z_p^n$

**Theorem 3.3.** [3] Characteristic property is transitive. That is, if $N$ is characteristic subgroup of $K$ and $K$ is characteristic subgroup of $G$, then $N$ is characteristic subgroup of $G$.

**Theorem 3.4.** Number of characteristic subgroup of a group $Z_p^n \times Z_p^n$ are $\tau(p^n)$ and its characteristic subgroup lattice is isomorphic to subgroup lattice of group $Z_p^n$.

**Proof:**

**Case 1:** When subgroup of group $Z_p^n \times Z_p^n$ which is isomorphic group $Z_{p_1^n} \times Z_{p_2^n}$ where $1 \leq \alpha_1 < \alpha_2 \leq n$

If possible there exist a characteristic subgroup $H$ from group $Z_p^n \times Z_p^n$ which is isomorphic group $Z_{p_1^n} \times Z_{p_2^n}$ where $1 \leq \alpha_1 < \alpha_2 \leq n$

By using theorem 3.2, then there exists a characteristic subgroup $K$ of order $p$ from subgroup $H$.

Now $K$ is characteristic subgroup of $H$ and $H$ is characteristic subgroup of $Z_p^n \times Z_p^n$, by use of theorem 3, we conclude that $K$ is a characteristic subgroup of $Z_p^n \times Z_p^n$. By use of theorem 3.1, $K$ is not a characteristic subgroup of $Z_p^n \times Z_p^n$, which contraction with fact that there exist a characteristic subgroup $H$ from group $Z_p^n \times Z_p^n$ which is isomorphic group $Z_{p_1^n} \times Z_{p_2^n}$ where $1 \leq \alpha_1 < \alpha_2 \leq n$.

**Case 2:** When subgroup of group $Z_p^n \times Z_p^n$ which is isomorphic $Z_{p_1^n} \times Z_{p_2^n}$ where $0 \leq \alpha \leq n$

From [2], there is exactly one subgroup from group $Z_p^n \times Z_p^n$ which is isomorphic to $Z_{p_1^n} \times Z_{p_2^n}$. This subgroup must be characteristic subgroup. Hence there exist one subgroup for each $\alpha$, therefore total number of characteristic subgroups of group $Z_p^n \times Z_p^n$ are $n+1$ or $\tau(p^n)$. These subgroups are $< (p^{n-i},0),(0,p^{n-i}) >$ where $i = 0, 1, 2, \ldots, n$

Its characteristic subgroup lattice is as follows:-

$< (0,0) > < (p^{n-1},0),(0,p^{n-1}) > < (p^{n-2},0),(0,p^{n-2}) > \ldots < (1,0),(0,1) > = Z_{p^n} \times Z_{p^n}$

Subgroup lattice of group $Z_p^n$ is as follows:-

$< 0 > < p^{n-1} > < p^{n-2} > \ldots < 1 > = Z_p^n$

Let as define a mapping $f$ from a set of characteristic subgroup of group $Z_p^n \times Z_p^n$ to set of subgroups of $Z_p^n$ such that $f(< (p^{n-i},0),(0,p^{n-i}) >) = < p^{n-i} >$. This mapping $f$ also preserves subset property means $< (p^{n-i},0),(0,p^{n-i}) > < (p^{n-j},0),(0,p^{n-j}) > \Leftrightarrow f(< (p^{n-i},0),(0,p^{n-i}) >) \subseteq f(< (p^{n-j},0),(0,p^{n-j}) >)$

Hence characteristic subgroup lattice of group $Z_p^n \times Z_p^n$ is isomorphic to subgroup lattice of group $Z_p^n$

**Theorem 3.5.** Number of characteristic subgroup of a group $Z_n \times Z_n$ are $\tau(n)$ and its characteristic subgroup lattice is isomorphic to subgroup lattice of group $Z_n$.


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**Proof**: We know that $NC(Z_n \times Z_n) = \prod_{i=1}^{r} NC(Z_{p_i^{a_i}} \times Z_{p_i^{a_i}})$ where $n = p_1^{a_1} p_2^{a_2} p_3^{a_3} \ldots p_r^{a_r}$. Hence $NC(Z_n \times Z_n) = \prod_{i=1}^{r} \tau(p_i^{a_i}) = \tau(n)$.

If $LC(G)$ for characteristic subgroup lattice of $G$, then $LC(Z_n \times Z_n) \cong LC(Z_{p_1^{a_1}} \times Z_{p_2^{a_2}} \times Z_{p_3^{a_3}} \times \ldots \times Z_{p_r^{a_r}})$ is the direct product of corresponding subgroup lattices (Suzuki[5]).

From theorem 3.4, we have $LC(Z_{p_1^{a_1}} \times Z_{p_2^{a_2}} \times Z_{p_3^{a_3}} \times \ldots \times Z_{p_r^{a_r}}) \cong LC(Z_{p_1^{a_1}})$, where $LC(Z_{p_1^{a_1}})$ denotes subgroup lattice of group $Z_{p_1^{a_1}}$.

Hence, $LC(Z_n \times Z_n) \cong LC(Z_{p_1^{a_1}}) \times LC(Z_{p_2^{a_2}}) \times \ldots \times LC(Z_{p_r^{a_r}}) \cong L(Z_n)$.

**4. Conclusion**

In this paper, we have conclude that Number of characteristic subgroup of a group $Z_n \times Z_n$ are $\tau(n)$ and its characteristic subgroup lattice is isomorphic to subgroup lattice of group $Z_n$.

**REFERENCES**