Fixed Point Theorems for Kannan Contractions and Weakly Contractive Mappings on a Modular Metric Space Endowed with a Graph

Prerna Pathak¹, Aklesh Pariya², V. H. Badshah³ and Nirmala Gupta⁴

¹,³School of Studies in Mathematics, Vikram University, Ujjain (M.P.), India
Email: prernapathak20@yahoo.com

²Department of Mathematics, Medi-caps University, Indore (M.P.), India
Email: akleshpariya3@yahoo.co.in

⁴Department of Mathematics, Govt. Girls Degree College, Ujjain (M.P.), India
Email: gupta.nirmala70@gmail.com
Corresponding author. Email: prernapathak20@yahoo.com

Received 13 June 2017; accepted 29 June 2017

Abstract. The notion of a modular metric spaces were introduced by Chistyakov [5, 6]. Abdou and Khamsi [1] gave the analog of Banach contraction principle in modular metric spaces. More recently, Alfuraidan [3] gave generalization of Banach contraction principle on a modular metric space endowed with a graph which is the modular metric version of Jachymski [8] fixed point results.

In this paper, we generalize and prove some fixed point results for Kannan contraction and weakly contractive mappings in a modular metric space endowed with a graph. The result of this paper is new and improving the previously known result in modular metric spaces endowed with a graph.

Keywords: Modular metric spaces, common fixed point, connected graph, Banach contraction, Kannan contraction.

AMS Mathematics Subject Classification (2010): 47H09, 46B20, 47H10, 47E10

1. Introduction
The existence of fixed points for single valued mappings in partially ordered metric spaces was initially considered by Ran and Reurings [15]. Fixed point theorems for monotone single valued mappings in a metric space endowed with a partial ordering have been widely investigated. Recently, many results appeared giving sufficient condition for f to be a PO if (X, d) is endowed with a partial ordering ≼. These results are the hybrid of two fundamental and useful theorems in fixed point theory, Banach Contraction Principle and the Knaster-Tarski theorem (see[7]). Jachymski [8] obtain some useful result for mappings defined on a complete metric spaces endowed with a graph instead of partial ordering. Bojor [4] proved fixed point result for Kannan mappings in metric spaces endowed with a graph. Samreen and Kamran [16] proved fixed point theorems for
weakly contractive mappings on a metric space endowed with a graph. After that many researchers have investigated in this direction by weakly contractive condition and analyzing connectivity condition of graph.

The notion of modular spaces was introduce by Nakan o [13] and was intensively develop by Koshi and Shimogaki [11], Yamamuro [17] and by Musielak and Orlicz [12]. Recently, Aghanians and Nourozi [2] discuss the existence and uniqueness of the fixed point for Banach and Kannan contraction defined on modular spaces endowed with a graph.

The notion of modular metric spaces was introduce by Chistyakov [5,6]. Further Abdou and Khamsi [1] gave the analog of Banach contraction principle in modular metric spaces. More recently, Alfuraidan [3] gave generalization of Banach contraction principle on a modular metric space endowed with a graph which is the modular metric version of Jachymski [8] fixed point results for mappings on a metric space with a graph.

Ran and Reurings [15] proved the following fixed points result.

**Theorem 1.1.** [15] Let \((X,\preceq)\) be a partially ordered set such that every pair \(x, y \in X\) has an upper and lower bound. Let \(d\) be a metric on \(X\) such that \((X, d)\) is a complete metric space. Let \(f : X \to X\) be a continuous monotone (either order preserving or order reversing) mapping. Suppose that the following condition hold:

1. There exist a \(k \in (0,1)\) with 
   \[ d(f(x), f(y)) \leq kd(x, y), \quad \text{for all } x \preceq y. \]
2. There exist an \(x_0 \in X\) with \(x_0 \preceq f(x_0)\) or \(x_0 \succeq f(x_0)\).

Then \(f\) is a Picard operator (PO), that is, \(f\) has a unique fixed point \(x_* \in X\) and for each \(x \in X\), \(\lim_{n \to \infty} f^n x = x_*\).

Nieto et al. in [14], proved the following fixed point theorem.

**Theorem 1.2.** [14] Let \((X,\preceq)\) be a complete metric spaces endowed with a partial ordering \(\preceq\). Let \(f : X \to X\) be an order preserving mapping such that there exists a \(k \in [0,1)\) with 

\[ d(f(x), f(y)) \leq kd(x, y), \quad \text{for all } x \preceq y. \]

Assume that one of the following conditions holds:

1. \(f\) is continuous and there exists an \(x_0 \in X\) with \(x_0 \preceq f(x_0)\) or \(x_0 \succeq f(x_0)\);
2. \((X, d, \preceq)\) is such that for any non decreasing \((x_n)_{n \in \mathbb{N}}\), if \(x_n \to x\), then \(x_n \preceq x\) for \(n \in \mathbb{N}\), and there exist an \(x_0 \in X\) with \(x_0 \preceq f(x_0)\);
3. \((X, d, \preceq)\) is such that for any non-decreasing \((x_n)_{n \in \mathbb{N}}\), if \(x_n \to x\), then \(x_n \succeq x\) for \(n \in \mathbb{N}\), and there exist an \(x_0 \in X\) with \(x_0 \succeq f(x_0)\);

then \(f\) has a fixed point. Moreover, if \((X, \preceq)\) is such that every pair of elements of \(X\) has an upper or a lower bound, then \(f\) is a PO.

Jachymski [9] obtained the contraction principle for mappings on a metric spaces endowed with a graph.

**Theorem 1.3.** [9] Let \((X, d)\) be a complete metric space and let the triplet \((X, d, G)\) have the following property:
Fixed Point Theorems for Kannan Contractions and Weakly Contractive Mappings on a Modular Metric Space Endowed with a Graph

(P) for any sequence \((x_n)_{n\in\mathbb{N}}\) in X as \(n\to\infty\) and \((x_n,x_{n+1})\in E(G)\), then \((x_n,x)\in E(G)\), for all \(n\). Let \(f:X\to X\) be a G-contraction. Then the following statements hold:
1. \(F_f \neq \emptyset\) if and only if \(X_f \neq \emptyset\);  
2. if \(X_f \neq \emptyset\) and G is weakly connected, then \(f\) is a Picard operator, i.e. \(F_f = \{x^*\}\) and sequence \(\{f^n(x)\} \to x^*\) as \(n \to \infty\), for all \(x \in X\);  
3. for any \(x \in X\), \(f\) is a G-Kannan mapping. We suppose that:
   (i) there is a subsequence \((x_{k_n})_{n \in \mathbb{N}}\) with \((x_{k_n},x) \in E(G)\) for \(n \in \mathbb{N}\),
   (ii) there is a subsequence \((x_{k_n})_{n \in \mathbb{N}}\) with \((x_{k_n},x) \in E(G)\) for \(n \in \mathbb{N}\).
Then \(T\) is a PO.

Samreen and Kamran [16] proved fixed point theorem for weakly contractive mappings on a metric space endowed with a graph.

Theorem 1.4. [4] Let \((X,d)\) be a complete metric space endowed with a graph \(G\) and \(T:X\to X\) be a G-Kannan mapping. We suppose that:
(i) \(G\) is weakly T-connected;  
(ii) for any \((x_n)_{n \in \mathbb{N}}\in X\), if \(x_n \to x\) and \((x_n,x_{n+1})\in E(G)\) for \(n \in \mathbb{N}\)
then there is a subsequence \((x_{k_n})_{n \in \mathbb{N}}\) with \((x_{k_n},x) \in E(G)\) for \(n \in \mathbb{N}\).
Then \(T\) is a PO.


Theorem 1.5. [16] Let \((X,d)\) be a completed metric space endowed with a graph \(G\) and \(f\) be a weakly G-contractive mapping from \(X\) into \(X\). Suppose that the following condition holds.
(i) \(G\) satisfies property \((p')\),
(ii) there exist some \(x_0 \in X\) and \(f|_{[x_0]}\) has a unique fixed point \(\xi \in [x_0]\) and \(f^n(x) \to \xi\) for any \(x \in [x_0]\).
Then \(f|_{[x_0]}\) has a unique fixed point \(\xi \in [x_0]\) and \(f^n(x) \to \xi\) for any \(x \in [x_0]\).

Alfuraidan [3] proved the contraction principle for mappings on a modular metric space with a graph.

Theorem 1.6. [2] Let \(X\) be a \(\rho\)-complete modular space endowed with a graph \(G\) and the triple \((X,\rho,G)\). Moreover, this fixed point is unique if \(k < \frac{1}{2}\) and \(X\) satisfies the following condition For all \(x,y \in X\), there exists a \(z \in X\) such that \((x,z),(y,z)\in E(G)\). Then a Kannan \(G-\rho\) contraction \(f:X\to X\) has a fixed point if and only if \(G_f \neq \emptyset\).

Theorem 1.7. [3] Let \((X,\omega)\) be a modular metric space with a graph \(G_\omega\). Suppose that \(\omega\) is a convex regular modular metric which satisfies the \(\Delta_2\) – type condition. Assume that \(M = V(G_\omega)\) is a nonempty \(\omega\) – bounded, \(\omega\) – complete subset of \(X_\omega\) and the triple \((M,d_\omega,G_\omega)\) has property (P) Let \(T:M\to M\) be \(G_\omega\)-contraction map and \(M_T := \{x \in M; (x,Tx)\in E(G_\omega)\}\). If \((x_0,T(x_0))\in E(G_\omega)\), then the following statement holds:
(i) For any \(x \in M\), \(T|_{[x]}\) has a fixed point.
Prerna Pathak, Aklesh Pariya, V. H. Badshah and Nirmala Gupta

(ii) If \( G_\omega \) is weakly connected, then \( T \) has a fixed point in \( M \).

(iii) If \( M' := \bigcup\{[x]_{G_\omega}: x \in M_T\} \), then \( T|_{M'} \) has a fixed point in \( M \).

2. Basic definition and preliminaries

Let \( X \) be a nonempty set. Throughout this paper for a function \( \omega : (0, \infty) \times X \times X \rightarrow (0, \infty) \) will be written as \( \omega_\lambda(x, y) = \omega(\lambda, x, y) \) for all \( \lambda > 0 \) and \( x, y \in X \).

**Definition 2.1.** [5, 6] Let \( X \) be a non-empty set. A function \( \omega : (0, \infty) \times X \times X \rightarrow [0, \infty] \) is said to be a metric modular on \( X \) if it satisfies the following three axioms:

(i) \( \omega_\lambda(x, y) = 0 \) for all \( \lambda > 0 \) if and only if \( x = y \);

(ii) \( \omega_\lambda(x, y) = \omega_\lambda(y, x) \) for all \( \lambda > 0 \) and \( x, y \in X \);

(iii) \( \omega_{\lambda+\mu}(x, y) \leq \omega_\lambda(x, z) + \omega_\mu(z, y) \) for all \( \lambda, \mu > 0 \) and \( x, y, z \in X \).

If instead of (i), we have only the condition

\[ \omega_\lambda(x, x) = 0 \] for all \( \lambda > 0 \) and \( x \in X \).

Then \( \omega \) is said to be a (metric) pseudo modular on \( X \). A modular \( \omega \) on \( X \) is said to be regular if the following weaker version of (i) is satisfied:

\[ x = y \text{ if and only if } \omega_\lambda(x, y) = 0 \text{ for some } \lambda > 0. \]

Finally \( \omega \) is said to be convex if for \( \lambda, \mu > 0 \) and \( x, y, z \in X \), it satisfies the inequality

\[ \omega_{\lambda+\mu}(x, y) = \frac{\lambda}{\lambda + \mu} \omega_\lambda(x, z) + \frac{\mu}{\lambda + \mu} \omega_\mu(z, y). \]

Note that for a pseudo modular \( \omega \) on a set \( X \) and any \( x, y \in X \), the function \( \lambda \rightarrow \omega_\lambda(x, y) \) is non increasing on \((0, \infty)\). Indeed, if \( 0 < \mu < \lambda \), then

\[ \omega_\lambda(x, y) \leq \omega_{\lambda-\mu}(x, x) + \omega_\mu(x, y) = \omega_\mu(x, y) \]

**Definition 2.2.** Let \( X_\omega \) be a modular metric space.

1. The sequence \( (x_n)_{n \in \mathbb{N}} \) in \( X_\omega \) is said to be convergent to \( x \in X_\omega \) if

\[ \omega_\lambda(x_n, x) \rightarrow 0 \text{ as } n \rightarrow \infty \text{ for all } \lambda > 0. \]

2. The sequence \( (x_n)_{n \in \mathbb{N}} \) in \( X_\omega \) is said to be Cauchy if

\[ \omega_\lambda(x_m, x_n) \rightarrow 0 \text{ as } n \rightarrow \infty \text{ for all } \lambda > 0. \]

3. A subset \( C \) of \( X_\omega \) is said to be closed if the limit of the convergent sequence of \( C \) always belong to \( C \).

4. A subset \( C \) of \( X_\omega \) is said to be complete if any Cauchy sequence in \( C \) is a convergent sequence and its limit in \( C \).

5. A subset \( C \) of \( X_\omega \) is said to be bounded if for all \( \lambda > 0 \)

\[ \delta_\omega(C) = \sup\{\omega_\lambda(x, y); x, y \in C\} < \infty. \]

We will use following notations and terminology of graph theory (see [3]) related to the rest of our result.

Let \( (X, \omega) \) be a modular metric space and \( M \) be a non empty subset of \( X_\omega \). Let \( \Delta \) denote the diagonal of the Cartesian product \( M \times M \). Consider a directed graph \( G_\omega \) such that the set \( V(G_\omega) \) of its vertices coincide with \( M \), and the set \( E(G_\omega) \) of its edges contain all loops, i.e. \( E(G_\omega) \supseteq \Delta \). We assume \( G_\omega \) has no parallel edges (arcs), so we can identify \( G_\omega \) with the pair \( (V(G_\omega), E(G_\omega)) \). Our graph theory notation and terminology are standard and can be found in all graph theory books, like [14]. Moreover, we may treat \( G_\omega \) as a weighted graph (see [10]) by assigning to each edge the distance between its vertices.
Fixed Point Theorems for Kannan Contractions and Weakly Contractive Mappings on a Modular Metric Space Endowed with a Graph

By $G^{-1}$ we denote the conversion of a graph $G$, i.e., the graph obtained from $G$ by reversing the direction of edges. Thus we have

$$E(G^{-1}) = \{(y, x) | (x, y) \in E(G)\}.$$ 

A diagraph $G$ is called an oriented graph if whenever $(u, v) \in E(G)$, then $(v, u) \notin E(G)$. The letter $\tilde{G}$ denotes the undirected graph obtain from $G$ by ignoring the direction of edges.

Actually, it will be more convenient for us to treat $\tilde{G}$ as a directed graph for which the set of its edges is symmetric. Under this convention,

$$E(\tilde{G}) = E(G) \cup E(G^{-1}).$$

We call $(V', E')$ a sub graph of $V \subseteq V(\tilde{G}), E' \subseteq E(\tilde{G})$, and for any edge $(x, y) \in E'$, $x, y \in V'$.

If $x$ and $y$ are vertices in a graph $G$, then a (directed) path in $G$ from $x$ to $y$ of length $N$ is a sequence $(x_i)_{i=1}^N$ of $N + 1$ vertices such that $x_0 = x, x_N = y$ and $(x_{n-1}, x_n) \in E(G)$ for $t = 1, \ldots, N$. A graph $G$ is connected if there is a directed path between any two vertices. $G$ is a weakly connected if $\tilde{G}$ is connected. If $G$ is such that $E(\tilde{G})$ is symmetric and $x$ is a vertex in $G$, then the sub graph $G_x$ consisting of all edges and vertices which are contained in some path beginning at $x$ is called the component of $G$ containing $x$. In this case $V(G_x) = [x]_G$, where $[x]_G$ is the equivalence class of the following relation $\mathcal{R}$ defined on $V(G)$ by the rule:

$$y \mathcal{R} z \text{ if there is a (directed) path in } G \text{ from } y \text{ to } z.$$ 

Clearly $G_x$ is connected.

**Definition 2.3.** [3] Let $(X, \omega)$ be a modular metric space and $M$ be a non empty subset of $X_\omega$. A mapping $T : M \to M$ is called

(i) $G_\omega^-$ contraction if $T$ preserve edges of $G_\omega$, i.e.,

$$\forall x, y \in M \ ((x, y) \in E(G_\omega) \Rightarrow (T(x), T(y)) \in E(G_\omega)),$$

and if there exists a constant $\alpha \in [0, 1)$ such that

$$\omega_1(T(x), T(y)) \leq \alpha \omega_1(x, y) \text{ for any } (x, y) \in E(G_\omega).$$

(ii) $(\varepsilon, \alpha) - G_\omega^-$ uniformly locally contraction if $T$ preserve edges of $G_\omega$ and there exists a Constant $\alpha \in [0, 1)$ such that for any $(x, y) \in E(G_\omega)$

$$\omega_1(T(x), T(y)) \leq \alpha \omega_1(x, y) \text{ whenever } \omega_1(x, y) < \varepsilon.$$

**Definition 2.4.** [3] A point $x \in M$ is called a fixed point of $T$ whenever $x = T(x)$. The set of fixed points of $T$ will be denoted by Fix($T$).

Now we introduce the $G_\omega$ Kannan contraction and weakly $G_\omega$ contractive mappings in a modular metric space endowed with a graph as follows.

**Definition 2.5.** Let $(X, \omega)$ be a modular metric space with a graph $G_\omega$. A mapping $T : M \to M$ is called

(1) $G_\omega^-$ Kannan contraction if $T$ preserve the edges of $G_\omega$, i.e., for all $x, y \in M \ ((x, y) \in E(G_\omega) \Rightarrow (T(x), T(y)) \in E(G_\omega))$

and if there exists positive number $k \in (0, 1/2)$ such that
Prerna Pathak, Aklesh Pariya, V. H. Badshah and Nirmala Gupta

$$\omega(Tx, Ty) \leq k(\omega(x, x) + \omega(Ty, y))$$ for any \(x, y \in M\) with \((x, y) \in E(G_\omega)\).

(2) weakly \(G_\omega\) contractive if \(T\) preserve the edges of \(G_\omega\),
i.e., for all \(x, y \in M\) \((x, y) \in E(G_\omega) \Rightarrow (Tx, Ty) \in E(G_\omega))
and \(\omega(Tx, Ty) \leq \omega(x, y) - \phi(\omega(x, y))\)
whenever \(\phi: [0, \infty) \to [0, \infty)\) is continuous non decreasing such that \(\phi\) is positive on \((0, \infty)\) and \(\phi(0) = 0\).

Our first result can be seen as an extension of Jachymski [9] fixed point theorems to modular metric spaces. As Jachymski [8] did, we introduce the following property.

Lemma 2.1. [16]

Let \(X, d\) be a metric space and \(T: X \to X\) be a weakly \(G\)-contractive map. Then for any \(x \in X\) and \(y \in \text{cl}\{x\}\) we have

$$\lim_{n \to \infty} d(T^n x, T^n y) = \lim_{n \to \infty} r(T^n x, T^n y) = 0.$$ 

Proposition 2.2. [16]

Let \((X, d)\) be a metric space and \(T\) be a weakly \(G\)-contractive mapping from \(X\) into \(X\). Let there exist \(x_0 \in X\) such that \(Tx_0 \in [x_0]_G\) then the sequence \(\{T^n x_0\}\) is Cauchy.

3. Main results

Theorem 3.1. Let \((X, \omega)\) be a modular metric space with a graph \(G_\omega\). Suppose that \(\omega\) is a convex regular modular metric which satisfies the \(\Delta_2\) - type condition. Assume that \(M = V(G_\omega)\) is a nonempty \(\omega\) - bounded, \(\omega\) - complete subset of \(X_\omega\) and the triple \((M, d_\omega, G_\omega)\) has property (P). Let \(T: M \to M\) be Kannan contraction map and \(M_T := \{x \in M; (x, Tx) \in E(G_\omega)\}\).

If \((x_0, T(x_0)) \in E(G_\omega)\), then the following statements hold:

(i) For any \(x \in M_T, T_{\{x\}}G_\omega\), has a fixed point.
(ii) If \(G_\omega\) is weakly connected, then \(T\) has a fixed point in \(M\).
(iii) If \(M' = \cup\{x\}_{G_\omega}; x \in M_T\), then \(T|_{M'}\) has a fixed point in \(M\).

Proof (i): As \((x_0, T(x_0)) \in E(G_\omega)\) and \((y_0, T(y_0)) \in E(G_\omega)\) then \(x_0, y_0 \in M_T\). Since \(T\) is a Kannan contraction, there exists a constant \(k \in (0, \frac{1}{2})\) such that \((T(x_0), T(y_0)) \in E(G_\omega)\)
and \(\omega_1(Tx_0, Ty_0) \leq k[\omega_1(x_0, x_0) + \omega_1(y_0, y_0)]\) \((3.1.1)\)
By induction we can construct a sequence \(\{x_n\}\) such that \(x_{n+1} = Tx_n\) and \((x_n, x_{n+1}) \in E(G_\omega)\)

$$\omega_1(x_{n+1}, x_n) = \omega_1(Tx_n, Tx_{n-1})$$
$$\omega_1(x_{n+1}, x_n) \leq k[\omega_1(x_{n+1}, x_n) + \omega_1(x_{n-1}, x_{n-1})]$$
$$\leq k[\omega_1(x_{n+1}, x_{n}) + \omega_1(x_{n}, x_{n-1})]$$
$$\omega_1(x_{n+1}, x_n) \leq \frac{k}{(1-k)} \omega_1(x_{n}, x_{n-1})$$
where \(\alpha = \frac{k}{(1-k)} < 1\)
$$\omega_1(x_{n+1}, x_n) \leq \alpha \omega_1(x_{n}, x_{n-1})$$

82
Fixed Point Theorems for Kannan Contractions and Weakly Contractive Mappings on a Modular Metric Space Endowed with a Graph

So by induction, we construct a sequence \( \{x_n\} \) such that \((x_{n+1}, x_n) \in E(G_\omega)\) and 
\[ \omega_1(x_{n+1}, x_n) \leq \alpha^n \omega_1(x_0, x_1) \] 
for any \( n \geq 1 \). Then by lemma 2.2 ,
\[ \Rightarrow \{x_n\} \] is \( \omega \)-Cauchy. Since \( M \) is \( \omega \)-complete, therefore \( \{x_n\} \) is \( \omega \)-convergence to some point \( \in M \).

By property (P), \((x_n, x) \in E(G_\omega)\) for all \( n \) and hence
\[ \omega_1(x_{n+1}, T(x)) = \omega_1(Tx_n, Tx) \leq k(\omega_1(Tx_n, x_n) + \omega_1(Tx, x)) \]
Taking limit \( n \to \infty \) both sides we get
\[ \omega_1(x, Tx) \leq k(\omega_1(x, x) + \omega_1(Tx, x)) \]  
i.e. \( \omega_1(x, Tx) \leq k\omega_1(Tx, x) \) which is a contradiction.

Hence \( \omega_1(x, Tx) = 0 \).

Therefore \( x = Tx \).

i.e. \( x \) is a fixed point of \( T \).

As \((x_0, x) \in E(G_\omega)\), we have \( x \in [x_0]_{G_\omega} \).

**Uniqueness.** Let \( x \) and \( y \) be two fixed point of \( T \).

Consider \( \omega_1(x, y) = \omega_1(Tx, Ty) \leq k[\omega_1(x, Tx) + \omega_1(y, Ty)] \)
\[ \leq k[\omega_1(x, x) + \omega_1(y, y)] \]
This gives
\[ \omega_1(x, y) = 0 \Rightarrow x = y. \]

Hence point is unique.

(ii) Since \( M_T \neq \emptyset \), there exists an \( x_0 \in M_T \) and since \( G_\omega \) is weakly connected, then 
\( [x_0]_{G_\omega} = M \) and by M and by (i), mapping \( T \) has a fixed point.

(iii) It follows easily from (i) and (ii).

**Theorem 3.2.** Let \((X, \omega)\) be a modular metric space with a graph \( G_\omega \). Suppose that \( \omega \) is a convex regular modular metric which satisfies the \( \Delta_2^- \) type condition. Assume that \( M = V(G_\omega) \) is a nonempty \( \omega \)-bounded, \( \omega \)-complete subset of \( X_{\omega} \) and the triple \((M, d_{\omega}, G_\omega)\) has property (P). Let \( T : M \to M \) be weak contraction mapping and \( M_T := \{x \in M; (x, Tx) \in E(G_\omega)\} \).

If \((x_0, T(x_0)) \in E(G_\omega)\), then the following statements hold:

(iv) For any \( x \in M_T, T|_{[x]_{G_\omega}} \), has a fixed point.

(v) If \( G_\omega \) is weakly connected, then \( T \) has a fixed point in \( M \).

(vi) If \( M' = U\{[x]_{G_\omega}; x \in M_T\} \), then \( T|_{M'} \) has a fixed point in \( M \).

**Proof:** As \((x_0, T(x_0)) \in E(G_\omega)\) and \((y_0, T(y_0)) \in E(G_\omega)\) then \(x_0, y_0 \in M_T\). Since \( T \) is a weak contraction, there exists a constant \( k \in (0, \frac{1}{2}) \) such that \((T(x_0), T(y_0)) \in E(G_\omega)\) and
\[ \omega_1(Tx_0, Ty_0) \leq \omega_1(x_0, y_0) - \Psi(\omega_1(x_0, y_0)) \]
By induction we can construct a sequence \( \{x_n\} \) such that \( x_{n+1} = Tx_n \) and 
\[ (x_n, x_{n+1}) \in E(G_\omega) \]
\[ \omega_1(x_{n+1}, x_n) = \omega_1(Tx_n, Tx_{n-1}) \]
\[ \omega_1(x_{n+1}, x_n) \leq \omega_1(x_n, x_{n-1}) - \Psi(\omega_1(x_n, x_{n-1})) \]
\[ \omega_1(x_{n+1}, x_n) \leq \omega_1(x_n, x_{n-1}) \]

Similarly
Prerna Pathak, Aklesh Pariya, V. H. Badshah and Nirmala Gupta

\[ \omega_t(x_{n+1}, x_{n+2}) = \omega_t(Tx_{n+1}, Tx_{n}) \]
\[ \omega_t(x_{n+2}, x_{n+3}) \leq \omega_t(x_{n+1}, x_{n}) - \Psi(\omega_t(x_{n+1}, x_{n})) \]
\[ \omega_t(x_{n+2}, x_{n+1}) \leq \omega_t(x_{n+1}, x_{n}) \]
\[ \omega_t(x_{n+3}, x_{n+2}) = \omega_t(Tx_{n+2}, Tx_{n+1}) \]
\[ \omega_t(x_{n+3}, x_{n+2}) \leq \omega_t(x_{n+1}, x_{n}) - \Psi(\omega_t(x_{n+1}, x_{n})) \]

Hence in general
\[ \omega_t(x_{i+1}, x_i) \leq \omega_t(x_i, x_{i-1}) - \Psi(\omega_t(x_i, x_{i-1})); \quad \forall i = 1, 2, 3 \ldots n \]

Since \( \Psi \) is non decreasing and this shows that \( \{x_i\}_{i=1}^n \) is a \( \omega \)-cauchy sequence
\[ \omega_t(x_{n+1}, x_n) \leq \omega_t(x_n, x_{n-1}) \leq \ldots \leq \omega_t(x_1, x_0) \]

Since \( \omega_t(x_{n+1}, x_n) \) is non increasing sequence of non-negative real number bounded below by 0, thus convergent.

Taking limit as \( n \to \infty \),
\[ \lim_{n \to \infty} \omega_t(x_{n+1}, x_n) = 0; \quad \forall i = 1, 2, 3 \ldots n. \]

**Uniqueness:** Let \( x \) and \( y \) be two fixed point of \( T \).

Consider \( \omega_t(x, y) = \omega_t(Tx, Ty) \leq \omega_t(x, y) - \psi \omega_t(x, y) \]

This gives
\[ \omega_t(x, y) = 0 \iff x = y. \]

Hence point is unique.

(ii) Since \( M_T \neq \emptyset \), there exists an \( x_0 \in M_T \) and since \( G_\omega \) is weakly connected, then \( [x_0]_{G_\omega} = M \) and by \( M \) and by (i), mapping \( T \) has a fixed point.

(iii) It follows easily from (i) and (ii).

**REFERENCES**


84
Fixed Point Theorems for Kannan Contractions and Weakly Contractive Mappings on a Modular Metric Space Endowed with a Graph