

Short Communication

The Maximum Degree, the Independence Number and Some Hamiltonian Properties of a Graph

Rao Li

Department of Computer Science, Engineering, and Mathematics,
University of South Carolina Aiken, Aiken, SC 29801, USA.

E-mail: raol@usca.edu

Received 15 August 2025; accepted 29 September 2025

Abstract. In this note, we present sufficient conditions involving the maximum degree and the independence number for some Hamiltonian properties of a graph.

Keywords: The maximum degree, the independence number, Hamiltonian graph, traceable graph

AMS Mathematics Subject Classification (2010): 05C45

1. Introduction

We consider only finite undirected graphs without loops or multiple edges. Notation and terminology not defined here follow those in [1]. Let $G = (V(G), E(G))$ be a graph with n vertices and e edges. We use δ and Δ to denote the minimum and maximum degrees of a graph G , respectively. A subset S of $V(G)$ in G is independent if there is no edge between any pair of distinct vertices in S . A maximum independent set in a graph G is an independent set of maximum cardinality. The independence number of a graph G , denoted $\alpha(G)$, is the size of a maximum independent set in G . For disjoint vertex subsets X and Y of $V(G)$, we define $E(X, Y)$ as $\{e : e = xy \in E, x \in X, y \in Y\}$. We use $K_{r,s}$ to denote a complete bipartite graph in which the two partition sets have cardinalities of r and s , respectively. A cycle C in a graph G is called a Hamilton cycle of G if C contains all the vertices of G . A graph G is called Hamiltonian if G has a Hamilton cycle. A path P in a graph G is called a Hamilton path of G if P contains all the vertices of G . A graph G is called traceable if G has a Hamilton path. In this note, we present sufficient conditions involving the maximum degree and the independence number for Hamiltonian graphs and traceable graphs. The main results are as follows.

Theorem 1.1. Let G be a k -connected graph ($k \geq 2$) with $n \geq 3$ vertices, e edges, and maximum degree Δ . If $e \geq (n - k - 1)\Delta$, then G is Hamiltonian or G is $K_{k, k+1}$.

Theorem 1.2. Let G be a k -connected graph ($k \geq 1$) with $n \geq 9$ vertices, e edges, and maximum degree Δ . If $e \geq (n - k - 2)\Delta$, then G is traceable or G is $K_{k, k+2}$.

2. Lemmas

We will use the following results as lemmas in our proofs of Theorem 1 and Theorem 2. Lemma 2.1 and Lemma 2.2 below are from [2].

Lemma 2.1 [2]. Let G be a k -connected graph of order $n \geq 3$. If $\alpha \leq k$, then G is Hamiltonian.

Lemma 2.2 [2]. Let G be a k -connected graph of order n . If $\alpha \leq k + 1$, then G is traceable.

Lemma 2.3 below is from [4].

Lemma 2.3 [4]. Let G be a balanced bipartite graph of order $2n$ with bipartition (A, B) . If $d(x) + d(y) \geq n + 1$ for any $x \in A$ and any $y \in B$ with $xy \notin E$, then G is Hamiltonian.

Lemma 2.4 below is from [3].

Lemma 2.4 [3]. Let G be a 2-connected bipartite graph with bipartition (A, B) , where $|A| \geq |B|$. If each vertex in A has degree at least s and each vertex in B has degree at least t , then G contains a cycle of length at least $2\min(|B|, s + t - 1, 2s - 2)$.

3. Proofs

Proof of Theorem 1.1. Let G be a k -connected graph ($k \geq 2$) with $n \geq 3$ vertices, e edges, and the maximum degree Δ satisfying the conditions in Theorem 1.1. Suppose G is not Hamiltonian. Then we have that $n \geq 2\delta + 1 \geq 2k + 1$ otherwise $\delta \geq k \geq n/2$ and G is Hamiltonian. In addition, Lemma 2.1 implies that $\alpha \geq k + 1$. Let I be any maximum independent set in G . Then $|I| = \alpha < n$. Thus

$$\sum_{u \in I} d(u) = |E(I, V - I)| \leq \sum_{v \in V - I} d(v).$$

Since $\sum_{u \in I} d(u) + \sum_{v \in V - I} d(v) = 2e$, we have that

$$\sum_{u \in I} d(u) \leq e \leq \sum_{v \in V - I} d(v).$$

Notice that $d(v) \leq \Delta$ for each vertex $v \in V - I$. We have that

$$(n - k - 1) \Delta \leq e \leq \sum_{v \in V - I} d(v) \leq (n - \alpha) \Delta \leq (n - k - 1) \Delta.$$

Then

$$(n - k - 1) \Delta = e = \sum_{v \in V - I} d(v) = (n - \alpha) \Delta = (n - k - 1) \Delta.$$

Thus $d(v) = \Delta$ for each $v \in V - I$, $|I| = \alpha = k + 1$, and $\sum_{v \in V - I} d(v) = e$ which implies that $\sum_{u \in I} d(u) = e$ and G is a bipartite graph with partition sets I and $V - I$.

Since $n \geq 2k + 1$, we have that $|V - I| \geq k$. If $|V - I| \geq k + 2$, then $V - I$ is an independent set with size at least $k + 2$, contradicting to the fact that I is a maximum independent set in G with size $k + 1$. Thus

$|V - I| = k + 1$ or $|V - I| = k$. If $|V - I| = k + 1$, then Lemma 2.3 implies that G is Hamiltonian, a contradiction. If $|V - I| = k$, then G is $K_{k, k+1}$.

This completes the proof of Theorem 1.1.

Proof of Theorem 1.2. Let G be a k -connected graph ($k \geq 1$) with $n \geq 9$ vertices, e edges, and the maximum degree Δ satisfying the conditions in Theorem 1.2. Suppose G is not

The Maximum Degree, the Independence Number and Some Hamiltonian Properties of a Graph

traceable. Then we have that $n \geq 2\delta + 2 \geq 2k + 2$ otherwise $\delta \geq k \geq (n - 1)/2$ and G is traceable. In addition, Lemma 2.2 implies that $\alpha \geq k + 2$. Let I be any maximum independent set in G . Then $|I| = \alpha < n$. Thus

$$\sum_{u \in I} d(u) = |E(I, V - I)| \leq \sum_{v \in V - I} d(v).$$

Since $\sum_{u \in I} d(u) + \sum_{v \in V - I} d(v) = 2e$, we have that

$$\sum_{u \in I} d(u) \leq e \leq \sum_{v \in V - I} d(v).$$

Notice that $d(v) \leq \Delta$ for each vertex $v \in V - I$. We have that

$$(n - k - 2) \Delta \leq e \leq \sum_{v \in V - I} d(v) \leq (n - \alpha) \Delta \leq (n - k - 2) \Delta.$$

Then

$$(n - k - 2) \Delta = e = \sum_{v \in V - I} d(v) = (n - \alpha) \Delta = (n - k - 2) \Delta.$$

Thus $d(v) = \Delta$ for each $v \in V - I$, $|I| = \alpha = k + 2$, and $\sum_{v \in V - I} d(v) = e$ which implies that $\sum_{u \in I} d(u) = e$ and G is a bipartite graph with partition sets I and $V - I$. Since $n \geq 2k + 2$, we have that $|V - I| \geq k$. If $|V - I| \geq k + 3$, then $V - I$ is an independent set with size at least $k + 3$, contradicting to the fact that I is a maximum independent set in G with size $k + 2$.

Thus $|V - I| = k + 2$, $|V - I| = k + 1$, or $|V - I| = k$. If $|V - I| = k + 2$, then $k \geq 3$ and Lemma 2.3 implies that G is Hamiltonian and thereby G is traceable, a contradiction. If $|V - I| = k + 1$, then $k \geq 3$ and Lemma 2.4 implies that G has a cycle of length at least $(n - 1)$ and thereby G is traceable, a contradiction. If $|V - I| = k$, then G is $K_{k, k+2}$.

This completes the proof of Theorem 1.2.

4. Conclusion

In this note, we present new sufficient conditions that involve the maximum degree and the independence number for Hamiltonian and traceable graphs.

Acknowledgements. The author would like to thank the referee for his/her suggestions and comments which helped to improve the original manuscript.

Conflict of interest. The paper is written by a single author so there is no conflict of interest.

Authors' Contributions. It is a single-author paper. So, full credit goes to the author.

REFERENCES

1. J. A. Bondy and U. S. R. Murty, *Graph Theory with Applications*, Macmillan, London and Elsevier, New York (1976).
2. C. Chvátal and P. Erdős, A note on Hamiltonian circuits, *Discrete Mathematics* 2 (1973) 111-113.
3. B. Jackson, Long cycles in bipartite graphs, *Journal of Combinatorial Theory Series B*, 38 (1985) 118-131.
4. J. Moon and L. Moser, On Hamiltonian bipartite graphs, *Israel J. Math.*, 1 (1963) 163-165.