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Numerical Solution of Singular Boundary Value Problems by Galerkin's Method Utilizing Taylor Wavelets

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Abstract. Singular boundary value problems (SBVPs) describe various models with applications in engineering and other areas. Generally, obtaining the analytic solutions of such kind of problems is a challenge due to the singularity involved in the governing equations. For this reason, numerical methods are very essential and of considerable importance. Here, I proposed the numerical solution of SBVPs by the Galerkin method using Taylor wavelets. The paper also provides illustrative examples to demonstrate the efficiency and precision of the method.

Keywords: Numerical solution; Taylor wavelet; Function approximation; Galerkin's method; Singular boundary value problems; Euler wavelet

AMS Mathematics Subject Classification (2010): 65T60, 97N40, 30E25

1. Introduction

Mathematical modeling often employs differential equations, especially within the realms of science and engineering. Various challenges in mathematical physics frequently manifest as differential equations, which can be classified as either ordinary or partial differential equations [1]. Singular boundary value problems (SBVPs) occur frequently in applied mathematics and engineering. The numerical treatment of such type of problems has always been a complicated and difficult task due to the singular behaviour that occurs at a point. Hence, the singular boundary value problems have attracted much interest and have been investigated by many researchers.

"Wavelets" have been a very popular topic of conversations in several scientific and engineering gatherings these days. Some of the researchers have decided that wavelets are new basis for representing continuous functions, as a technique for time–frequency analysis, and as a new mathematical subject. Of course, "wavelets" are a versatile tool with very rich mathematical content and great potential for applications. However, wavelet analysis is a numerical concept which allows one to represent a function in terms of a set of base functions, called wavelets, which are localized both in location and scale [2].

The numerical approach facilitates the resolution of intricate problems using comparatively straightforward operations, offering a notable advantage over analytical methods due to its ease of implementation on contemporary computers. This advantage enables expedited solutions relative to those derived from analytical techniques. Galerkin's method is encompassed within a broader spectrum of numerical strategies, and Wavelet analysis signifies an emerging and promising area in the realm of applied and computational numerical research.

Recently, some of the researchers are solved various problems including SBVPs by numerical methods utilizing wavelets viz. Weighted Residual Method via Euler Wavelets [3], Wavelet based Galerkin Method [4], Hermite Wavelet Based Galerkin Method [5] etc.

The wavelet-Galerkin method offers significant advantages over both the finite difference and finite element methods, making it a widely utilized approach in various scientific and engineering fields. The Galerkin method is widely recognized in the field of applied mathematics for its convenience and practicality [6 - 7].

The research introduces the Galerkin method involves representing the solution using Taylor wavelets with unknown coefficients, and using the properties of Taylor wavelets in combination with the Galerkin method to calculate these coefficients and obtain a numerical solution for the SBVPs.

The outline of the paper is: Taylor wavelets and function approximation is given in section 2. In Section 3 is describes the Galerkin method based on Taylor wavelets for the numerical solution of SBVPs. The numerical implementation of the method is given in section 4. Finally, Section 5 the conclusions drawn from the proposed research.

2. Taylor wavelets and function approximation

Wavelets constitute a family of functions constructed from dilation and translation of a single function called the mother wavelet. When the dilation parameter a and the translation parameter b varies continuously, we have the following family of continuous wavelets [8 - 9]:

$$\psi_{a,b}(t) = |a|^{\frac{-1}{2}} \psi\left(\frac{t - b}{a}\right), \; a, b \in R \;\&\; a \neq 0$$

If we restrict the parameters a & b to discrete values as

$$a = a_0^{-n} \;,\; b = m\, b_0\, a_0^{-n} \;;\; a_0 > 1, \;\; b_0 > 0$$

we have the following family of discrete wavelets

$$\psi_{n,m}(t) = |a_0|^{\frac{1}{2}} \psi\left(a_0^n t - m b_0\right), \; n, m \in Z$$

where $\psi_{n,m}$ form a wavelet basis for $L^2(R)$. In particular, when $a_0 = 2$ & $b_0 = 1$, then $\psi_{n,m}(t)$ forms an orthonormal basis.

Taylor wavelets are defined as follows:

$$\psi_{n,m}(t) = \begin{cases} 2^{\frac{k+1}{2}} \tilde{T}_m\left(2^{k+1} t - n + 1\right), & \frac{n-1}{2^{k-1}} \le t < \frac{n}{2^{k-1}} \\ 0, & Otherwise \end{cases} \quad (2.1)$$

with $\tilde{T}_m(t) = \sqrt{2m+1}\, T_m(t)$ (2.2)

The coefficient $\sqrt{2m+1}$ is for normality, k is any positive integer, $n = 1, 2, 3, \ldots\ldots, 2^{k-1}$ is an argument and T_m $(m = 0, 1, 2, 3 \ldots\ldots M-1)$ are the well known Taylor polynomials of oreder m which can be defined by $T_m(t) = t^m$.

Taylor polynomials form a complete basis over the interval $[0, 1)$.

For instance, for $k = 1$ and $M = 3$, we get the Taylor wavelet bases are as follows:

$\psi_{1,0}(t) = 1$,

$\psi_{1,1}(t) = \sqrt{3}\, t$,

$\psi_{1,2}(t) = \sqrt{5}\, t^2$ and so on.

Function approximation:

Suppose $u(t) \in L^2[0, 1)$ is expanded in terms of Fibonacci wavelets as:

$$y(t) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} c_{n,m} \psi_{n,m}(t) \quad (2.3)$$

Truncating the above infinite series, we get

$$y(t) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M} c_{n,m} \psi_{n,m}(t) \quad (2.4)$$

3. Method of solution

Now, consider SBVP in the following form,

$$y'' + p(t)\, y' + q(t)\, y = f(t) \quad (3.1)$$

With boundary conditions $y(a) = \alpha,\ y(b) = \beta$ (3.2)

Here the functions $p(t)$, $q(t)$ & $f(t)$ are analytic in $t \in (0, 1]$ and the functions $P(t)$ & $Q(t)$ are not analytic functions at $t = 0$ i.e. Singularity at $t = 0$.

Rewrite the Eq. (3.1) as $R(t) = y'' + p(t)\, y' + q(t)\, y - f(t)$ (3.3)

In cases where $R(x)$ is the residual of Eq. (3.1) equals zero, the exact solution is identified and the boundary conditions are satisfied.

The trial series solution of Eq. (3.1), within the range of $(0, 1]$, meets the specified boundary conditions and can be expanded to a modified Taylor wavelet by introducing unknown coefficients as follows:

$$y(t) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m}\psi_{n,m}(t) \quad (3.4)$$

The unknown coefficients $c_{n,m}$'s, which are to be determined,

The precision of the solution is improved by choosing higher degree Taylor wavelet polynomials.

Find the second derivative w.r.t. t of Eq. (3.4) in order to obtain the values of $y,\ y',\ y''$ and then substitute these values into Eq. (3.3). Solve for the unknown coefficients $c_{n,m}$'s by using weight functions as the assumed basis elements and integrating the boundary values along with the residual to achieve zero [10].

i.e. $$\int_0^1 \psi_{1,m}(t)\ R(t)\ dt = 0,\ m = 0, 1, 2, \ldots\ldots..$$

We can obtain a set of linear algebraic equations, which can be solved to find the unknown coefficients. Once determined these unknowns and then substitute them into the trial solution i.e. in Eq. (3.4), which gives the numerical solution for Eq. (3.1).

In order to calculate the precision of the TWGM on the test problems, the maximum absolute error as a measure of error and this is considered as:

$$E_{\max} = \max \left| y(t)_{exact} - y(t)_{numer} \right|,$$

$y(t)_{exact}$ and $y(t)_{numer}$ are exact and numerical solution.

4. Numerical implementation

Problem 4.1. Consider the second order differential equation of the form

$$y'' + \frac{1}{t}y' + y = t^2 - t^3 - 9t + 4,\quad 0 \le t \le 1 \quad (4.1)$$

With boundary conditions: $y(0) = 0,\ y(1) = 0$ (4.2)

Here, $p(t) = \frac{1}{t},\ q(t) = 1\ \&\ f(t) = t^2 - t^3 - 9t + 4$

At $t = 0$, $p(t)$ is not analytic. Therefore, the given equation is SBVP.

Now, the Eq. (4.1) is implemented according to the method outlined in section 3 in the following manner:

The residual of Eq. (4.1) can be written as:

$$R(t) = t\,y'' + y' + t\,y - \left(t^2 - t^3 - 9t + 4\right) \quad (4.3)$$

Choosing the weight function $w(t) = t(1-t)$ for Tayolr wavelet bases and they are satisfy the given boundary conditions Eq. (4.2), i.e. $\psi(t) = w(t) \times \psi(t)$

$$\psi_{1,0}(t) = \psi_{1,0}(t) \times t(1-t) = t(1-t)$$

$$\psi_{1,1}(t) = \psi_{1,1}(t) \times t(1-t) = \left(\sqrt{3}\,t\right) t(1-t)$$

$$\psi_{1,2}(t) = \psi_{1,2}(t) \times t(1-t) = \left(\sqrt{5}\,t^2\right) t(1-t)$$

Assuming the trail solution of Eq. (4.1) for $k = 1$ and $m = 2$ is given by

$$y(t) = c_{1,0}\,\psi_{1,0}(t) + c_{1,1}\,\psi_{1,1}(t) + c_{1,2}\,\psi_{1,2}(t) \qquad (4.4)$$

Then the Eq. (4.4) becomes

$$\begin{aligned} y(t) &= c_{1,0}\{t(1-t)\} + c_{1,1}\left\{\left(\sqrt{3}\,t\right)t(1-t)\right\} + \\ &\quad c_{1,2}\left\{\left(\sqrt{5}\,t^2\right)t(1-t)\right\} \\ \Rightarrow y(t) &= c_{1,0}\left(t - t^2\right) + c_{1,1}\left\{\sqrt{3}\left(t^2 - t^3\right)\right\} + \\ &\quad c_{1,2}\left\{\sqrt{5}\left(t^3 - t^4\right)\right\} \end{aligned} \qquad (4.5)$$

Differentiating Eq. (4.5) twice w.r.t. t and substitute the values of y, y', y'' in Eq. (4.3), we get the residual of Eq. (4.1). The "weight functions" are the same as the basis functions.

By the weighted Galerkin method, consider the following:

$$\int_0^1 \psi_{1,j}(t)\,R(t)\,dt = 0, \quad j = 0, 1, 2 \qquad (4.6)$$

For $j = 0, 1, 2$ in Eq. (4.6),

$$\text{i.e.} \quad \left.\begin{aligned} \int_0^1 \psi_{1,0}(t)\,R(t)\,dt &= 0 \\ \int_0^1 \psi_{1,1}(t)\,R(t)\,dt &= 0 \\ \int_0^1 \psi_{1,2}(t)\,R(t)\,dt &= 0 \end{aligned}\right\} \qquad (4.7)$$

From equation (4.7), a system of algebraic equations with unknown coefficients, namely $c_{1,0}$, $c_{1,1}$ and $c_{1,2}$. By solving this system, obtained the values of $c_{1,0} = -0.01783$, $c_{1,1} = 0.60841$, and $c_{1,2} = -0.01818$. Substituting these values into Eq. (4.5) gives the numerical solution. The numerical solution and the absolute errors are compared in table 1, whereas the numerical solution is presented in figure 1 with the exact solution of Eq (4.1) $y(t) = t^2 - t^3$ [3].

t	**Numerical solution**			**Exact solution**	**Absolute error**		
	FDM	**Ref [4]**	**TWGM**		**FDM**	**Ref [4]**	**TWGM**
0.1	-0.014709	0.010673	0.007843	0.009000	2.37e-02	1.67e-03	1.16e-03
0.2	-0.013726	0.033159	0.030849	0.032000	4.57e-02	1.16e-03	1.15e-03
0.3	-0.002584	0.063290	0.062877	0.063000	6.56e-02	2.90e-04	1.23e-04
0.4	0.015387	0.095881	0.095894	0.096000	8.06e-02	1.19e-04	1.06e-04
0.5	0.036564	0.125034	0.124975	0.125000	8.84e-02	3.40e-05	2.50e-05
0.6	0.056572	0.144429	0.143955	0.144000	8.74e-02	4.29e-04	4.50e-05

0.7	0.070066	0.147623	0.146981	0.147000	7.69e-02	6.23e-04	1.90e-05
0.8	0.070568	0.128350	0.127871	0.128000	5.74e-02	3.50e-04	1.29e-04
0.9	0.050294	0.080816	0.080789	0.081000	3.07e-02	1.84e-04	2.11e-04

Table 1. Comparison of numerical solution and absolute error in relation to the exact solution for problem 4.1

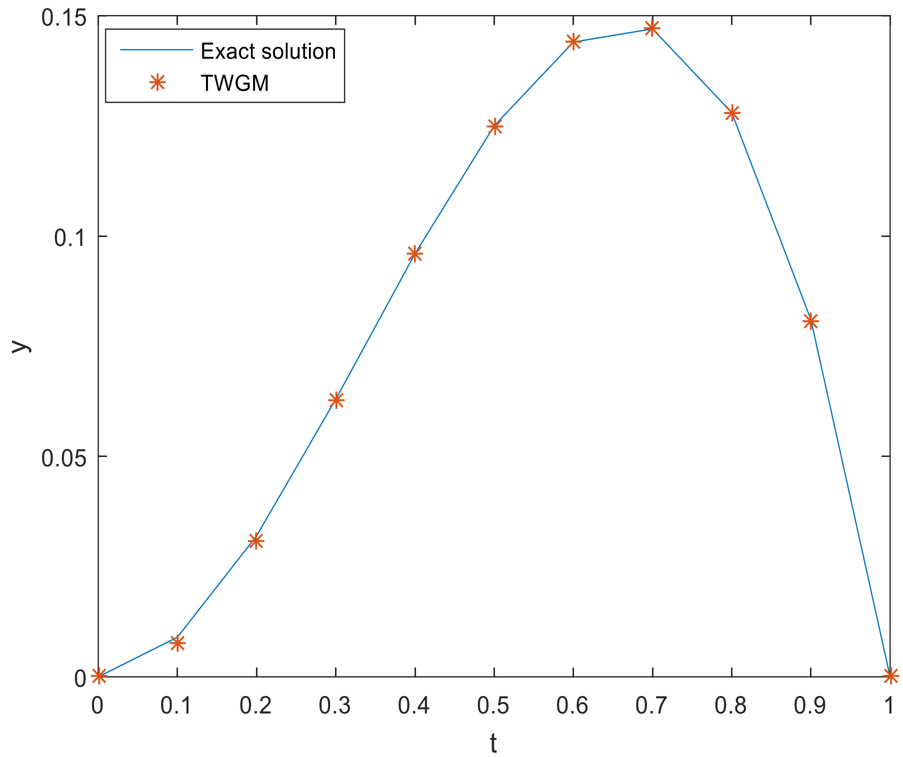

Figure 1. Comparison of numerical solution with exact solution of the problem 4.1.

Problem 4.2. Next, consider another differential equation of the form

$$y'' + \frac{8}{t}y' + ty = t^5 - t^4 + 44t^2 - 30t, \quad 0 \le t \le 1 \tag{4.8}$$

With boundary conditions: $y(0) = 0$, $y(1) = 0$ (4.9)

Here, $p(t) = \frac{8}{t}$, $q(t) = t$ & $f(t) = t^5 - t^4 + 44t^2 - 30t$

At $t = 0$, $p(t)$ is not analytic. Therefore, the given equation is SBVP.

In section 3 and in the above problem, the values of $c_{1,0} = 0.00034$, $c_{1,1} = -0.00097$ and $c_{1,2} = -0.44649$ are determined. The numerical solution is obtained by putting these values in Eq. (4.5). In table 2, the numerical solution and the absolute errors are compared, and figure 2 gives the comparison of the numerical solution with the exact solution of Eq. (4.8) is $y(t) = t^4 - t^3$ [11] depicted in figure 2.

Table 2. Numerical solution and absolute error are compared with exact solution of problem 4.2.

t	Numerical solution			Exact solution	Absolute error		
	Ref [4]	Ref [11]	TWGM		Ref [4]	Ref [11]	TWGM
0.1	-0.000823	-0.000937	-0.000883	-0.000900	7.70e-05	3.70e-05	1.70e-05
0.2	-0.004844	-0.006426	-0.006389	-0.006400	1.56e-03	2.60e-05	1.10e-05
0.3	-0.016861	-0.018899	-0.018901	-0.018900	2.04e-03	1.00e-06	1.00e-06
0.4	-0.037304	-0.038381	-0.038418	-0.038400	1.10e-03	1.90e-05	1.80e-05
0.5	-0.062986	-0.062482	-0.062514	-0.062500	4.86e-04	1.80e-05	1.40e-05
0.6	-0.087854	-0.086406	-0.086404	-0.086400	1.45e-03	6.00e-06	4.00e-06
0.7	-0.103744	-0.102944	-0.102909	-0.102900	8.44e-04	4.40e-05	9.00e-06
0.8	-0.101131	-0.102477	-0.102395	-0.102400	1.27e-03	7.70e-05	5.00e-06
0.9	-0.069880	-0.072976	-0.072888	-0.072900	3.02e-03	7.60e-05	1.20e-05

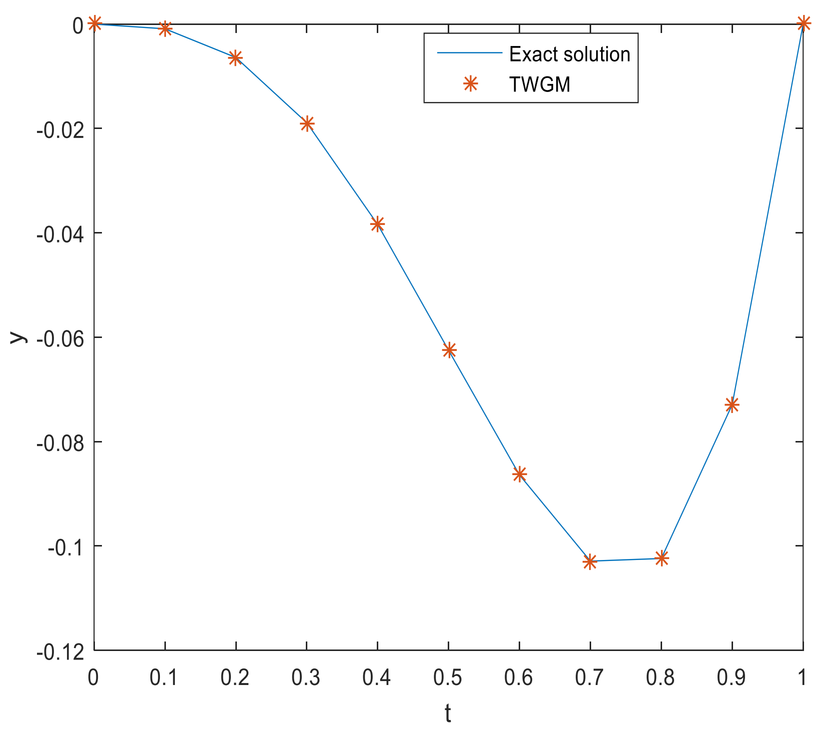

Figure 2. Numerical solution is presented with the exact solution of the problem 4.2.

5. Conclusion

The paper presents the numerical solution of SBVPs by Galerkin method using Taylor wavelets. The progress of new research in numerical analysis is significantly enhanced by this, proving advantageous for emerging researchers. The method introduced has been applied to some text problems, yielding results that are extremely adequate when compared to other existing numerical methods. The data presented in the above tables and figures shows that:

a) The proposed method gives numerical solutions that improve on those obtained by the finite difference method (FDM) and other existing methods (Ref [3] & [13]), close to the exact solution.

b) The error obtained this method is in particular better in comparison with FDM and the existing methods (Ref [3] & [13]).

Hence, utilising Taylor wavelets in the Galerkin method has been confirmed to be highly efficient in solving singular boundary value problems (SBVPs).

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