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Neutrosophic Power-Set and Neutrosophic Hyper-Structure of Neutrosophic Set of Three Types

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Abstract. This paper aims to further develop neutrosophic set theory as an extension of classical set theory, focusing on integrating the concept of indeterminacy and constructing mathematical frameworks. It introduces the neutrosophic number $(a + bI)$, a foundation for structures like neutrosophic algebra, sets, and operations. Previous works explored essential concepts such as neutrosophic subsets, operations, Cartesian products, functions, equivalence sets, and cardinality. The current study advances neutrosophic power sets and n^{th}-neutrosophic power sets, building on Smarandache's pioneering work. It systematically examines hyperfunctions, extra hyperfunctions, super hyperfunctions, and extra-super hyperfunctions. The paper is structured into six sections, addressing key topics like classical and neutrosophic power sets, cardinality, hyperfunctions, and their extensions. This research contributes to the theoretical enrichment and future directions for neutrosophic set theory.

Keywords: Neutrosophic Power-Set; n^{th}-Neutrosophic Power-Set; Equivalent Neutrosophic Set, Neutrosophic HyperFunction; Neutrosophic Extra HyperFunction; Super Neutrosophic HyperFunction; Extra Neutrosophic Super HyperFunction.

AMS Mathematics Subject Classification (2010): 03E99

1. Introduction

In 1965 [15], Zadeh introduced the concept of fuzzy set (FZ) as an extension of set theory (ST). At a later stage, specifically between 1983 and 1986, Atanassov introduced the concept of an intuitionistic fuzzy set (IFS) by introducing a non-degree membership function [10]. These works opened a new window for researchers to explore the various branches of mathematics based on set theories in light of the concepts of fuzzy set theory and intuitionistic fuzzy set theory. For example, but not limited to, the development of fuzzy matrix theory and applications [14, 19].

In another development, in 1995, Smarandache introduced the concept of the neutrosophic set to address the concept of indeterminacy. Of course, the concepts of fuzziness, intuitive fuzziness, and indeterminacy are concepts that arise from the existence of the universe around us, as well as from some cognitive issues that we do not have sufficient knowledge to judge.

In each scenario, this development relies on constructing a mathematical framework incorporating a codomain function for the degree membership function, encompassing degrees of falsehood, truth, and indeterminacy, along with operations defined on these degrees. This methodology forms a distinctive branch of neutrosophic set theory, as elaborated by various works. For instance, [18,22,23,24]. Another scenario involves generating a neutrosophic set from a classical set, utilizing the concept of indeterminacy. The objective of this paper, as well as preceding works, is to further develop and structure neutrosophic set theory as an extension of classical set theory, encompassing the concept of indeterminacy inherent to it.

This pivotal approach explored in neutrosophic algebra involves the concept of the neutrosophic number, denoted by,$(a + bI)$,which has been utilized extensively in neutrosophic structures, such as neutrosophic linear algebra, groups, rings, and number theory, for example [4,5,13,16]. This neutrosophic number facilitates the representation and analysis of neutrosophic integers, real, and complex numbers, enabling the investigation of their algebraic properties.

The concept of the neutrosophic number inspired us; the idea was to construct a neutrosophic set theory that generates from classical sets within a comparable framework.

In previous works, we introduced foundational concepts essential to neutrosophic set theory, including universal neutrosophic sets, empty neutrosophic sets, neutrosophic subsets, set equality, complements, powers, and their respective properties.
Furthermore, neutrosophic operations—such as union, intersection, difference, symmetric difference, and their generalizations—have been rigorously defined. Additional advancements include the neutrosophic Cartesian product, partial order relations, and corresponding properties [1,6,7,8].

In [2,3], we studied neutrosophic functions and their classifications, compositions, inverses, and properties. In [9], we conducted a comparative study of hyperfunctions, extra hyperfunctions, super hyperfunctions, and extra-super hyperfunctions about the nth-powerset. These explorations, proofs, and analyses aim to enrich the theoretical foundation of neutrosophic sets across three distinct types.

The present paper focuses on advancing neutrosophic power sets and nth-neutrosophic power sets in the context of the neutrosophic set theory of three types, extending the pioneering work of Smarandache [11, 25-29]. This study serves as a continuation of the previous construction of neutrosophic set theory while broadening its applicability.
The paper is organized into six sections:

1. Section One reviews the classical powerset and nth-powerset for any universal set, along with their properties under union and intersection. It also introduces the neutrosophic powerset and nth-neutrosophic powerset for three types of neutrosophic sets, supported by examples and analysis of their union and intersection properties.
2. Section Two expands on neutrosophic equivalent sets, including theories, examples, and definitions of neutrosophic finite, countable, infinite, uncountable sets, cardinality, and transfinite numbers. This section opens avenues for future research in neutrosophic cardinality and transfinite numbers.

3. Section Three explores neutrosophic hyperfunctions and their applications to neutrosophic subsets, extending classical hyperfunctions (as inspired by Marty's 1934 work on hypergroups) [17].
4. Section Four introduces the concept of extra neutrosophic hyperfunctions, outlining their properties and role within neutrosophic set theory.
5. Section Five discusses neutrosophic super-hyperfunctions and extra-hyperfunctions, analyzing their theoretical properties and implications.
6. Section Six presents the neutrosophic super hyperfunction and a neutrosophic extra hyperfunction, with some properties according to the three types of neutrosophic set theory.

   This paper represents a significant step in developing neutrosophic set theory as a robust mathematical framework with diverse applications across algebra and set theory.

2. Materials neutrosophic power-set and nth-neutrosophic power-set for a neutrosophic set of three types

The section contains two parts. Part one reviews Power-Set and n^{th}-Power-Set for any given universal set with union properties and the intersection of a power-indexed family of sets. Part two is devoted to the neutrosophic Power-Set and n^{th}-neutrosophic Power-Set of neutrosophic set theory of three types, which includes illustration examples and their properties under the union of a neutrosophic power indexed family of neutrosophic sets and their intersection, respectively.

1.2. Power-set and n^{th}-power-set for a set

Definition 1.1.2. [12,20,21,31] A set is a collection of well-defined objects called elements.

Definition 2.1.2. [12,20,21,31] Let H be a universal set. A set $P(H) = \{S: S \subseteq H\}$ is called the Power-Set of all subsets of a set H.

Theorem 1.2.2. [12,20,21,31] If H is a finite set of order n, then the order of $P(H)$ is equal to 2^n.

Theorem 2.1.2. [12,20,21,31] Let $\{A_{\alpha \in J}\}$ be a family of subset of H, then:

1. $\bigcap_{\alpha\in J} P(A_\alpha) = P\left(\bigcap_{\alpha\in J} A_\alpha\right)$, and
2. $\bigcup_{\alpha\in J} P(A_\alpha) \subset P\left(\bigcup_{\alpha\in J} A_\alpha\right)$.

Definition 3.1.2. [12,20,21,31] Let H be any set and n be any positive integer, then the set $P_n(H)$. The set of all n-elements of a subset of H with its order n.

Definition 4.1.2. [25-29] (n^{th} -Power set) Let H be a universe of discourse set, and $n \in \mathbb{Z}^+$. Define the n^{th} -Power set of a set H as follows:

$$
\begin{aligned}
P^n(H) &= P\left(P^{n-1}(H)\right),\\
&= P\left(P^{n-1}(P^{n-2}(H)\right)\\
&= P\left(P^{n-1}P^{n-2}(P^{n-3}(H)\right)\\
&\ \vdots
\end{aligned}
$$

$= P\left(P^{n-1}P^{n-2}P^{n-3}\cdots P^{1}(P^{0}(H))\right)$, where $P^{0}(H) = H$, and $P^{1}(H) = P(H)$, with the decreasing order relation of subsets, such as: $P^{0}(H) \subset P^{1}(H) \subset P^{2}(H) \cdots P^{n-1} \subset P^{n}$. If we excluded the empty set from $P(H)$, Then $P_{n}^{*}(H) = P^{n}(H) \backslash \emptyset$ defined in a similar way. The class $P^{n}(H)$ plays a crucial role in real-world problems.

Theorem 3.1.2. [25-29] Let H be a discrete finite set of 2 or more elements, and $n \geq 1$ is an integer. Then:$P^{0}(H) \subset P^{1}(H) \subset P^{2}(H) \cdots P^{n-1} \subset P^{n}$. For any subset A, we identify {A} with A.

2.2. Neutrosophic power-set and neutrosophic n^th-power-set for a set

In this subsection, we developed the Power-Set and n^th-Power set for a set to Neutrosophic Power-Set and Neutrosophic n^th-Power-Set to study the new concepts related to a neutrosophic set of three types.

Definition 1.2.2. [6] Let U be a universal set, then:

1. $U_{1}^{t}[I\,] = \{u_1 + u_2 I : u_1, u_2 \in U\}$ is a universal neutrosophic set of type-1,
2. $U_{2}^{t}[I] = \{uI \cup \{u\} : u \in U\}$ is a universal neutrosophic set of type-2, and
3. $U_{3}^{t}[I\,] = \{(u_1 + u_2 I) \cup \{u_1\} : u_1, u_2 \in U\}$ is a universal neutrosophic set of type-3, where I is an indeterminacy

Definition 2.2.2. [6] Let $\emptyset$ be the empty set, then:

1. $\emptyset_{1}^{t}[I] = \{u_1 + u_2 I : u_1, u_2 \in \emptyset\} = \emptyset$ is an empty neutrosophic set of type-1
2. $\emptyset_{2}^{t}[I] = \{uI \cup \{u\} : u \in \emptyset\} = \emptyset$ is an empty neutrosophic set of type-2, and
3. $\emptyset_{3}^{t}[I] = \{(u_1 + u_2 I) \cup \{u_1\} : u_1, u_2 \in \emptyset\} = \emptyset$ is an empty neutrosophic set of type-3, where I is an indeterminacy.

Definition 3.2.2. [6] Let $H_{i}^{t}[I\,]$, $i = 1,2,3,$ be three neutrosophic sets of three types, where H is any arbitrary classical set, either H is a finite set, then $H_{i}^{t}[I\,]$, $i = 1,2,3,$ are finite neutrosophic sets, the number of all neutrosophic elements is called the neutrosophic order and is denoted by $\psi\left(H_{\mathrm{i}}^{t}[I\,], i = 1,2,3\right)$, or H is an infinite set, and consequently, $H_{i}^{t}[I\,]$, $i = 1,2,3,$ have an infinite neutrosophic order.

Definition 4.2.2. [6] Let $H_{i}^{t}[I\,]$, $i = 1,2,3,$ be three neutrosophic sets of three types, where H is any arbitrary classical set. The neutrosophic Power-Set of three types, written $\Im\left(H_{i}^{t}[I]\right)$ and defined by $\Im\left(H_{i}^{t}[I]\right) = \left\{N_{i}^{t}[I\,] : \subseteq H_{i}^{t}[I\,], i = 1,2,3\right\}$.

Definition 5.2.2. (n^th-Neutrosophic Power-Set) Let $H_{i}^{t}[I]$, $i = 1,2,3,$ be three neutrosophic sets of three types, respectively. and $n \in \mathbb{Z}^{+}$. Define the n^{th} - Neutrosophic Power-Set of sets $H_{i}^{t}[I]$ as follows:

$$\Im^{n}\left(H_{i}^{t}[I]\right) = \Im\left(\Im^{n-1}\left(H_{i}^{t}[I]\right)\right),$$
$$= \Im\left(\Im^{n-1}\left(\Im^{n-2}\left(H_{i}^{t}[I]\right)\right)\right)$$
$$= \Im\left(\Im^{n-1}\Im^{n-2}\left(\Im^{n-3}\left(H_{i}^{t}[I]\right)\right)\right)$$

$\vdots$

$= \Im\left(\Im^{n-1}\Im^{n-2}\Im^{n-3}\cdots\Im^{1}\left(\Im^{0}(H_i^t[I])\right)\right)$, where $\Im^0\left(H_i^t[I]\right) = H_i^t[I]$, and $\Im^1\left(H_i^t[I]\right) = \Im\left(H_i^t[I]\right)$, with the decreasing order relation of subsets such as: $\Im^0\left(H_i^t[I]\right) \subset \Im^1 H_i^t[I] \subset \Im^2\left(H_i^t[I]\right)\cdots\Im^{n-1} \subset \Im^n\left(H_i^t[I]\right)$. If we exclude the empty set from $\Im\left(H_i^t[I]\right)$, we get $\Im_n^*\left(H_i^t[I]\right) = \Im^n\left(H_i^t[I]\right)\backslash \emptyset_{\mathrm{i}}^t[I]$ defined in a similar way.

Example 1.2.2. Let $H_1^t[I] = \{a + aI\}$, $H_2^t[I] = \{a, aI\}$, and $H_3^t[I] = \{a, a + aI\}$ be three neutrosophic sets of three types generated by $H = \{a\}$ is a singleton set. Then the 0-neutrosophic order of neutrosophic power sets $H_1^t[I], H_2^t[I]$, and $H_3^t[I]$are $\Im^0(H_1^t[I]) = \{a + aI\}$, $\Im^0(H_2^t[I]) = \{a, aI\}$, and $\Im^0(H_3^t[I]) = \{a, a + aI\}$, respectively. Moreover, the 1st-neutrosophic order of neutrosophic power sets of $\Im^0(H_1^t[I]), \Im^0(H_2^t[I])$, and $\Im^0(H_3^t[I])$ are given by: $\Im^1(H_1^t[I]) = \begin{Bmatrix}\{a + aI\}\\ \emptyset_1^t[I]\end{Bmatrix}, \Im^1(H_2^t[I]) = \begin{Bmatrix}\{a, aI\}\\ \emptyset_2^t[I]\end{Bmatrix}$, and $\Im^1(H_3^t[I]) = \begin{Bmatrix}\{a, a + aI\}\\ \emptyset_3^t[I]\end{Bmatrix}$. While the 2nd-neutrosophic order of neutrosophic power of $\Im^1(H_1^t[I])$, $\Im^1(H_2^t[I])$, and $\Im^1(H_3^t[I])$ are represented by: $\Im^2(H_1^t[I]) = \begin{Bmatrix}\{\emptyset_1^t[I], \{a + aI\}\}\\ \{\emptyset_1^t[I]\}, \{\{a + aI\}\}\\ \emptyset_1^t[I]\end{Bmatrix}$, $\Im^2(H_2^t[I]) = \begin{Bmatrix}\{\emptyset_2^t[I], \{a, aI\}\}\\ \{\emptyset_2^t[I]\}, \{\{a, aI\}\}\\ \emptyset_2^t[I]\end{Bmatrix}$, and $\Im^2(H_3^t[I]) = \begin{Bmatrix}\{\emptyset_3^t[I], \{a, a + aI\}\}\\ \{\emptyset_3^t[I]\}, \{\{a, a + aI\}\}\\ \emptyset_3^t[I]\end{Bmatrix}$**.** Also, the 3rd-neutrosophic order of neutrosophic power sets of $\Im^2(H_1^t[I]), \Im^2(H_2^t[I])$, and $\Im^2(H_3^t[I])$ are displayed by:

$$\Im^3(H_1^t[I]) = \Im\left(\Im^2(H_1^t[I])\right) = \Im\left(\begin{Bmatrix}\{\emptyset_1^t[I], \{a + aI\}\}\\ \{\emptyset_1^t[I]\}, \{\{a + aI\}\}\\ \emptyset_1^t[I]\end{Bmatrix}\right),$$

$$\Im^3(H_2^t[I]) = \Im\left(\Im^2(H_2^t[I])\right) = \Im\left(\begin{Bmatrix}\{\emptyset_2^t[I], \{a, aI\}\}\\ \{\emptyset_2^t[I]\}, \{\{a, aI\}\}\\ \emptyset_2^t[I]\end{Bmatrix}\right), \text{ and}$$

$$\Im^3(H_3^t[I]) = \Im\left(\Im^2(H_3^t[I])\right) = \Im\left(\begin{Bmatrix}\{\emptyset_3^t[I], \{a, a + aI\}\}\\ \{\emptyset_3^t[I]\}, \{\{a, a + aI\}\}\\ \emptyset_3^t[I]\end{Bmatrix}\right).$$

To calculate $\Im^3\left(H_{\mathrm{i}}^t[I]\right)$, and $\Im^4\left(H_{\mathrm{i}}^t[I]\right)$. We have the neutrosophic order of $\Im^3\left(H_{\mathrm{i}}^t[I]\right) = 2^4 = 16$ and $\Im^4\left(H_{\mathrm{i}}^t[I]\right) = 2^{16} = 65536$ elements by Theorem 1.2. In this case and beyond, we see the limitations of manual handling in classifying cases, and the role of the machine and algorithms comes to solve some of the required problems. If we exclude the empty set, we get $P_0^*(H) = \{a\}$.

Observation. If $H_i^t[I\,]$, $i = 1,2,3$, is a finite with order n, i.e.,$O\left(H_i^t[I\,]\right) = n$**,** then the order $\mathrm{O}\left(\Im\left(H_i^t[I]\right)\right) = 2^n$.

Theorem 1.2.2. Let $\{A[I]_{\alpha\in J}\}$ be a family of neutrosophic subsets of three types of $H_i^t[I]$, then:

1. $\bigcap_{\alpha\in J}\mathfrak{I}(A[I]_\alpha)=\mathfrak{I}\left(\bigcap_{\alpha\in J}A[I]_\alpha\right)$, and
2. $\bigcup_{\alpha\in J}\mathfrak{I}(A[I]_\alpha)\subset\mathfrak{I}\left(\bigcup_{\alpha\in J}A[I]_\alpha\right)$.

Proof. (1) Let $E[I]\in\bigcap_{\alpha\in J}\mathfrak{I}(A[I]_\alpha)\Leftrightarrow\forall\alpha\in J, E[I]\in\mathfrak{I}(A[I]_\alpha)$

$$\begin{aligned}&\Leftrightarrow\forall\alpha\in J, E[I]\subseteq A[I]_\alpha\\&\Leftrightarrow\forall\alpha\in J, E[I]\subseteq\textstyle\bigcap_\alpha A[I]_\alpha\\&\Leftrightarrow\forall\alpha\in J, E[I]\in\mathfrak{I}(\textstyle\bigcap_\alpha A[I]_\alpha).\end{aligned}$$

Hence, $\bigcap_{\alpha\in J}\mathfrak{I}(A[I]_\alpha)=\mathfrak{I}\left(\bigcap_{\alpha\in J}A[I]_\alpha\right)$.

(2). Let $E[I]\in\bigcup_{\alpha\in J}\mathfrak{I}(A[I]_\alpha)\Longrightarrow\exists\,\alpha\in J, E[I]\in\mathfrak{I}(A[I]_\alpha)$

$$\begin{aligned}&\Longrightarrow\exists\,\alpha\in J, E[I]\subseteq A[I]_\alpha\\&\Longrightarrow\exists\,\alpha\in J, E[I]\subseteq\textstyle\bigcup_\alpha A[I]_\alpha\\&\Longrightarrow\exists\,\alpha\in J, E[I]\subseteq\mathfrak{I}(\textstyle\bigcup_\alpha A[I]_\alpha).\end{aligned}$$

Therefore,

$\bigcup_{\alpha\in J}\mathfrak{I}(A[I]_\alpha)\subset\mathfrak{I}\left(\bigcup_{\alpha\in J}A[I]_\alpha\right)$.

The following illustrates that the equality of the second part in the theorem does not hold or commute.

Example 2.2.2. Let $H_1^t[I]=\begin{Bmatrix}1+1I,1+2I,\\2+1I,2+2I\end{Bmatrix}$ be a Neutrosophic Set of type-1 generated by $H=\{1,2\}$, and its Neutrosophic Power-Set:

$$\mathfrak{I}(H_1^t[I])=\left\{\begin{array}{c}\{\{1+1I\},\{1+2I\},\{2+1I\},\{2+2I\}\}\\\{\{1+1I\},\{2+1I\},\{2+2I\}\},\{\{1+2I\},\{2+1I\},\{2+2I\}\}\\\{\{1+1I\},\{1+2I\},\{2+1I\}\},\{\{1+1I\},\{1+2I\},\{2+2I\}\},\\\{\{1+2I\},\{2+1I\}\},\{\{1+2I\},\{2+2I\}\},\{\{2+1I\},\{2+2I\}\},\\\{\{1+1I\},\{1+2I\}\},\{\{1+1I\},\{2+1I\}\},\{\{1+1I\},\{2+2I\}\},\\\{1+1I\},\{1+2I\},\{2+1I\},\{2+2I\},\\\emptyset_1^t[I],\end{array}\right\}$$

Let $A[I]_1=\{1+1I\}$ and $A[I]_2=\{2+2I\}$, then $\bigcup_{\alpha\in J}\mathfrak{I}(A[I]_\alpha)=\{\emptyset_1^t[I],\{1+1I\},\{2+2I\}\}$ has three neutrosophic elements, while $\mathfrak{I}\left(\bigcup_{\alpha\in J}A[I]_\alpha\right)=\{\emptyset_1^t[I],\{1+1I\},\{2+2I\},\{\{1+1I\},\{2+2I\}\}\}$ has four neutrosophic elements.

3. Neutrosophic equivalent of sets

Section three discusses the nature of the prime-specific principle's contribution to neutrosophic equivalent sets of three types. Including relevant theories and examples, with a definition of neutrosophic finite (or denumerable) set, countable neutrosophic infinite set, neutrosophic infinite set, uncountable neutrosophic set, neutrosophic cardinality, and neutrosophic transfinite number. This content represents an expansion of our work in [1-3,6-8], and opens a new window for future work on the concepts of neutrosophic cardinality and neutrosophic transfinite number and their relationship to neutrosophic set theory of the three types.

Definition 1.3. Let $X_i^t[I]$ and $Y_i^t[I]$ be two neutrosophic sets of three types, then $X_i^t[I]$ is called a neutrosophic equivalent to $Y_i^t[I]$, written $X_i^t[I] \cong Y_i^t[I]$, if there exists a neutrosophic function $f_n^i\colon X_i^t[I] \longrightarrow Y_i^t[I]$ which is one-to-one and onto (or bijection). In this case, $X_i^t[I]$ and $Y_i^t[I]$ have the same neutrosophic cardinality. Also, we said that $X_i^t[I]$ and $Y_i^t[I]$ are equipotent. Otherwise, we said that they are not equivalent (or equipotent), written $X_i^t[I] \ncong Y_i^t[I]$.

Theorem 1.3. Let $X \cong \mathrm{Y}$ be two equivalent classical sets, then $X_i^t[I] \cong Y_i^t[I]$.
Proof: Assume that $X \cong \mathrm{Y}$, then there exists a classical function $g : \mathrm{X} \longrightarrow \mathrm{Y}$ which is a one-to-one and onto. Define a neutrosophic function $f_n^i\colon X_i^t[I] \longrightarrow Y_i^t[I]$. According to definition 1.2 in [2].

$$\begin{aligned} f_n^1(x) &= f_n^1(x_1) + f_n^1(x_2 I) \\ &= f_n^1(x_1) + f_n^1(x_2) f_n^1(I) \\ &= g(x_1) + g(x_2) I, \text{ where } f_n^1(I) = I,\ f_n^1(x_1) = g(x_1), f_n^1(x_2) = g(x_2), \end{aligned}$$

$$f_n^2(x) = \begin{cases} f_n^2(x) = g(x), \\ f_n^2(xI) = f_n^2(x) f_n^2(I) = g(x) I \end{cases}$$

$$f_n^3(x) = \begin{cases} f_n^3(x_1) \\ f_n^3(x_1) + f_n^3(x_2) f_n^3(xI) \end{cases}$$

where $f_n^3(I) = I, f_n^3(x_1) = g(x_1), f_n^3(x_2) = g(x_2)$. Now, by Theorems 1.2, 2.2, and 3.2, the neutrosophic function f_n^i is one-to-one, moreover according to Theorems 4.2,5.2, and 6.2 the neutrosophic function f_n^i is an onto. Therefore $X_i^t[I] \cong Y_i^t[I]$.

Theorem 2.3. The neutrosophic equivalent relation $\cong$ is a neutrosophic equivalence relation.
Proof. (1) $\cong$ is a neutrosophic reflexive relation, since there exists a one-to-one and onto neutrosophic identity function $\mathrm{id}_{\mathrm{in}}^{\mathrm{t}}\ X_i^t[I] \longrightarrow X_i^t[I]$ by Theorems 8.2, 9.2, and 10.2 in [2]
(2). $\cong$ is a neutrosophic symmetric relation. Suppose that $X_i^t[I] \cong Y_i^t[I]$, then there exists a bijection neutrosophic function $f_n^i\colon X_i^t[I] \longrightarrow Y_i^t[I]$ by Theorem 3.2 in [2]. Then there exists an inverse neutrosophic function $f_n^{i^{-1}}\colon Y_i^t[I] \longrightarrow X_i^t[I]$ is a one-to-one and onto neutrosophic function. Hence, $Y_i^t[I] \cong X_i^t[I]$.
(3). $\cong$ is a neutrosophic transitive relation. Suppose that $X_i^t[I] \cong Y_i^t[I]$ and $Y_i^t[I] \cong X_i^t[I]$.
Since $X_i^t[I] \cong Y_i^t[I] \Longrightarrow \exists\ f_n^i\colon X_i^t[I] \longrightarrow Y_i^t[I]$ which is one-to-one and onto, also,
Since $Y_i^t[I] \cong X_i^t[I] \Longrightarrow \exists\ g_n^i\colon Y_i^t[I] \longrightarrow Z_i^t[I]$ which is one-to-one and onto. By theorem $g_n^i \circ f_n^i\colon X_i^t[I] \longrightarrow Z_i^t[I]$ is a one-to-one and onto neutrosophic function, therefore $X_i^t[I] \cong Z_i^t[I]$, and consequently, the neutrosophic equivalent relation $\cong$ is a neutrosophic equivalence relation.

Example 1.3. Let $\mathbb{N} = \{0,1,2,\ldots\}$ be the set of natural numbers. Then the neutrosophic natural numbers of type-1 are given by:

$$\mathbb{N}_1^t[I]=\left\{\begin{matrix} 0, & 0+I, & 0+2I, & 0+3I, & \cdots \\ 1, & 1+I, & 1+2I, & 1+3I, & \cdots \\ 2, & 2+I & 2+2I & 2+3I, & \cdots \\ \vdots & \vdots & \vdots & \vdots & \cdots \end{matrix}\right\}.$$

And $\mathbb{N}_{even}=\{2,4,6,\ldots\}$ be the set of even natural numbers. Then the even neutrosophic natural number of type-1 is given by:

$$\mathbb{N}_{e1}^t[I]=\left\{\begin{matrix} 2+2I,2+4I,2+6I,\cdots \\ 4+2I,4+4I,4+6I,\cdots \\ 6+2I,6+4I,6+6I,\cdots \\ \vdots \end{matrix}\right\}$$

Let us define $g:\mathbb{N}\longrightarrow\mathbb{N}_{even}$ such that $g(x)=2x,\forall\, x\in\mathbb{N}$, where g is one-to-one and onto. So, by

Theorem 1.2. Deduce that $f_n^1\colon\mathbb{N}_1^t[I]\longrightarrow\mathbb{N}_{e1}^t[I]$ is a one-to-one and onto neutrosophic function and consequently, $\mathbb{N}_1^t[I]\cong\mathbb{N}_{e1}^t[I]$. If we treat with $\mathbb{N}_1^t[I]$and $\mathbb{N}_{e1}^t[I]$directly, it could be defined

$f_n^1\colon\mathbb{N}_1^t[I]\longrightarrow\mathbb{N}_{e1}^t[I]$ such that $f_n^1(x)=2x,\forall\, x\in\mathbb{N}_1^t[I]$. In this case, assume that

$$\begin{aligned} f_n^1(x)=f_n^1(y) &\Longrightarrow 2x=2y \\ &\Longrightarrow 2(x_1+x_2I)=2(y_1+y_2I) \\ &\Longrightarrow (x_1+x_2I)=(y_1+y_2I) \\ &\Longrightarrow x=y\ \Longrightarrow f_n^1 \text{ is a one-to-one.} \end{aligned}$$

If $x\in\mathbb{N}_{e1}^t[I]$, take $x=\frac{y}{2}$, $f_n^1(x)=f_n^1\left(\frac{y}{2}\right)=2\frac{y}{2}=y$. This means that f_n^1 is an onto and $\mathbb{N}_1^t[I]\cong\mathbb{N}_{e1}^t[I]$.

Definition 2.3. Let $X_1^t[I]$ be a neutrosophic set of type-1, then $X_1^t[I]$ is called a finite neutrosophic set. If it is a neutrosophic empty set or equivalent to the following neutrosophic set:

$$\mathcal{M}_1^t[I]=\left\{\begin{matrix} 1+1I,1+2I,\cdots,1+nI \\ 2+1I,2+2I,\cdots,2+nI \\ 3+1I,3+2I,\cdots,3+nI \\ \vdots \\ n+1I,n+2I,\cdots,n+nI \end{matrix}\right\}$$

For some $n\in\mathbb{Z}^+$. Then the neutrosophic order or cardinality, written by $\psi(X_1^t[I])=n$. Otherwise, the neutrosophic set is called an infinite neutrosophic set.

Definition 3.3. Let $X_2^t[I]$ be a neutrosophic set of type-2, then $X_2^t[I]$ is called a finite neutrosophic set. If it is a neutrosophic empty set or equivalent to the following neutrosophic set:

$$\mathcal{M}_2^t[I]=\left\{\begin{matrix} 1,1I \\ 2,2I \\ \vdots \\ n,nI \end{matrix}\right\}$$

For some $n\in\mathbb{Z}^+$. The neutrosophic order or cardinality, written by $\psi(X_2^t[I])=n$. Otherwise, the neutrosophic set is called an infinite neutrosophic set.

Example 2.3. Consider $X_2^t[I] = \begin{Bmatrix} 1,1I \\ 2,2I \end{Bmatrix}$ and $Y_2^t[I] = \begin{Bmatrix} a, aI \\ b, bI \end{Bmatrix}$ are two neutrosophic sets of type-2. Then there exists a one-to-one and onto correspondence between them, $f_n^2(1) = a, f_n^2(1I) = f_n^2(aI), f_n^2(2) = b,$ and $f_n^2(2I) = bI$. In this case, two finite neutrosophic sets are equivalent (or equipotent) if and only if they contain the same neutrosophic cardinality.

Definition 4.3. Let $X_i^t[I]$ be a neutrosophic set of three types, then $X_i^t[I]$ is called a countably infinite neutrosophic set. If there exists a neutrosophic bijection $f_n^i: X_i^t[I] \longrightarrow \mathbb{Z}^{+}{}_i^t[I]$.In this case, the neutrosophic transfinite number, written by $\psi(X_2^t[I]) = \aleph$.

Definition 5.3. An infinite neutrosophic set that is not countably infinite is called an uncountable neutrosophic set.

Definition 6.3. A countable neutrosophic set is either a finite neutrosophic set or a countably infinite neutrosophic set.

4. Neutrosophic hyperfunction and inverse neutrosophic hyperfunction of one neutrosophic variable

Section four is divided into two parts. In the first part, we explored the concept of the neutrosophic hyperfunction of a single variable across three types of neutrosophic sets, providing examples for better understanding. We also delved into neutrosophic hyperfunctions applied to neutrosophic subsets under various neutrosophic operations, including unions, intersections, differences, and subsets. This work builds upon classical hyperfunction, as discussed in our review paper [9]. The term 'Hyper' stems from Marty's 1934 work on Hypergroups, where he extended the codomain of binary operations from H to P(H).

1.4. Neutrosophic hyperfunction of one neutrosophic variable

Definition 1.1.4. Let $H_1^t[I]$ be a neutrosophic set of type-1 generated by H and $\mathfrak{I}\,(H_1^t[I])$ be a Neutrosophic Power-Set of $H_1^t[I]$. A Function $f_h^1: H_1^t[I] \longmapsto \mathfrak{I}\,(H_1^t[I])$ is called a Neutrosophic HyperFunction of type-1, if for all $x \in H_1^t[I]$, then there exists a neutrosophic subset $A_1^t[I]_{\delta\in J}$ such that $f_h^1(x) = A_1^t[I]_{\delta\in J}$.

Observation. The codomain of neutrosophic hyperfunction includes the empty set. If we consider $\mathfrak{I}^*\,(H_1^t[I]) = \mathfrak{I}\,(H_1^t[I])\backslash\emptyset_1^t[I]$. Then the codomain of a hyperfunction $f_h^1: H_1^t[I] \longmapsto \mathfrak{I}^*\,(H_1^t[I])$ does not include the neutrosophic empty set.
Now, we will define the neutrosophic hyperfunction induced by the classical hyperfunction.

Definition 2.1.4. Let $H_1^t[I]$ be a neutrosophic set of type-1 generated by H and $\mathfrak{I}\,(H_1^t[I])$ be a neutrosophic powerset of $H_1^t[I]$. A Function $f_h^1: H_1^t[I] \longmapsto \mathfrak{I}\,(H_1^t[I])$ is called a neutrosophic hyperfunction generated by a classical hyperfunction $g^h: H \longmapsto P(H)$, if it satisfies the following property:

$$\begin{aligned} f_h^1(x) &= f_h^1\big((x_1 + x_2 I)\big) \\ &= f_h^1(x_1) + f_h^1(x_2 I) \\ &= f_h^1(x_1) + f_h^1(x_2) f_h^1(\mathrm{I}). \end{aligned}$$

where $f_h^1(\mathrm{I}) = I$, $f_h^1(x_1) = g^h(x_1)$, and $f_h^1(x_2) = g^h(x_2)$.

Example 1.1.4. Let $H_1^t[I] = \begin{Bmatrix} 1+1I, 1+2I, \\ 2+1I, 2+2I \end{Bmatrix}$ be a neutrosophic set of type-1. The neutrosophic powerset

$$\Im(H_1^t[I]) = \begin{Bmatrix} \big\{\{1+1I\},\{1+2I\},\{2+1I\},\{2+2I\}\big\} \\ \big\{\{1+1I\},\{2+1I\},\{2+2I\}\big\},\big\{\{1+2I\},\{2+1I\},\{2+2I\}\big\} \\ \big\{\{1+1I\},\{1+2I\},\{2+1I\}\big\},\big\{\{1+1I\},\{1+2I\},\{2+2I\}\big\}, \\ \big\{\{1+2I\},\{2+1I\}\big\},\big\{\{1+2I\},\{2+2I\}\big\},\big\{\{2+1I\},\{2+2I\}\big\}, \\ \big\{\{1+1I\},\{1+2I\}\big\},\big\{\{1+1I\},\{2+1I\}\big\},\big\{\{1+1I\},\{2+2I\}\big\}, \\ \{1+1I\},\{1+2I\},\{2+1I\},\{2+2I\} \\ \emptyset_1^t[I] \end{Bmatrix}$$

Consider $g^h: H \longmapsto P(H)$ such that $g^h(1) = \{2\}$, and $g^h(2) = \{1,2\}$. Then the neutrosophic hyperfunction of type one $f_h^1: H_1^t[I] \longmapsto \Im\,(H_1^t[I])$ is given by:

$$\begin{aligned} f_h^1(1+1I) &= f_h^1(1) + f_h^1(1I) \\ &= f_h^1(1) + f_h^1(1) f_h^1(I) \\ &= g^h(1) + g^h(1) f_h^1(I) \\ &= \{2\} + \{2\}\mathrm{I} \\ &= \{2\} + \{2I\} \\ &= \{2+2I\}. \end{aligned}$$

$$\begin{aligned} f_h^1(1+2I) &= f_h^1(1) + f_h^1(2I) \\ &= f_h^1(1) + f_h^1(2) f_h^1(I) \\ &= g^h(1) + g^h(2)\mathrm{I} \\ &= \{2\} + \{1,2\}I \\ &= \{2\} + \{1I, 2I\} \\ &= \{2+1I, 2+2I\}. \end{aligned}$$

$$\begin{aligned} f_h^1(2+1I) &= f_h^1(2) + f_h^1(1I) \\ &= f_h^1(2) + f_h^1(1) f_h^1(I) \\ &= \{1,2\} + \{2\}I \\ &= \{1,2\} + \{2I\} \\ &= \{1+2I, 2+2I\}. \end{aligned}$$

$$\begin{aligned} f_h^1(2+2I) &= f_h^1(2) + f_h^1(2I) \\ &= f_h^1(2) + f_h^1(2) f_h^1(I) \\ &= \{1,2\} + \{1,2\}I \\ &= \{1,2\} + \{1I, 2I\} \\ &= \{1+1I, 1+2I, 2+1I, 2+2I\} \\ &= H_1^t[I]. \end{aligned}$$

According to definition 1.4, we can be defined $f_h^1 : H_1^t[I] \longmapsto \mathfrak{I}\,(H_1^t[I])$ by the following choice of $\mathfrak{I}\,(H_1^t[I])$.
$f_h^1(1+1I) = \{2+1I\}, f_h^1(1+2I) = \{\{1+1I\},\{1+2I\}\}$,
$f_h^1(2+1I) = \{1+2I\}$, and $f_h^1(2+2I) = \{\{1+1I\},\{2+1I\},\{2+2I\}\}$.

Definition 3.1.4. Let $H_2^t[I]$ be a neutrosophic set of type-2 generated by H and $\mathfrak{I}\,(H_2^t[I])$ be a neutrosophic powerset of $H_2^t[I]$. A Function $f_h^2 : H_2^t[I] \longmapsto \mathfrak{I}\,(H_2^t[I])$ is called a neutrosophic hyperfunction of type-2, if for all $x \in H_2^t[I]$, then there exists a neutrosophic subset $A_2^t[I]_{\delta\in J}$ such that $f_h^2(x) = A_2^t[I]_{\delta\in J}$.

Definition 4.1.4. Let $H_2^t[I]$ be a neutrosophic set of type-2 generated by H and $\mathfrak{I}\,(H_2^t[I])$ be a neutrosophic powerset of $H_2^t[I]$. A Function $f_h^2 : H_2^t[I] \longmapsto \mathfrak{I}\,(H_2^t[I])$ is called a neutrosophic hyperfunction of type-2 generated by a classical hyperfunction $g^h : H \longmapsto P(H)$, if it satisfies the following property:

$$f_h^2(x) = \begin{cases} g^h\,(x), \text{determinate} - \text{part} \\ f_h^2(xI) = f_h^2(x) f_h^2(\mathrm{I}), \text{indeterminate} - \text{part} \end{cases}$$

where $f_h^2(\mathrm{x}) = g^h\,(x), f_h^1(I) = I$.

Example 2.1.4. Let $H_2^t[I] = \begin{Bmatrix} 1, 1I, \\ 2, 2I \end{Bmatrix}$ be a neutrosophic set of type-2 generated by $H = \{1,2\}$ with its neutrosophic powerset:

$$\mathfrak{I}(H_2^t[I]) = \begin{Bmatrix} \{\{1\},\{1I\},\{2\},\{2I\}\} \\ \{\{1\},\{1I\},\{2\}\},\{\{1\},\{1I\},\{2I\}\},\{\{1\},\{2\},\{2I\}\},\{\{1I\},\{2\},\{2I\}\} \\ \{\{1I\},\{2\}\},\{\{1I\},\{2I\}\},\{\{2\},\{2I\}\}, \\ \{\{1\},\{1I\}\},\{\{1\},\{2\}\},\{\{1\},\{2I\}\}, \\ \{1\},\{1I\},\{2\},\{2I\} \\ \emptyset_2^t[I] \end{Bmatrix}$$

Consider $g^h : H \longmapsto P(H)$ such that $g^h(1) = \{2\}$, and $g^h(2) = \{1,2\}$. Then the neutrosophic hyperfunction of type-2 $f_h^2 : H_2^t[I] \longmapsto \mathfrak{I}(H_2^t[I])$ is given by:
$f_h^2(1) = f_h^2(1) = g^h(1) = \{2\}$, $f_h^2(1I) = f_h^2(1) f_h^2(I) = g^h(1) f_h^2(\mathrm{I}) = \{2\}I = \{2I\}, f_h^2(2) = g^h(2) = \{1,2\}$, and $f_h^2(2I) = f_h^2(2) f_h^2(\mathrm{I}) = g^h(2) f_h^2(\mathrm{I}) = \{1,2\}I = \{1I, 2I\}$.

Definition 5.1.4. Let $H_3^t[I]$ be a neutrosophic set of type-3 generated by H and $\mathfrak{I}\,(H_3^t[I])$ be a neutrosophic powerset of $H_3^t[I]$. A Function $f_h^3 : H_3^t[I] \longmapsto \mathfrak{I}\,(H_3^t[I])$ is called a neutrosophic hyperfunction of type-3,

Definition 6.1.4. Let $H_3^t[I]$ be a neutrosophic set of type-3 generated by H and $\mathfrak{I}\,(H_3^t[I])$ be a neutrosophic powerset of $H_3^t[I]$. A Function $f_h^3 : H_3^t[I] \longmapsto \mathfrak{I}\,(H_3^t[I])$ is called a Neutrosophic HyperFunction of type-3, if generated by a classical hyperfunction $g^h : H \longmapsto P(H)$ such that

$$f_h^3(x) = \begin{cases} f_h^3(x_1) \\ f_h^3(x_1) + f_h^3(x_2 I) \end{cases}$$

where $f_h^3(x_1) = g^h(x_1), f_h^3(x_2 I) = f_h^3(x_2) = g^h(x_2) f_h^3(I) = I.$,

Example 4.1.4. Let $H_1^t[I] = \begin{Bmatrix} h_1 + h_1 I, h_1 + h_2 I, \cdots + h_1 + h_{\mathrm{n}} I \\ h_2 + h_1 I, h_2 + h_2 I, \cdots + h_2 + h_{\mathrm{n}} I \\ \vdots \\ h_{\mathrm{n}} + h_1 I, h_{\mathrm{n}} + h_2 I, \cdots + h_{\mathrm{n}} + h_{\mathrm{n}} I \end{Bmatrix}$ be a neutrosophic set of type one, which is generated by $H = \{h_1, h_2, \dots, h_n\}$ and $\Im(H_1^t[I])$ be the neutrosophic powerset of $H_1^t[I]$. Then the neutrosophic hyperfunction of type-1 $f_h^1 : H_1^t[I] \longmapsto \Im(H_1^t[I])$ defined by

$$f_h^1(h) = \begin{cases} \emptyset_1^t[I]., if\ h \notin H_1^t[I] \\ H_1^t[I] - \{h\}, if\ h \in H_1^t[I] \end{cases}$$

Or $f_h^1(h) = H_1^t[I] - \{h\}$, for all $h \in H_1^t[I]$ is a neutrosophic hyperfunction of type one. Also, if we consider

$H_2^t[I] = \begin{Bmatrix} h_1, h_1 I \\ h_2, h_2 I, \\ \vdots \\ h_{\mathrm{n}}, h_{\mathrm{n}} I \end{Bmatrix}$, and $f_h^2 : H_2^t[I] \longmapsto \Im(H_2^t[I])$ such that $f_h^2(h) =$
$\begin{cases} \emptyset_2^t[I]., if\ h \notin H_2^t[I] \\ H_2^t[I] - \{h\}, if\ h \in H_2^t[I] \end{cases}$

Or $f_h^2(h) = H_2^t[I] - \{h\}$, for all $h \in H_2^t[I]$ is a neutrosophic hyperfunction of type two. The following theorem gives us the main properties of neutrosophic subsets of $H_i^t[I]$ under neutrosophic operations, union, intersection, difference, and subset between any two subsets of $H_i^t[I]$.

Theorem 1.1.4. Let $H_i^t[I]$ be a neutrosophic set of three types generated by a classical set H and $\Im(H_i^t[I])$ is a power neutrosophic set of them. If $f_h^i : H_i^t[I] \longmapsto \Im(H_i^t[I])$ is a neutrosophic hyperfunction and $A_i^t[I], B_i^t[I] \subset H_i^t[I]$. Then:

1. $f_h^i(A_i^t[I] \cup B_i^t[I]) = f_h^i(A_i^t[I]) \cup f_h^i B_i^t[I]$,
2. $f_h^i(A_i^t[I] \cap B_i^t[I]) \subset f_h^i(A_i^t[I]) \cap f_h^i(B_i^t[I])$
3. $f_h^i(A_i^t[I]) - f_h^i(B) \subset f_h^i(A_i^t[I] - B_i^t[I])$, and
4. If $A_i^t[I] \subset B_i^t[I]$, then $f_h^i(A_i^t[I]) \subset f_h^i(B_i^t[I])$.

Proof. (1). Assume that $E[I]_{\gamma \in J} \in f_h^i(A_i^t[I] \cup B_i^t[I])$, for some γ.
$\Leftrightarrow \exists\ x \in (A_i^t[I] \cup B_i^t[I])$ such that $f_h^i(x) = E[I]_{\gamma \in J}$ for some γ.
$\Leftrightarrow \exists\ x \in A_i^t[I] \ni f_h^i(x) = E[I]_{\gamma \in J}$ for some γ $\vee \exists\ x \in B_i^t[I] \ni f_h^i(x) =$
$E[I]_{\gamma \in J}$ for some γ.
$\Leftrightarrow E[I]_{\gamma \in J} \in f_h^i(A_i^t[I]) \vee E[I]_{\gamma \in J} \in f_h^i(B_i^t[I])$.
$\Leftrightarrow E[I]_{\gamma \in J} \in \left(f_h^i(A_i^t[I]) \cup f_h^i(B_i^t[I])\right)$. Therefore $f_h^i(A_i^t[I] \cup B_i^t[I]) = f_h^i(A_i^t[I]) \cup$
$f_h^i B_i^t[I]$.
(2). Suppose that $E[I]_{\gamma \in J} \in f_h^i(A_i^t[I] \cap B_i^t[I])$.

$\Longrightarrow \exists\, x \in \left(A_i^t[I] \cap B_i^t[I]\right) \ni f_h^i(x) = E[I]_{\gamma\in J}$ for all γ.
$\Longrightarrow \left(\exists\, x \in A_i^t[I] \ni f_h^i(x) = E[I]_{\gamma\in J}\right.$ for all $\left.\gamma\right)$
$\wedge \left(\exists\, x \in B_i^t[I] \ni f_h^i(x) = E[I]_{\gamma\in J}\right.$ for all $\left.\gamma\right)$
$\Longrightarrow \left(E[I]_{\gamma\in J} \in f_h^i(A_i^t[I])\right) \wedge \left(E[I]_{\gamma\in J} \in f_h^i(B_i^t[I])\right)$
$\Longrightarrow E[I]_{\gamma\in J} \in \left(f_h^i(A_i^t[I]) \cap f_h^i(B_i^t[I])\right)$
$\Longrightarrow f_h^i\left(A_i^t[I] \cap B_i^t[I]\right) \subset f_h^i\left(A_i^t[I]\right) \cap f_h^i\left(B_i^t[I]\right)$.
(3). Let $E[I]_{\gamma\in J} \in \left(f_h^i\left(A_i^t[I]\right) - f_h^i(B)\right)$
$\Longrightarrow E[I]_{\gamma\in J} \in f_h^i\left(A_i^t[I]\right)$, for some γ and $E[I]_{\gamma\in J} \notin f_h^i\left(B_i^t[I]\right)$, for all γ
$\Longrightarrow \exists\, x \in A_i^t[I] \ni f_h^i(x) = E[I]_{\gamma\in J}$, for some γ and $f_h^i(x) \notin f_h^i\left(B_i^t[I]\right)$, for all γ
$\Longrightarrow E[I]_{\gamma\in J} \in \left(f_h^i(A_i^t[I] - f_h^i\left(B_i^t[I]\right)\right)$
$\Longrightarrow f_h^i\left(A_i^t[I]\right) - f_h^i(B) \subset f_h^i\left(A_i^t[I] - B_i^t[I]\right)$.
(4). Suppose that $A_i^t[I] \subset B_i^t[I]$ and $E[I]_{\gamma\in J} \in f_h^i(A_i^t[I]$.
$\Longrightarrow \exists\, x \in A_i^t[I] \ni f_h^i(x) = E[I]_{\gamma\in J}$, for some γ
$\Longrightarrow \exists\, x \in B_i^t[I] \ni f_h^i(x) = E[I]_{\gamma\in J}$, for some γ
$\Longrightarrow E[I]_{\gamma\in J} \in f_h^i\left(B_i^t[I]\right)$
$\Longrightarrow f_h^i\left(A_i^t[I]\right) \subset f^h\left(B_i^t[I]\right)$.

Corollary 2.1.4. Let $H_i^t[I]$ be a neutrosophic set of three types generated by a classical set H and $\Im\left(H_i^t[I]\right)$ is a power neutrosophic set of them and $A_i^t[I], B_i^t[I] \subset H_i^t[I]$. If $g^h: H \longmapsto P(H)$ a classical hyperfunction, then $f_h^i: H_i^t[I] \longmapsto \Im\left(H_i^t[I]\right)$ is a neutrosophic HyperFunction that satisfies the following properties:

1. $f_h^i\left(A_i^t[I] \cup B_i^t[I]\right) = f_h^i\left(A_i^t[I]\right) \cup f_h^i B_i^t[I]$,
2. $f_h^i\left(A_i^t[I] \cap B_i^t[I]\right) \subset f_h^i\left(A_i^t[I]\right) \cap f_h^i\left(B_i^t[I]\right)$,
3. $f_h^i\left(A_i^t[I]\right) - f_h^i(B) \subset f_h^i\left(A_i^t[I] - B_i^t[I]\right)$, and
4. If $A_i^t[I] \subset B_i^t[I]$, then $f_h^i\left(A_i^t[I]\right) \subset f_h^i\left(B_i^t[I]\right)$.

Proof. By theorem 1.4 and theorem 1.3 in [11].

Example 5.1.4. Let $H_1^t[I] = \begin{Bmatrix} a + aI, a + bI, \\ b + aI, b + bI \end{Bmatrix}$, $H_2^t[I] = \begin{Bmatrix} \mathrm{a}, \mathrm{a}I, \\ \mathrm{b}, \mathrm{b}I \end{Bmatrix}$ and $H_3^t[I] = \begin{Bmatrix} a, a + aI, a + bI, \\ b, b + aI, b + bI \end{Bmatrix}$ be the neutrosophic sets of three types generated by H, $A_1^t[I] = \{a + aI\}$, $B_1^t[I] = \{b + bI\}$, $A_2^t[I] = \{a, aI\}$, $B_2^t[I] = \{b, bI\}$, $A_3^t[I] = \{a, a + aI\}$, and $B_3^t[I] = \{b, b + bI\}$. There are six neutrosophic subsets of $H_1^t[I]$, $H_2^t[I]$, and $H_3^t[I]$ respectively. Consider the classical hyperfunction $g^h: H \longmapsto P(H)$ such that

$$g^h(h) = \begin{cases} \{a\}, if\ h \in H \\ \emptyset, if\ h \notin H \end{cases}$$

Then the neutrosophic hyperfunction generated by g^h are given by:

$$\begin{aligned} f_h^1(A_1^t[I]) = f_h^1(\mathrm{a} + \mathrm{a}\mathrm{I}) &= f_h^1(\mathrm{a}) + f_h^1(\mathrm{a}I) \\ &= f_h^1(\mathrm{a}) + f_h^1(\mathrm{a}) f_h^1(I) \end{aligned}$$

$$= g(\mathrm{a}) + g(\mathrm{a})I$$
$$= \{a\} + \{a\}I = \{a\} + \{aI\} = \{a + aI\} = A_1^t[I].$$
$$f_h^1(B_1^t[I]) = f_h^1(\mathrm{b} + \mathrm{bI}) = f_h^1(\mathrm{b}) + f_h^1(\mathrm{b}I)$$
$$= f_h^1(\mathrm{b}) + f_h^1(\mathrm{b}) f_h^1(I)$$
$$= g(\mathrm{b}) + g(\mathrm{b})I$$
$$= \{a\} + \{a\}I = \{a\} + \{aI\} = \{a + aI\}.$$

$f_h^1(A_1^t[I]) \cap f_h^1(B_1^t[I]) = \{a + aI\}$, and $A_1^t[I] \cap B_1^t[I] = \emptyset_1^t[I]$. So, $f_h^1(A_1^t[I] \cap B_1^t[I]) = f_h^1(\emptyset_1^t[I]) = \emptyset_1^t[I]$. That is $f_h^i\big(A_i^t[I] \cap B_i^t[I]\big) \subset f_h^i\big(A_i^t[I]\big) \cap f_h^i\big(B_i^t[I]\big)$. And,

$$f_h^2(A_2^t[I]) = f_h^2(\{\{\mathrm{a}, \mathrm{aI}\}) = \{f_h^2(a), f_h^2(aI)\}$$
$$= \{f_h^2(a), f_h^2(a) f_h^2(I)\}$$
$$= \{g(\mathrm{a}) + g(\mathrm{a})I\}$$
$$= \{\{a\}, \{a\}I\} = \{\{a\}, \{aI\}\} = \{a, aI\}.$$
$$f_h^2(B_2^t[I]) = f_h^2(\{\{\mathrm{b}, \mathrm{bI}\}) = f_h^2(\mathrm{b}), f_h^2(bI)\}$$
$$= \{f_h^2(b), f_h^2(b) f_h^2(I)\}$$
$$= \{g(b), g(b) f_h^2(I)\}$$
$$= \{\{a\}, \{a\}I\} = \{\{a\}, \{aI\}\} = \{a, aI\}.$$ We get,

$f_h^2(A_2^t[I]) \cap f_h^2(B_2^t[I]) = \{a, aI\}$, and $A_2^t[I] \cap B_2^t[I] = \emptyset_2^t[I]$. hence, $f_h^2(A_2^t[I] \cap B_2^t[I]) = f_h^1(\emptyset_2^t[I]) = \emptyset_2^t[I]$. We note that, $f_h^i\big(A_i^t[I] \cap B_i^t[I]\big) \subset f_{nh}^i\big(A_i^t[I]\big) \cap f_{nh}^i\big(B_i^t[I]\big)$. Moreover, we can define neutrosophic hyperfunction without deducing from classical hyperfunction, such that

$$f_h^i(\mathrm{h}) = \begin{cases} \{a + aI\}, if\ h \in H_i^t[I] \\ \emptyset_i^t[I], if\ h \notin H_i^t[I] \end{cases}$$

To get the same result. The next theorem represents the properties of the neutrosophic family of subsets of $H_i^t[I]$.

Theorem 3.1.4. Consider the neutrosophic hyperfunction $f_h^i: H_i^t[I] \longmapsto \Im\big(H_i^t[I]\big)$, then for any family $\{A[I]_{\rho\in J}\}$ of the neutrosophic subsets of $H_i^t[I]$, we have

1. $f_h^i\big(\bigcup_{\rho\in J} A[I]_\rho\big) = \bigcup_{\rho\in J} f_h^i(A[I]_\rho)$, and
2. $f_h^i\big(\bigcap_{\rho\in J} A[I]_\rho\big) \subset \bigcap_{\rho\in J} f_h^i(A[I]_\rho)$.

Proof. (1) Suppose that $E[I]_{\rho\in J} \in \bigcup_{\rho\in J} f_{nh}^i(A[I]_\rho) \Leftrightarrow \exists \rho \in J, E[I]_{\rho\in J} \in f_h^i(A[I]_\rho)$

$$\Leftrightarrow \exists \rho \in J, h \in H_i^t[I] \ni f_h^i(h) = E[I]_{\rho\in J}.$$
$$\Leftrightarrow \exists \rho \in J, h \in \bigcup_{\rho\in J} A_\rho \ni f_h^i(h) = E[I]_{\rho\in J}$$
$$\Leftrightarrow f_h^i(h) = E[I]_{\rho\in J} \in f_h^i\big(\bigcup_{\rho\in J} A_\rho\big).$$

Hence
$f_h^i\big(\bigcup_{\rho\in J} A[I]_\rho\big) = \bigcup_{\rho\in J} f_h^i(A[I]_\rho)$.

(2). Suppose that $E[I]_{\rho\in J} \in f_h^i\big(\bigcap_{\rho\in J} A[I]_\rho\big) \Longrightarrow \exists h \in \bigcap_{\rho\in J} A[I]_\rho \ni f_h^i(h) = E[I]_{\rho\in J}$

$$\Longrightarrow \exists h, \forall \rho \in A[I]_\rho \ni f_h^i(h) = E[I]_{\rho\in J}$$
$$\Longrightarrow \exists h, \forall \rho, E[I]_{\rho\in J} \in f_h^i(A[I]_\rho)$$

$\Longrightarrow E[I]_{\rho\in J} \in \bigcap_{\rho\in J} f_h^i(A[I]_\rho)$. Therefore, $f_h^i\left(\bigcap_{\rho\in J} A[I]_\rho\right) \subset \bigcap_{\rho\in J} f_h^i(A[I]_\rho)$.

Corollary 4.1.4. Let $H_i^t[I]$ be a neutrosophic set of three types generated by a classical set H and $\Im\left(H_i^t[I]\right)$ is a power neutrosophic set of them and $\{A[I]_{\rho\in J}\}$ is any family of neutrosophic subsets of $H_i^t[I]$.
If $g^h: H \longmapsto P(H)$ a classical hyperfunction, then $f_h^i: H_i^t[I] \longmapsto \Im\left(H_i^t[I]\right)$ is a neutrosophic hyperfunction that satisfies the following properties:

1. $f_h^i\left(\bigcup_{\rho\in J} A[I]_\rho\right) = \bigcup_{\rho\in J} f_h^i(A[I]_\rho)$, and
2. $f_h^i\left(\bigcap_{\rho\in J} A[I]_\rho\right) \subset \bigcap_{\rho\in J} f_h^i(A[I]_\rho)$.

Proof. By theorem 3.4 and theorem 2.3 in [9].

Theorem 5.1.4. Let $f_h^i: H_i^t[I] \longmapsto \Im\left(H_i^t[I]\right)$ be a neutrosophic hyperfunction and $A_i^t[I], B_i^t[I] \subset H_i^t[I]$, then $f_h^i: H_i^t[I] \longmapsto \Im\left(H_i^t[I]\right)$ is a one-to-one neutrosophic hyperfunction if and only if, $f_h^i\left(A_i^t[I] \cap B_i^t[I]\right) = f_h^i\left(A_i^t[I]\right) \cap f_h^i\left(B_i^t[I]\right)$.

Proof. Let $f_h^i: H_i^t[I] \longmapsto \Im\left(H_i^t[I]\right)$ be a neutrosophic hyperfunction and $A_i^t[I], B_i^t[I] \subset H_i^t[I]$ Suppose that f_h^i is a one-to-one neutrosophic hyperfunction. To prove that

$$f_h^i\left(A_i^t[I] \cap B_i^t[I]\right) = f_h^i\left(A_i^t[I]\right) \cap f_h^i\left(B_i^t[I]\right).$$

Let $E[I] \in f_h^i\left(A_i^t[I] \cap B_i^t[I]\right) \Leftrightarrow \exists x \in \left(A_i^t[I] \cap B_i^t[I]\right) \ni f_h^i(x) = E[I]$,
$\Leftrightarrow \left(\exists x \in A_i^t[I] \ni f_h^i(h) = E[I]\right) \wedge \left(\exists x \in B_i^t[I] \ni f_h^i(h) = E[I]\right)$
$\Leftrightarrow \left(E[I] \in f_h^i\left(A_i^t[I]\right)\right) \wedge \left(E[I] \in f_h^i\left(B_i^t[I]\right).\right)$
$\Leftrightarrow E[I] \in \left(f_h^i\left(A_i^t[I]\right) \cap f_h^i\left(B_i^t[I]\right)\right)$. Conversely, suppose that $f_h^i\left(A_i^t[I] \cap B_i^t[I]\right) = f_h^i\left(A_i^t[I]A\right) \cap f_h^i\left(B_i^t[I]\right)$. To show that the neutrosophic hyperfunction
$f_h^i: H_i^t[I] \longmapsto \Im\left(H_i^t[I]\right)$ is a one-to-one. Let $x, y \in H_i^t[I]$ with $x \neq y$ such that $f_h^i(x) = f_h^i(y) = E[I]$. Consider $A_i^t[I] = x$ and $B_i^t[I] = y$, we deduced that
$f_h^i\left(A_i^t[I]\right) \cap f_h^i\left(B_i^t[I]\right) = f_h^i(x) \cap f^h(y) = E[I] \neq f_h^i\left(A_i^t[I] \cap B_i^t[I]\right) = f_h^i\left(\emptyset_i^t[I]\right)$.

Definition 7.1.4. Let $f_h^i: H_i^t[I] \longmapsto \Im\left(H_i^t[I]\right)$ be a neutrosophic hyperfunction and $g_h^i: \Im\left(H_i^t[I]\right)I \longmapsto \Im^2\left(H_i^t[I]\right)$ be a neutrosophic hyper-function. Then, the neutrosophic composition of the neutrosophic hyperfunction:
$g_h^i \circ f_h^i: H_i^t[I] \longmapsto \Im^2\left(H_i^t[I]\right)$ such that $\left(g_h^i \circ f_h^i\right)(h) = g^h\left(f_h^i(h)\right), \forall\, h \in H_i^t[I]$.

Definition 8.1.4. Let $H_i^t[I]$ be a neutrosophic set of three types, $n \in \mathbb{Z}^+$, and $\Im^n\left(H_i^t[I]\right)$ is n^{th} $-$Neutrosophic power set of a set $H_i^t[I]$. Then there exists a sequence of neutrosophic hyperfunctions $f_{h_j}^i, j = 1,2, \dots, n$.

$f_{h_1}^i: H_i^t[I] \to \Im(H_i^t[I]), f_{h_2}^i: \Im(H_i^t[I]) \longrightarrow \Im^2(H_i^t[I]), f_{h_3}^i: \Im^2(H_i^t[I]) \longrightarrow \Im^3(H_i^t[I]), \ldots,$
and
$f_{h_n}^i: \Im^{n-1}(H_i^t[I]) \longrightarrow \Im^2(H_i^t[I])$ such that

$$\begin{aligned}(f_{h_n}^i \circ f_{h_{n-1}}^i \circ \ldots \circ f_{h_2}^i \circ f_{h_1}^i)(x) &= (f_{h_n}^i \circ f_{h_{n-1}}^i \circ \ldots \circ f_{h_2}^i)(f_{h_1}^i(x)) \\ &= (f_{h_n}^i \circ f_{h_{n-1}}^i \circ \ldots)\left(f_{h_2}^i(f_{h_1}^i(x))\right) \\ &=\vdots \\ &= (f_{h_n}^i)\left(f_{h_{n-1}}^i\left(\ldots f_{h_2}^i(f_{h_1}^i(x))\right)\right), \forall\, x \in H_i^t[I].\end{aligned}$$

2.4. Inverse neutrosophic hyperfunction of one variable

Definition 1.2.4. Let $H_i^t[I]$ be a neutrosophic set of three types, and $\Im(H_i^t[I])$ be the power set of $H_i^t[I]$. A function $f_h^{i^{-1}}: \Im(H_i^t[I]) \longmapsto H_i^t[I]$ is called the inverse neutrosophic hyperfunction, if for all $E[I] \in \Im(H_i^t[I])$, then there exists an element $x \in H_i^t[I]$ such that $f_h^{i^{-1}}(E[I]) = x$. Also, we can define the inverse neutrosophic hyperfunction induced by the inverse hyperfunction.

Definition 2.1.4. Let $H_i^t[I]$ be a universal set, and $\Im(H_i^t[I])$ be the power set of $H_i^t[I]$. A function $f_h^{i^{-1}}: \Im(H_i^t[I]) \longmapsto H_i^t[I]$ is called the inverse neutrosophic hyperfunction induced by the inverse hyperfunction $f_h^{-1}: P(H) \longmapsto H$.

Example 1.2.4. Let $H_1^t[I]$ and $\Im(H_1^t[I])$ be a neutrosophic set of type one and its neutrosophic power-set, an example 1.3. Define some of the inverse neutrosophic hyperfunctions of type-1 $f_h^{1^{-1}}: \Im(H_1^t[I]) \longmapsto H_1^t[I]$ which are given by:

1. $f_h^{1^{-1}}(A[I]) = \begin{cases} H_1^t[I] - A[I], if\ A[I] \neq \emptyset_1^t[I] \\ 2 + 2I, if\ A[I] = \emptyset_1^t[I] \end{cases}$. Or
2. $f_h^{1^{-1}}(A[I]) = \begin{cases} the\ smallest\ neutrosphic\ element\ of\ A[I], if\ A[I] \neq \emptyset_1^t[I] \\ 1 + 2I, if\ A[I] = \emptyset_1^t[I] \end{cases}$. Or
3. $f_h^{1^{-1}}(A[I]) = \begin{cases} the\ largest\ neutrosphic\ element\ of\ A[I], if\ A[I] \neq \emptyset_1^t[I] \\ 1 + 1I, if\ A[I] = \emptyset_1^t[I] \end{cases}$

   For all $A[I] \in \Im(H_1^t[I])$.

Theorem 1.2.4. Consider an infinite neutrosophic set $H_i^t[I]$ of three types generated by any infinite universal set H, then there exists a denumerable (finite) neutrosophic set $A_i^t[I]$ such that $A_i^t[I] \subset H_i^t[I]$.
Proof. Method-1. Let $H_i^t[I]$ be an infinite neutrosophic set of three types generated by any infinite universal set H, and $\Im(H_i^t[I])$ is a neutrosophic power set of a set $H_i^t[I]$. By Theorem 1.2.3 in [11]. Then there is a denumerable set $A \subset H$ by Theorems 3.1, 3.2, and 3.3 in [11]. We have $A_i^t[I] \subset H_i^t[I], i = 1,2,3$. Moreover, A is a denumerable set, which implies that there exists a bijection mapping or function $f_c: \mathrm{A} \longmapsto \{1,2,3, \ldots, n\}$, for some $n \in \mathbb{Z}^+$. According to Theorems 1.2, 2.2, and 3.2 in [14], we have

$f_1^i: A_i^t[I] \longmapsto \begin{Bmatrix} h_1 + h_1 I, h_1 + h_2 I, \cdots + h_1 + h_\mathrm{n} I \\ h_2 + h_1 I, h_2 + h_2 I, \cdots + h_2 + h_\mathrm{n} I \\ \vdots \\ h_\mathrm{n} + h_1 I, h_\mathrm{n} + h_2 I, \cdots + h_\mathrm{n} + h_\mathrm{n} I \end{Bmatrix}$, $f_3^i: A_i^t[I] \longmapsto \begin{Bmatrix} h_1, h_1 I \\ h_2, h_2 I, \\ \vdots \\ h_\mathrm{n}, h_\mathrm{n} I \end{Bmatrix}$, and

$f_2^i: A_i^t[I] \longmapsto \begin{Bmatrix} h_1, h_1 + h_1 I, h_1 + h_2 I, \cdots + h_1 + h_\mathrm{n} I \\ h_2, h_2 + h_1 I, h_2 + h_2 I, \cdots + h_2 + h_\mathrm{n} I \\ \vdots \\ h_\mathrm{n}, h_\mathrm{n} + h_1 I, h_\mathrm{n} + h_2 I, \cdots + h_\mathrm{n} + h_\mathrm{n} I \end{Bmatrix}$ are neutrosophic bijections mappings. Hence $A_i^t[I]$is a neutrosophic denumerable subset of $H_i^t[I], i = 1,2,3$.

Method-2. By a similar argument to Theorem 1.2.3 in [9]. Let $H_i^t[I]$ be an infinite neutrosophic set of three types generated by any infinite universal set H, and $\Im\left(H_i^t[I]\right)$ is a neutrosophic power-set of a set $H_i^t[I]$. Define the inverse neutrosophic hyperfunction $f_n^{i^{-1}}: \Im\left(H_i^t[I]\right) \longmapsto H_i^t[I]$ by $f_n^{i^{-1}}\left(H_i^t[I]\right) = x$, where it depends on the types of the neutrosophic set and is complete by the same argument.

Example 1.2.4. Let $H_1^t[I]$ and $\Im(H_1^t[I])$be a neutrosophic set of type one and its neutrosophic power-set, an example 1.3. Define some of the inverse neutrosophic hyperfunctions of type-1 $f_h^{1^{-1}}: \Im(H_1^t[I]) \longmapsto H_1^t[I]$ which are given by:

1. $f_h^{1^{-1}}(A[I]) = \begin{cases} H_1^t[I] - A[I], if\ A[I] \neq \emptyset_1^t[I] \\ 2 + 2I, if\ A[I] = \emptyset_1^t[I] \end{cases}$. Or
2. $f_h^{1^{-1}}(A[I]) = \begin{cases} the\ smallest\ neutrosphic\ element\ of\ A[I], if\ A[I] \neq \emptyset_1^t[I] \\ 1 + 2I, if\ A[I] = \emptyset_1^t[I] \end{cases}$.

   Or
3. $f_h^{1^{-1}}(A[I]) = \begin{cases} the\ largest\ neutrosphic\ element\ of\ A[I], if\ A[I] \neq \emptyset_1^t[I] \\ 1 + 1I, if\ A[I] = \emptyset_1^t[I] \end{cases}$

For all $A[I] \in \Im(H_1^t[I])$.

Theorem 1.2.4. Consider an infinite neutrosophic set $H_i^t[I]$ of three types generated by any infinite universal set H, then there exists a denumerable (finite) neutrosophic set $A_i^t[I]$ such that $A_i^t[I] \subset H_i^t[I]$.

Proof. Method-1. Let $H_i^t[I]$ be an infinite neutrosophic set of three types generated by any infinite universal set H, and $\Im\left(H_i^t[I]\right)$ is a neutrosophic power set of a set $H_i^t[I]$. By Theorem 1.2.3 in [11]. Then there is a denumerable set $A \subset H$ by Theorems 3.1, 3.2, and 3.3 in [11]. We have $A_i^t[I] \subset H_i^t[I], i = 1,2,3$. Moreover, A is a denumerable set, which implies that there exists a bijection mapping or function $f_c: \mathrm{A} \longmapsto \{1,2,3, \dots, n\}$, for some $n \in \mathbb{Z}^+$. According to Theorems 1.2, 2.2, and 3.2 in [14], we have

$$f_1^i: A_i^t[I] \longmapsto \begin{Bmatrix} h_1 + h_1 I, h_1 + h_2 I, \cdots + h_1 + h_{\mathrm{n}} I \\ h_2 + h_1 I, h_2 + h_2 I, \cdots + h_2 + h_{\mathrm{n}} I \\ \vdots \\ h_{\mathrm{n}} + h_1 I, h_{\mathrm{n}} + h_2 I, \cdots + h_{\mathrm{n}} + h_{\mathrm{n}} I \end{Bmatrix}, f_3^i: A_i^t[I] \longmapsto \begin{Bmatrix} h_1, h_1 I \\ h_2, h_2 I, \\ \vdots \\ h_{\mathrm{n}}, h_{\mathrm{n}} I \end{Bmatrix}, \text{ and}$$

$$f_2^i: A_i^t[I] \longmapsto \begin{Bmatrix} h_1, h_1 + h_1 I, h_1 + h_2 I, \cdots + h_1 + h_{\mathrm{n}} I \\ h_2, h_2 + h_1 I, h_2 + h_2 I, \cdots + h_2 + h_{\mathrm{n}} I \\ \vdots \\ h_{\mathrm{n}}, h_{\mathrm{n}} + h_1 I, h_{\mathrm{n}} + h_2 I, \cdots + h_{\mathrm{n}} + h_{\mathrm{n}} I \end{Bmatrix} \text{ are neutrosophic bijections}$$

mappings. Hence $A_i^t[I]$is a neutrosophic denumerable subset of $H_i^t[I], i = 1,2,3$.
Method-2. By a similar argument to Theorem 1.2.3 in [9]. Let $H_i^t[I]$ be an infinite neutrosophic set of three types generated by any infinite universal set H, and $\Im\left(H_i^t[I]\right)$ is a neutrosophic power set of a set $H_i^t[I]$. Define the inverse neutrosophic hyperfunction ${f_n^i}^{-1}: \Im\left(H_i^t[I]\right) \longmapsto H_i^t[I]$ by ${f_n^i}^{-1}\left(H_i^t[I]\right) = x$, where it depends on the types of the neutrosophic set and is complete by the same argument.

5. Neutrosophic extra hyperfunction of one variable

In this section, we introduce the neutrosophic extra hyperfunction of the neutrosophic set of three types, along with their properties, as an extension of the work in [9].

Definition 1.5. Let $H_i^t[I]$ be a neutrosophic set of three types generated by any universal set H, and $\Im\left(H_i^t[I]\right)$ is a neutrosophic power set of $H_i^t[I]$. A function $f_{eh}^i: \Im\left(H_i^t[I]\right) \longmapsto \Im\left(H_i^t[I]\right)$ is called an extra neutrosophic hyperfunction, if for all $A[I]_{\gamma \in J} \in \Im\left(H_i^t[I]\right)$, then there exists δ, $B[I]_{\delta \in I} \in \Im\left(H_i^t[I]\right)$ such that $f_{eh}^i(A[I]_{\gamma \in J}) = B[I]_{\delta \in I}$. This extra neutrosophic hyperfunction includes the empty set.

Observation. If $\Im^*\left(H_i^t[I]\right) = \Im\left(H_i^t[I]\right) \backslash \emptyset_i^t[I]$, then $f_{eh}^i: \Im^*\left(H_i^t[I]\right) \longmapsto \Im^*\left(H_i^t[I]\right)$ does not contain a neutrosophic empty set. Also, we can define extra neutrosophic hyperfunction induced by neutrosophic hyperfunction.

Definition 2.5. Let $H_i^t[I]$ be a neutrosophic set of three types generated by any universal set H, and $\Im\left(H_i^t[I]\right)$ is a neutrosophic power set of $H_i^t[I]$. If $f_h^i: H_i^t[I] \longmapsto \Im\left(H_i^t[I]\right)$ is a neutrosophic hyperfunction, then
$f_{eh}^i: \Im^*\left(H_i^t[I]\right) \longmapsto \Im^*\left(H_i^t[I]\right)$ is an extra hyperfunction induced by f_h^i. The following theorem is a generalization of Theorem 1.4 in [9].

Theorem 1.5. Let $H_i^t[I]$ be a neutrosophic set of three types generated by any universal set H, and $\Im\left(H_i^t[I]\right)$ is a neutrosophic power set of $H_i^t[I]$. If $f_h^i: H_i^t[I] \longrightarrow H_i^t[I]$ is a one-to-one neutrosophic function, then $f_{eh}^i: \Im\left(H_i^t[I]\right) \longmapsto \Im\left(H_i^t[I]\right)$ is a one-to-one neutrosophic extra hyperfunction.

Proof. There are two probabilities:

Probablity-1. If $H_i^t[I] = \emptyset_i^t[I]$, $i = 1,2,3$ then $\Im(H_i^t[I]) = \{\emptyset_i^t[I]\}$. This means that the neutrosophic extra hyperfunction $f_{eh}^i: \Im(H_i^t[I]) \mapsto \Im(H_i^t[I])$ is a one-to-one (or neutrosophic bijective), because the domain of extra hyperfunction consists of one neutrosophic element.

Probablity-2. Suppose that $H_i^t[I] = \emptyset_i^t[I]$, then $\Im(H_i^t[I])$ has at least two neutrosophic elements. Let's say $A_i^t[I]$ and $B_i^t[I]$, and $A_i^t[I] \neq A_i^t[I]$, for any $i = 1,2,3$. Then there exists a neutrosophic element $x \in A_i^t[I]$ and $x \notin B_i^t[I] \Longrightarrow f_h^i(x) \in f_h^i(A_i^t[I])$ and $f_h^i(x) \notin f_h^i(B_i^t[I])$. Since$f_h^i$ is a one-to-one, we get

$f_h^i(A_i^t[I]) \neq f_h^i(B_i^t[I])$, therefore, $f_{eh}^i(A_i^t[I]) \neq f_{eh}^i(B_i^t[I])$. Hence f_{eh}^i is a one-to-one.

Theorem 2.5. Let $H_i^t[I]$ be a neutrosophic set of three types, and $\Im(H_i^t[I])$ be the neutrosophic power set of $H_i^t[I]$. If $f_h^i: H_i^t[I] \longrightarrow H_i^t[I]$ is a neutrosophic function, then the extra neutrosophic hyperfunction $f_{eh}^i: \Im(H_i^t[I]) \mapsto \Im(H_i^t[I])$ preserving the elementary neutrosophic set operations as follows:

1. $f_{eh}^i(\bigcup_{\rho\in J} A_i^t[I]_\rho) = \bigcup_{\rho\in J} f_{eh}^i(A_i^t[I]_\rho)$,
2. $f_{eh}^i(A_i^t[I]_\rho - B_i^t[I]_\rho) = f_{eh}^i(A_i^t[I]_\rho) - f_{eh}^i(B_i^t[I]_\rho)$.

Proof (1). Suppose that $f_{eh}^i(E_i^t[I]_\rho) \in f_{eh}^i(\bigcup_{\rho\in J} A_i^t[I]_\rho)$

$\Leftrightarrow E_i^t[I]_\rho \in \bigcup_{\rho\in J} A_i^t[I]_\rho$

$\Leftrightarrow E_i^t[I]_\rho \in A_i^t[I]_\rho$, for some $\rho \in J$.

$\Leftrightarrow f_{eh}^i(E_i^t[I]_\rho) \in f_{eh}^i(A_i^t[I]_\rho)$, for some $\rho \in J$.

$\Leftrightarrow f_{eh}^i(E_i^t[I]_\rho) \in \bigcup_{\rho\in J} f_{eh}^i(A_i^t[I]_\rho)$, for some $\rho \in J$. Hence,

$f_{eh}^i(\bigcup_{\rho\in J} A_i^t[I]_\rho) = \bigcup_{\rho\in J} f_{eh}^i(A_i^t[I]_\rho)$.

(2). Consider $f_{eh}^i(E_i^t[I]_\rho) \in f_{eh}^i(\bigcap_{\rho\in J} A_i^t[I]_\rho)$

$\Leftrightarrow E_i^t[I]_\rho \in \bigcap_{\rho\in J} A_i^t[I]_\rho$

$\Leftrightarrow E_i^t[I]_\rho \in A_i^t[I]_\rho$, for all $\rho \in J$.

$\Leftrightarrow f_{eh}^i(E_i^t[I]_\rho) \in \bigcap_{\rho\in J} f_{eh}^i(A_i^t[I]_\rho)$, for all $\rho \in J$. Therefore, $f_{eh}^i(\bigcap_{\rho\in J} A_i^t[I]_\rho) = \bigcap_{\rho\in J} f_{eh}^i(A_i^t[I]_\rho)$.

(3). Assume that $f_{eh}^i(E_i^t[I]_\rho) \in f_{eh}^i(A_i^t[I]_\rho - B_i^t[I]_\rho)$

$\Leftrightarrow E_i^t[I]_\rho \in (A_i^t[I]_\rho - B_i^t[I]_\rho)$

$$\Leftrightarrow E_i^t[I]_\rho \in A_i^t[I]_\rho \wedge E_i^t[I]_\rho \notin B_i^t[I]_\rho$$

$$\Leftrightarrow f_{eh}^i(E_i^t[I]_\rho) \in f_{eh}^i(A_i^t[I]_\rho) \wedge f_{eh}^i(E_i^t[I]_\rho) \in f_{eh}^i(B_i^t[I]_\rho)$$

$\Leftrightarrow f_{eh}^i(E_i^t[I]_\rho) \in \left(f_{eh}^i(A_i^t[I]_\rho) - f_{eh}^i(B_i^t[I]_\rho)\right)$. We deduced that,

$f_{eh}^i(A_i^t[I]_\rho - B_i^t[I]_\rho) = f_{eh}^i(A_i^t[I]_\rho) - f_{eh}^i(B_i^t[I]_\rho)$.

6. Neutrosophic super hyperfunction and neutrosophic extra super hyperfunction of one variable

In this section, we introduce a neutrosophic super hyperfunction and a neutrosophic extra hyperfunction, along with some of their properties according to the neutrosophic set theory of three types. In future work, we aim to develop this research further.

Definition 1.6. Let $H_i^t[I]$ be a neutrosophic set of three types, and $\mathfrak{I}^n\left(H_i^t[I]\right)$ be the neutrosophic power set of $H_i^t[I]$. A function $f_{sh'}^i : H_i^t[I] \longmapsto \mathfrak{I}^n\left(H_i^t[I]\right)$ is called a neutrosophic super hyperfunction, if for all $h \in H_i^t[I]$, then there exists a neutrosophic subset element $E[I] \in \mathfrak{I}^n\left(H_i^t[I]\right)$ such that the function $f_{sh}^i(h) = E[I], n \geq 2$.

Definition 2.6. Let $H_i^t[I]$ be a neutrosophic set of three types, and $\mathfrak{I}^n\left(H_i^t[I]\right)$ be the neutrosophic power set of $H_i^t[I]$. If $f^s : H \longmapsto P^n(H)$ is a classical super-hyperfunction, then $f_{sh'}^i : H_i^t[I] \longmapsto \mathfrak{I}^n\left(H_i^t[I]\right)$ is a neutrosophic super-hyperfunction induced by f^s.
A function $f_{sh'}^i : H_i^t[I] \longmapsto \mathfrak{I}^n\left(H_i^t[I]\right)$ is called a neutrosophic super-hyperfunction, if for all $h \in H_i^t[I]$, then there exists a neutrosophic subset element $E[I] \in \mathfrak{I}^n\left(H_i^t[I]\right)$ such that the function $f_{sh}^i(h) = E[I], n \geq 2$. The following theorem

Theorem 1.6. Let $f_{sh'}^i : H_i^t[I] \longmapsto \mathfrak{I}^n\left(H_i^t[I]\right)$ be the neutrosophic super hyperfunction, $A_i^t[I], B_i^t[I] \subset H_i^t[I]$. then:

1. $f_{sh}^i\left(A_i^t[I] \cup B_i^t[I]\right) = f_{sh}^i\left(A_i^t[I]\right) \cup f_{sh}^i\left(B_i^t[I]\right)$,
2. $f_{sh}^i\left(A_i^t[I] \cap B_i^t[I]\right) \subset f_{sh}^i\left(A_i^t[I]\right) \cap f_{sh}^i\left(B_i^t[I]\right)$,
3. $f_{sh}^i\left(A_i^t[I]\right) - f_{sh}^i\left(B_i^t[I]\right) \subset f_{sh}^i\left(\left(A_i^t[I]\right) - \left(B_i^t[I]\right)\right)$, and
4. If $A_i^t[I] \subset B_i^t[I]$, then $f_{sh}^i\left(A_i^t[I]\right) \subset f_{sh}^i\left(B_i^t[I]\right)$.

Proof. By the same argument as Theorem 1.1.4. in [9]. The following theorem is a generalization of Theorem 2.5.

Theorem 2.6. Let $f_{sh'}^i : H_i^t[I] \longmapsto \mathfrak{I}^n\left(H_i^t[I]\right)$ be the neutrosophic super hyperfunction, for any neutrosophic family $\left\{A[I]_{\rho \in J}\right\}$ of the neutrosophic subsets of $H_i^t[I]$, then:

1. $f_{sh}^i\left(\bigcup_{\rho \in J} A[I]_\rho\right) = \bigcup_{\rho \in J} f_{sh}^i(A[I]_\rho)$, and
2. $f_{sh}^i\left(\bigcap_{\rho \in J} A[I]_\rho\right) \subset \bigcap_{\rho \in J} f^s(A[I]_\rho)$.

Proof. By a similar method to Theorem 2.5.

Theorem 9.3. Let $f_{sh'}^i : H_i^t[I] \longmapsto \mathfrak{I}^n\left(H_i^t[I]\right)$ be the neutrosophic super hyperfunction, and $A_i^t[I], B_i^t[I] \subset H_i^t[I]$. Then f_{sh}^i is a one-to-one neutrosophic super hyperfunction if and only if, $f_{sh}^i\left(A_i^t[I] \cap B_i^t[I]\right) \subset f_{sh}^i\left(A_i^t[I]\right) \cap f_{sh}^i\left(B_i^t[I]\right)$.
Proof. By a similar method to Theorem 2.3.

Definition 6.3. Let $H_i^t[I]$ be a neutrosophic set of three types, and $\mathfrak{I}^n\left(H_i^t[I]\right)$ be the neutrosophic power set of $H_i^t[I]$. Then $f_{esh'}^i : \mathfrak{I}^m\left(H_i^t[I]\right) \longmapsto \mathfrak{I}^n\left(H_i^t[I]\right)$ is called a neutrosophic extra super hyperfunction or a neutrosophic n-super hyperfunction, if for all $A[I]_{\gamma\in J} \in \mathfrak{I}^m\left(H_i^t[I]\right)$, then there exists a neutrosophic element $B[I]_{\delta\in I} \in \mathfrak{I}^n\left(H_i^t[I]\right)$ such that $f_{sh}^i\left(A[I]_{\gamma\in J}\right) = B[I]_{\delta\in I}$, where $m, n \geq 0$.

7. Conclusion

In this article, we develop the power set and nth-power set into neutrosophic power sets and nth-neutrosophic power sets based on a neutrosophic set of three types. Moreover, we present neutrosophic equivalent sets, neutrosophic hyperfunctions, and inverse neutrosophic hyperfunctions, along with their properties. There are three types of neutrosophic functions: hyperfunction, neutrosophic super hyperfunction, and extra neutrosophic super hyperfunction. In future work, we will continue to extend these concepts and develop three kinds of neutrosophic sets.

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REFERENCES

1. A.M. Al-Odhari, Some results of neutrosophic relations, *Neutrosophic Knowledge,* 5 (2024) 77-86.
2. A.M. Al-Odhari, Advanced analysis of neutrosophic functions: a detailed examination of three types within neutrosophic set theory, *Neutrosophic Optimization and Intelligent Systems*, 5 (2025) 38-47.
3. A.M. Al-Odhari, Invertible neutrosophic functions on neutrosophic set theory of three types, *Neutrosophic Knowledge,* 6 (2025) 21-29.
4. A.M. Al-Odhari, A review study on some properties of the structure of neutrosophic ring, *Neutrosophic Sets and Systems*, 54 (2023) 139-156.
5. A.M. Al-Odhari, Axiomatic of neutrosophic groups, *Sana'a University Journal of Applied Sciences and Technology*, 2(2) (2024) 205-214.
6. A.M. Al-Odhari, Basic introduction of neutrosophic set theory, *Plithogenic Logic and Computation*, 2 (2024) 20-28.
7. A.M. Al-Odhari, On the generalization of neutrosophic set operations:testing proofs by examples, *HyperSoft Set Methods in Engineering*, 2 (2024) 72-82.
8. A.M. Al-Odhari, Some aspects of neutrosophic set theory, *Neutrosophic Optimization and Intelligent Systems*, 4 (2024) 31-38.
9. A.M. Al-Odhari, A brief comparative study on hyperstructure, super hyperstructure, and n- super superhyperstructure, *Neutrosophic Knowledge*, 6 (2025) 38-49.
10. K.T. Atanassov, Intuitionstic fyzzy sets, *Fyzzy Sets and Systems*, 20 (1986) 87-96.
11. T. Fujita and F. Smarandache, A concise introduction to hyperfuzzy, hyperneutrosophic, hyperplithogenic, HyperSoft, and HyperRough Sets with Practical Examples, 80 (2025) 609-631.
12. P.R. Halmos, *Naive Set Theory*, Princeton, N.J.,Van Nostrand, 1960.

13. V.W. Kandasamy and F. Smarandache, Neutrosophic Rings, Hexis, Phoenix, Arizona, 2006.
14. M. Kavitha, K. Rameshwar and P. Murugadas, convergence and reduction of transitive neutrosophic soft matrices, *Annals of Pure and Applied Mathematics*, 30(2) (2024) 137-148.
15. L.A. Zadeh, Fuzzy sets, information and control, 8 (1965) 338-353.
16. P. Maity, Real neutrosophic matrix and real nilpotent neutrosophic matrix, *Annals of Pure and Applied Mathematics*, 30(2) (2024) 87-101.
17. F. Marty, Sur une Généralisation de la Notion de Groupe, Huitième Congrès des Mathématiciens Scand, (1934) 45-49.
18. M. Pal, *Neutrosophic Matrix and Neutrosophic Fuzzy Matrix*, in *Recent Developments of Fuzzy Matrix Theory and Applications*, Springer, Cham, (2024) 381–423.
19. M. Pal, *Recent Developments of Fuzzy Matrix Theory and Applications*, Springer, Cham, 2024.
20. C.C. Pinter, *A Book of Set Theory*, Mineola, New York: Dover Publications, INC, Retrieved from www.doverpublications.com 2014.
21. J.J. Rotman, *Introduction to Modern Set Theory*. Retrieved from https://www.people.vcu.edu/~clarson/roitman-set-theory.pdf 2011.
22. F. Smarandache, *A Unifying Field in Logics: Neutrosophic Logic*, Rehoboth: American Research Press(1999).
23. F. Smarandache, Neutrosophy, A new branch of philosophy, *Multiple-Valued Logic*, 8(3) (2002) 297-384.
24. F. Smarandache, Neutrosophic set, a generalization of the intuitionistic fuzzy sets, *Inter. J. Pure Appl. Math*, 24 (2005) 287-297.
25. F. Smarandache, Superhyperalgebra and neutrosophic superhyperalgebra, in nidus idearum, *Brussels,* 2 (2016) 107-108.
26. F. Smarandache, Hyperuncertain, superuncertain, and superhyperuncertain sets/logics/probabilities/statistics, *Critical Review*, 14 (1017) 10-19.
27. F. Smarandache, Introduction to neutro algebraic structures and antialgebraic structures, *Advances of Standard and Nonstandard Neutrosophic*, (2019) 240-265.
28. F. Smarandache, Introduction to superhyperalgebra and neutrosophic superhyperalgebra, *Journal of Algebraic Hyperstructures and Logical Algebras,* (2022) 1-8.
29. F. Smarandache, Superhyperfunction, superhyperstructure, neutrosophic superhyperfunction and neutrosophic superhyperstructure: current understanding and future directions, *Neutrosophic Systems with Applications*, 12 (2023) 68-76.
30. F. Smarandache, Foundation of superhyperstructure & neutrosophic superhyperstructure, *Neutrosophic Sets and Systems,* 63 (2024) 367-381.
31. W.A. Weiss, *An Introduction to Set Theory*, University of Toronto. Retrieved from https://www.math.toronto.edu/weiss/set_theory.pdf 2008.