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Nonlinear Wave Propagation in Bubbly Viscoelastic Liquid with Heat Transfer

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Abstract. A mathematical model for the analysis of viscosity, elasticity and heat transfer effects between the bubble and the second-grade liquid was developed to investigate the propagation of nonlinear waves in second-grade liquids containing gas bubbles. The standard perturbation method obtains a two-dimensional nonlinear wave equation in the bubbly liquid. The approximate solution is obtained using a semi-analytical method. The results show that the shock wave dissipates faster under the influence of heat transfer, which dominates the other dissipating factors. while the dispersion of the wave is affected by the elasticity of the liquid. The result may be applied in biomedical and engineering applications.

Keywords: Shock wave, second grade liquid, bubbly liquid, KdV-Burger equation

AMS Mathematics Subject Classification (2010): 35Q79

1. Introduction

It is known that motions in viscoelastic liquid with gas bubbles are important in many fields of engineering and industry, such as waste treatment, composites processing, boiling, bubble columns, cell imaging technique, and plastic foam processing [1, 2, 3]. Researchers like van Wijngaarden [5] developed a mathematical model for an incompressible and inviscid bubbly liquid flow. The effect of relative motion between bubbles and liquid in the relaxation process was considered in [5]. The heat transfer phenomenon and its effect between bubbles and viscous liquid were analysed by Watanabe & Prosperetti in [6] based on the equations derived in [7].

Weakly non-linear wave equations in bubbly liquid flow were initially studied theoretically in a novel work by van Wijngaarden in [8], who derived the Korteweg–de–Vries (KdV) and Korteweg–de–Vries–Burgers (KdV–Burgers) equations from the set of basic equations for bubbly liquid flows. Systematic studies of these complicated wave phenomena are therefore important for understanding physical behaviour and enhancing their applications. One of such important problems is the investigation of non-linear waves in bubble–viscoelastic liquid mixtures. The study of bubble in non-Newtonian fluids is still at an early stage compared to the understanding of bubbles dynamics in Newtonian fluids. Korteweg–de Vries and Korteweg–de-Vries-Burgers equation, incorporating the effect of heat transfer was derived in [9]. Non-linear waves in a viscous liquid with gas bubbles in two and three-dimensional cases were considered in [10]. Linear wave analysis for liquid with small gas bubble fraction were studied in [11]. Including high order terms in the derivation of non-linear evolution equations with respect to the small parameter was obtained in [12] which gives an exact description of non-linear waves.

A numerical method that reproduces the measured waveform of a shock wave in a bubbly liquid has been presented in [13]. The effects of the viscoelasticity in slightly-compressible bubbly viscoelastic liquid flow have been systematically investigated in an isothermal situation in [14]. Shock propagation in a polydisperse bubbly liquid was analysed in [15]. Momentum transfer from a shock wave to a bubbly medium is numerically simulated in [16]. Modelling bubble clusters in compressible liquids is considered in [17]. A theoretical analysis of thermal effect inside bubbles for weakly non-linear pressure waves in bubbly liquids is considered in [18]. A detailed reviewed of shock wave models for bubble clusters in cavitating flows is treated in [19].

Researches mainly focused on shock waves propagation in Newtonian liquids only. While it is established that most engineering and industrial mixtures, and liquids are non-Newtonian in nature[20]. Bubbly viscoelastic liquid is important in depolymerisation, angioplasty, targeted gene delivery etc. Due to the differences between Newtonian and non-Newtonian flows, one cannot rely on Newtonian liquid to gain better understanding into such systems. Viscoelastic constitutive equations need to be considered in modelling liquid rheology and bubble dynamics accurately.

The aim of this paper is to study the long weakly non-linear waves in viscoelastic liquid with gas bubbles incorporating the effect of heat transfer between the bubble and liquid. Reduction perturbation method [21] will be used for the derivation of the non-linear wave equation. This work is organized as follows: Section 1 covers the introduction of the paper, Model formulation is presented in Section 2. Derivation of both the linear and non-linear equations are carried out in Section 3 and 4. Result and discussion are presented in Section 5, conclusion is given in Section 6.

2. Formulation

In this study we considered the mixture with an averaged density, pressure and velocity. There are no interaction coalescence, formation and destruction of bubbles. The number of spherical bubbles in a unit mass of mixture is constant; N. There is no mass transfer between the bubble and liquid, and the buoyancy effect is not taking in to consideration. The pressure inside a bubble is uniformly and equally distributed. The inertia for the bubbly system is provided by liquid surrounding the bubble. The viscoelasticity of the liquid is considered at the boundary between the bubble and liquid.

The equations for the derivation of two-dimensional non-linear wave equation in bubbly viscoelastic flow are [10]

$$\frac{\partial \rho}{\partial t} + \nabla \bullet (\rho \mathbf{v}) = 0, \tag{1}$$

$$\rho \left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \bullet \nabla \mathbf{v} \right) + \nabla p = 0, \tag{2}$$

where $\rho(x, y, t)$ is the density of the bubble, $p(x, y, t)$ is the pressure of the mixture,

$\mathbf{v}(x,y,t) = (v^{(1)}(x,y,t), v^{(2)}(x,y,t))$ is the velocity of the mixture and $\nabla=$ $(\partial/\partial x, \partial/\partial y)$ is gradient operator.

2.1. Equation of motion of bubbles in an incompressible second grade liquid

From the continuity equation, the velocity in radial form, $v_r(r,t)$ induced in the liquid around the bubble is

$$v_r(r,t) = \frac{R^2(t)\dot{R}(t)}{r^2}. \tag{3}$$

The dot represents derivative with respect to time. The radial momentum equation in viscoelastic liquid is [22]

$$\rho_l\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r}\right) = -\frac{\partial p}{\partial r} + \frac{\partial \tau_{rr}}{\partial r} + \frac{2\tau_{rr}-\tau_{\theta\theta}-\tau_{\phi\phi}}{r}, \tag{4}$$

where ρ_l is the liquid density, and p is the total pressure, $\tau_{\theta\theta}$, $\tau_{\phi\phi}$ and τ_{rr} are stress components of the liquid. Assuming $\tau_{rr} = -\tau_{\theta\theta} - \tau_{\phi\phi}$, substituting into (4) enables us to write the bubble dynamics equation as

$$\left(R\ddot{R} + \frac{3}{2}\dot{R}^2\right) = \frac{1}{\rho_l}\left(p_l - p_0 + \int_r^\infty \left(\frac{\partial \tau_{rr}}{\partial r} + \frac{3\tau_{rr}}{r}\right)dr\right). \tag{5}$$

The gas pressure inside the bubble and the pressure outside the bubble are related by the equation [23]

$$p_l = p_g - \frac{2\sigma}{R} + \tau_{rr}(R) \tag{6}$$

where $R = R(x,y,t)$ is the bubble radius, $p_g(x,y,t)$ is the pressure of the gas, x and y are coordinates and t is time,

σ is the surface tension. The mixture density can be expressed as [24]

$$\rho = \rho_l(1-\alpha) + \alpha\rho_g, \qquad \alpha = V\rho, \quad V = \frac{4}{3}\pi N R^3, \tag{7}$$

where ρ_l, ρ_g, ρ are densities of the liquid, gas bubble and mixture respectively, α is the gas volume fraction in the unit mass of the mixture.

A second grade viscoelastic liquid is considered in this work, which has applications in science and engineering[25]. It is give as

$$\tau_{rr} = \mu A_1 + \lambda_1 A_2 + \lambda_2 A_1^2, \tag{8}$$

where

$$A_1 = D + D^T, \quad A_2 = \frac{dA_1}{dt} + A_1 D + D^T A_1,$$
$$D = \frac{\partial v_r}{\partial r}, \quad \frac{dA_1}{dt} = \frac{\partial A_1}{\partial t} + v_r.\nabla,$$

and A_1 and A_2 are the Rivlin–Ericksen tensors, The first term is Newtonian liquid, while second term represent the non-linear property of a purely viscous liquid. The memory effects of the external phase is contained in the last term, which represents the viscoelastic nature of the liquid. $\lambda_1, \lambda_2, \mu$ are the relaxation, retardation times and dynamic viscosity of the liquid. Substituting (6) and (8) into (5) with the viscoelastic effect only at the bubble boundary, we have

$$\rho_l\left(R\ddot{R} + \frac{3}{2}\dot{R}^2\right) = p_g - p_0 - \frac{2\sigma}{R} - \mu\frac{4\dot{R}}{R} + \lambda_1\left(\frac{6\dot{R}^2}{R^2} - \frac{4\ddot{R}}{R}\right) + \lambda_2\frac{8\dot{R}^2}{R^2}. \tag{9}$$

The heat transfer equation is [26]

$$\frac{dp_g}{dt} + \frac{3\gamma p_g}{R}\dot{R} + \frac{3(\gamma-1)}{R}q = 0, \tag{10}$$

where

$$q = \mathrm{Nu}\varpi_g\frac{T_g - T_0}{2R}, \qquad \frac{T_g}{T_0} = \frac{p_g}{p_0}\left(\frac{R}{R_0}\right)^3, \tag{11}$$

$$\mathrm{Nu} = \begin{cases} \sqrt{\mathrm{Pe}} & for \quad \mathrm{Pe} \geq 100 \\ 100 & for \quad \mathrm{Pe} < 100, \end{cases} \tag{12}$$

where Nu is the Nusselt number characterizing the inter-phase heat transfer,Pe is the Peclet number, T_g is the absolute temperature of the gas in the bubble, T_0 is he initial temperature in the liquid, γ is a polytropic exponent, ϖ_g is the adiabatic indexes for the gas, q is the heat transfer rate or heat flux from liquid to gas bubble per unit area of the phase interface respectively. Let the deviation of the bubble radius from the unperturbed radius be small, that is

$$R(x,y,t) = R_0 + \varphi(x,y,t), \quad R_0 = \text{constant} \tag{13}$$

where $|\varphi(x,y,t)| < R_0$, and R_0 is the radius of the bubble in the unperturbed state. The averaged density of the mixture; (7) is expressed by the following relation [27]

$$\rho = \frac{\rho_l}{1 - V\rho_g + V\rho_l}, \qquad \alpha = V\rho, \quad V = \frac{4}{3}\pi N R^3. \tag{14}$$

The series expansion of (14) using (13) up to order $\varphi^2(x,y,t)$ is

$$\rho = \rho_0 - \gamma_1\varphi(x,y,t) + \gamma_2\varphi(x,y,t)^2, \tag{15}$$

$$\rho_0 = \frac{\rho_l}{(1+V_0\rho_l)}, \quad \gamma_1 = \frac{3V_0\rho_l^2}{(1+V_0\rho_l)^2},$$

$$\gamma_2 = \frac{3\rho_l^2(1+2V_0\rho_l)}{R_0^2(1+V_0\rho_l)^3}, \quad V_0 = \frac{4}{3}\pi N R_0^3.$$

Substituting (10) and (15) into (9), (1) and (2) to obtain the following system of equations

$$-\gamma_1\frac{\partial\varphi}{\partial t} + 2\gamma_2\varphi\frac{\partial\varphi}{\partial t} + \rho_0\nabla\bullet\mathbf{v} - \gamma_1\varphi\nabla\bullet\mathbf{v} - \gamma_1\mathbf{v}\bullet\nabla\varphi = 0, \tag{16}$$

$$\rho_0\left(\frac{\partial\mathbf{v}}{\partial t} + \mathbf{v}\bullet\nabla\mathbf{v}\right) - \gamma_1\varphi\frac{\partial\mathbf{v}}{\partial t} + \nabla p = 0, \tag{17}$$

$$\begin{aligned} & p(x,y,\ t) - \ p_0 + \frac{2\sigma}{R_0} + \frac{\varphi p}{R_0} - \frac{3Hn\varphi\varphi_t p}{R_0^2} + \frac{3nHp\varphi_t}{R_0} + Hp_t \\ & + \left(\frac{6Hn\sigma}{R_0^2} - \frac{2H\sigma}{R_0^2} + \frac{4\mu}{R_0}\right)\varphi_t + \left(\frac{4\lambda_1}{R_0} + \frac{4H\mu}{R_0} + \rho R_0\right)\varphi_{tt} \\ & - \frac{3p_0\varphi^2}{R_0^2} \ + \frac{2p_0\varphi}{R_0} + \left(\frac{3\rho}{2} + \frac{12n\mu}{R_0^2} - \frac{6\lambda_1}{R_0^2} - \frac{8\lambda_2}{R_0^2} - \frac{4H\mu}{R_0^2}\right)\varphi_t^2 \\ & + \left(HR_0\rho + \frac{4H\lambda_1}{R_0}\right)\varphi_{ttt} + \left(\ 2\rho - \frac{4H\mu}{R_0^2}\right)\varphi\varphi_{tt} \\ & + \left(H4\rho \ + \frac{12Hn\lambda_1}{R_0^2} + H3\rho n - \frac{18H\lambda_1}{R_0^2} - \frac{18H\lambda_2}{R_0^2}\right)\varphi_t\varphi_{tt} \\ & + \left(H\rho - \frac{4H\lambda_1}{R_0^2}\right)\varphi\varphi_{ttt} + \left(\frac{4H\sigma}{R_0^3} - \frac{12Hn\sigma}{R_0^3}\right)\varphi\varphi_t = 0, \end{aligned} \tag{18}$$

where

$$H = \frac{2R_0^2 p_0}{3\varpi_g \mathrm{Nu}(n-1)T_0}. \tag{19}$$

H describes the heat transfer phenomena of the second grade liquid.

3. Linear case

Linearising the system of equations (16) - (18), we have a linear wave equation

$$\varphi_{tt} - c_0^2\nabla\varphi, \quad c_0^2 = \frac{2p_0}{\gamma_1 R_0}, \tag{20}$$

where c_0 is the speed of wave in the mixture. Introducing dimensionless variables

$$x = \lambda x^*, \quad y = \lambda y^*, \quad t = \lambda/c_0 t^*, \quad \mathbf{v} = c_0\mathbf{v}^*,$$

$$\rho = \rho_0 \rho^*, \quad p = p_0 p^* + p_0 - \frac{2\sigma}{R_0}, \quad \varphi = R_0 \varphi^*, \tag{21}$$

equation (16) in non-dimensional form is

$$\varphi_t + \varphi \nabla \bullet \mathbf{v} + \mathbf{v} \bullet \nabla \varphi - \frac{2\gamma_2 R_0}{\gamma_1} \varphi_t - \frac{\rho_0}{\gamma_1 R_0} \nabla \bullet \mathbf{v} = 0, \tag{22}$$

$$\frac{\rho_0}{\gamma_1 R_0} (\mathbf{v}_t + \mathbf{v} \bullet \nabla \mathbf{v}) - \varphi \nabla \mathbf{v} + \frac{p_0}{\gamma_1 c_0^2 R_0} \nabla p = 0, \tag{23}$$

$$p + \beta p_t + \varphi p + \beta_0 \varphi_t p - \beta_0 \varphi \varphi_t p + \beta_1 \varphi + \beta_2 \varphi_t + \beta_3 \varphi_{tt}$$
$$+\beta_4 \varphi_{ttt} + \beta_5 \varphi \varphi_t - 3\varphi^2 + \beta_6 \varphi_t^2 + \beta_7 \varphi \varphi_{tt} + \beta_8 \varphi_t \varphi_{tt} + \beta_9 \varphi \varphi_{ttt} = 0, \tag{24}$$

where

$$\beta = \frac{Hc_0}{\lambda}, \quad \beta_0 = \frac{3nHc_0}{\lambda}, \quad \beta_1 = \left(3 - \frac{2\sigma}{p_0 R_0}\right),$$
$$\beta_2 = \left(\frac{3nHc_0}{\lambda} - \frac{2H\sigma c_0}{p_0 \lambda R_0} + \frac{4\mu c_0}{p_0 \lambda}\right), \beta_3 = \left(\frac{4\lambda_1 c_0}{p_0 \lambda} + \frac{4H\mu c_0}{p_0 \lambda} + \frac{\rho R_0^2 c_0}{p_0 \lambda}\right),$$
$$\beta_4 = \left(\frac{HR_0 \rho c_0 R_0}{p_0 \lambda} + \frac{4H\lambda_1 c_0}{p_0 \lambda}\right),$$
$$\beta_5 = \left(\frac{4Hc_0 \sigma}{p_0 \lambda R_0} - \frac{12Hc_0 n\sigma}{p_0 \lambda R_0} - \left(1 - \frac{2\sigma}{p_0 R_0}\right) \frac{3Hnc_0}{\lambda}\right),$$
$$\beta_6 = \left(\frac{3\rho c_0^2 R_0^2}{2p_0 \lambda^2} + \frac{12c_0^2 \mu n}{p_0 \lambda^2} - \frac{6c_0^2 \lambda_1}{p_0 \lambda^2} - \frac{8c_0^2 \lambda_2}{p_0 \lambda^2} - \frac{4c_0^2 H\mu}{p_0 \lambda^2}\right),$$
$$\beta_7 = \left(\frac{2\rho c_0 R_0^2}{p_0 \lambda} - \frac{4H\mu c_0}{p_0 \lambda}\right),$$
$$\beta_8 = \left(\frac{H4\rho R_0^2 c_0^2}{p_0 \lambda^2} + \frac{12Hnc_0^2 \lambda_1}{p_0 \lambda^2} + \frac{3Hc_0^2 \rho n R_0^2}{p_0 \lambda^2} - \frac{18Hc_0^2 \lambda_1}{p_0 \lambda^2} - \frac{18c_0^2 H\lambda_2}{p_0 \lambda^2}\right),$$
$$\beta_9 = \left(\frac{H\rho c_0 R_0^2}{\lambda p_0} - \frac{4Hc_0 \lambda_1}{p_0 \lambda}\right).$$

4. Non-linear case

Suppose the waves propagate along the x-axis and transverse in the y direction. We shall use the perturbation method to obtain the non-linear evolution equation for the waves in the bubbly second grade liquid with heat transfer, We scale the independent variables as follows

$$\xi = \varepsilon^{1/2}(x - t), \quad \tau = \varepsilon^{3/2} t, \quad \delta = \varepsilon \kappa y. \tag{25}$$

We consider perturbation of the wave propagating in the x direction and the transverse variables y are 'slower' than x.

Thus, the average inter-bubble distance is less than the characteristic wavelength and the radius of bubbles in the unperturbed state and $\varepsilon = R_0 \lambda^{-1}$. The parameter δ indicates perturbation in y direction. Substituting (25) into (22) - (24) and elimination $\varepsilon^{1/2}$ in (22) and (23), we get

$$\gamma_1(\varepsilon \varphi_\tau - \varphi_\xi) + \gamma_1 \varphi \left(v_\xi^{(1)} + \varepsilon^{1/2} \kappa v_\delta^{(2)}\right) + \gamma_1 \left(v^{(1)} \varphi_\xi + \varepsilon^{1/2} \kappa v^{(2)} \varphi_\delta\right)$$
$$-2\gamma_2 R_0 \varphi(\varepsilon \varphi_\tau - \varphi_\xi) - \frac{\rho_0}{R_0} \left(v_\xi^{(1)} + \varepsilon^{1/2} \kappa v_\delta^{(2)}\right) = 0, \tag{26}$$

$$\frac{\rho_0}{\gamma_1 R_0} \left(\varepsilon v_\tau^{(1)} - v_\xi^{(1)} + v^{(1)} v_\xi^{(1)} + \varepsilon^{1/2} \kappa v^{(2)} v_\delta^{(1)}\right)$$
$$-\varphi \left(\varepsilon v_\tau^{(1)} - v_\xi^{(1)}\right) + \frac{1}{\beta_1} p_\xi = 0, \tag{27}$$

$$\frac{\rho_0}{\gamma_1 R_0} \left(\varepsilon v_\tau^{(2)} - v_\xi^{(2)} + v^{(1)} v_\xi^{(2)} + \varepsilon^{1/2} \kappa v^{(2)} v_\delta^{(2)}\right)$$

$$-\varphi\left(\varepsilon v_\tau^{(2)} - v_\delta^{(2)}\right) + \varepsilon^{1/2}\frac{\kappa}{\beta_1}p_\delta = 0, \tag{28}$$

$$\begin{aligned}
&p + \beta\left(\varepsilon^{\frac{3}{2}}p_\tau - \varepsilon^{\frac{1}{2}}p_\xi\right) + \varphi p + \beta_0\left(\varepsilon^{\frac{3}{2}}\varphi_\tau - \varepsilon^{\frac{1}{2}}\varphi_\xi\right)p \\
&-\beta_0\varphi\left(\varepsilon^{\frac{3}{2}}\varphi_\tau - \varepsilon^{\frac{1}{2}}\varphi_\xi\right)p + \beta_1\varphi + \beta_2\left(\varepsilon^{\frac{3}{2}}\varphi_\tau - \varepsilon^{\frac{1}{2}}\varphi_\xi\right) \\
&+\beta_3\left(\varepsilon^3\varphi_{\tau\tau} - 2\varepsilon^2\varphi_{\tau\xi} + \varepsilon\varphi_{\xi\xi}\right) \\
&+\beta_4\left(\varepsilon^{\frac{9}{2}}\varphi_{\tau\tau\tau} - 3\varepsilon^{\frac{7}{2}}\varphi_{\tau\tau\xi} + 3\varepsilon^{\frac{5}{2}}\varphi_{\tau\xi\xi} - \varepsilon^{\frac{3}{2}}\varphi_{\xi\xi\xi}\right) \\
&+\beta_5\varphi\left(\varepsilon^{\frac{3}{2}}\varphi_\tau - \varepsilon^{\frac{1}{2}}\varphi_\xi\right) - 3\varphi^2 \\
&+\beta_6\left(\varepsilon^{\frac{3}{2}}\varphi_\tau - \varepsilon^{\frac{1}{2}}\varphi_\xi\right)^2 + \beta_7\varphi\left(\varepsilon^3\varphi_{\tau\tau} - 2\varepsilon^2\varphi_{\tau\xi} + \varepsilon\varphi_{\xi\xi}\right) \\
&+\beta_8\left(\varepsilon^{\frac{3}{2}}\varphi_\tau - \varepsilon^{\frac{1}{2}}\varphi_\xi\right)\left(\varepsilon^3\varphi_{\tau\tau} - 2\varepsilon^2\varphi_{\tau\xi} + \varepsilon\varphi_{\xi\xi}\right) \\
&+\beta_9\varphi\left(\varepsilon^{\frac{9}{2}}\varphi_{\tau\tau\tau} - 3\varepsilon^{\frac{7}{2}}\varphi_{\tau\tau\xi} + 3\varepsilon^{\frac{5}{2}}\varphi_{\tau\xi\xi} - \varepsilon^{\frac{3}{2}}\varphi_{\xi\xi\xi}\right) = 0.
\end{aligned} \tag{29}$$

Let the solution of (26) - (29) be in the form

$$\begin{aligned}
p &= \varepsilon p_1 + \varepsilon^2 p_2 + \cdots \\
\varphi &= \varepsilon\varphi + \varepsilon^2\varphi_2 + \cdots \\
v^{(1)} &= \varepsilon v_1^{(1)} + \varepsilon^2 v_2^{(1)} + \cdots \\
v^{(2)} &= \varepsilon^{3/2}v_1^{(2)} + \varepsilon^{5/2}v_2^{(2)} + \cdots
\end{aligned} \tag{30}$$

Substituting (30) into (26) - (29), we obtained for the first order of $O(\varepsilon)$

$$\varphi_{1\xi} + \frac{\rho_0}{\gamma_1 R_0}v_\xi^{(1)} = 0, \quad -\frac{\rho_0}{\gamma_1 R_0}v_\xi^{(1)} + \frac{1}{\beta_1}p_{1\xi} = 0, \quad p_1 = -\beta_1\varphi_1, \tag{31}$$

which is integrated to give

$$\varphi_1(\xi,\delta,\tau) = \frac{\rho_0}{\gamma_1 R_0}v^{(1)}, \quad -\frac{\rho_0}{\gamma_1 R_0}v^{(1)} + \frac{1}{\beta_1}p_1 = 0, \quad p_1 = -\beta_1\varphi_1. \tag{32}$$

The integration constants are assumed to be zero. Substitute (30) into (26) - (29) and equating the expressions at ε^2 in (26), (27) and (29), and at $\varepsilon^{3/2}$ in (28), we obtain the following system of equations with $H = \beta_2 = O(\varepsilon^2)$

$$\begin{aligned}
&\gamma_1\varphi_{1\tau} - \gamma_1\varphi_{2\xi} + \gamma_1\varphi_1 v_\xi^{(1)} + \gamma_1 v_1^{(1)}\varphi_{1\xi} + 2\gamma_2 R_0 v_\xi^{(1)}\varphi_1\varphi_{1\xi} \\
&-\frac{\rho_0}{R_0}v_{2\xi}^{(1)} - \frac{\rho_0}{R_0}\kappa v_{1\delta}^{(2)} = 0, \\
&\frac{\rho_0}{\gamma_1 R_0}\left(v_{1\tau}^{(1)} - v_{2\xi}^{(1)} + v_1^{(1)}v_{1\xi}^{(1)}\right) + \varphi_1 v_{1\xi}^{(1)} + + \frac{1}{\beta_1}p_{2\xi} = 0, \\
&p_2 + \beta_1 H\varphi_{1\xi} - \beta_1\varphi_1^2 + \beta_1\varphi_2 - \beta_2\varphi_{1\xi} + \beta_3\varphi_{1\xi\xi} - 3\varphi_{1\xi}^2 = 0.
\end{aligned} \tag{33}$$

Eliminating p_2 and the velocities in (33), we have

$$\varphi_{1\tau} + \left(\frac{\gamma_2}{\gamma_1}R_0 - \frac{\gamma_1 R_0}{\rho_0} - 1\right)\varphi_1\varphi_{1\xi} - \left(\frac{\beta_2}{2\beta_1} - \frac{H}{2}\right)\varphi_{1\xi\xi} \tag{34}$$

$$-\frac{6}{2\beta_1}\varphi_{1\xi}\varphi_{1\xi\xi} + \frac{\beta_3}{2\beta_1}\varphi_{1\xi\xi\xi} - \frac{\rho_0}{2\gamma_1 R_0}\kappa = 0, \tag{35}$$

$$-\frac{\rho_0}{\gamma_1 R_0}v_{1\xi}^{(2)} - \kappa\varphi_{1\delta} = 0, \tag{36}$$

which can be written as

$$\left(\varphi_{1\tau}+\left(\frac{\gamma_2}{\gamma_1}R_0-\frac{\gamma_1 R_0}{\rho_0}-1\right)\varphi_1\varphi_{1\xi}-\left(\frac{\beta_2}{2\beta_1}-\frac{H}{2}\right)\varphi_{1\xi\xi} -\frac{6}{2\beta_1}\varphi_{1\xi}\varphi_{1\xi\xi}+\frac{\beta_3}{2\beta_1}\varphi_{1\xi\xi\xi}\right)_\xi+\frac{\kappa^2}{2}\varphi_{1\delta\delta}=0. \quad (37)$$

Equivalently, using (32), equation (37) can be written in terms of pressure as

$$\left(p_{1\tau}+Ap_1p_{1\xi}-Bp_{1\xi\xi}+Cp_{1\xi\xi\xi}\right)_\xi+Dp_{1\delta\delta}=0 \quad (38)$$

$$A=\frac{1}{\beta_1}\left(\frac{\gamma_1R_0}{\rho_0}-\frac{\gamma_2R_0}{\gamma_1}\right),\quad B=\frac{\beta_2}{2\beta_1},\quad C=\frac{\beta_3}{2\beta_1},\quad D=\frac{\kappa^2}{2} \quad (39)$$

Equation (38) is a two-dimensional Korteweg–de-Vries Burgers for the description of non-linear waves motion in a viscoelastic liquid with gas bubbles incorporating the effect of heat transfer. If $B=0$, we have Kadomtsev-Petviashvili equation(KP) which is two-dimensional case of Korteweg–de-Vries equation

$$\left(p_{1\tau}+Ap_1p_{1\xi}+Cp_{1\xi\xi\xi}\right)_\xi+Dp_{1\delta\delta}=0. \quad (40)$$

In the case of dissipation, the main influence of non-linear waves is governed by the perturbation of the Burgers equation, while the Korteweg–de Vries equation corresponds to the influence of dispersion of the waves. The dispersion and dissipation relation of the system shall be analysed. Equations (38) and (40) admit analytical solitary wave solutions.

5. Result and discussion

Solutions to (2+1)-KdVB equation (38) are derived in [28] using Adomian decomposition method [29]. The analysis on the nature of shock wave is done using the initial condition [30]

$$p_0(\xi,0,\tau)=0.5\left(1-\tanh\frac{|\xi|-25-\tau}{5}\right) \quad (41)$$

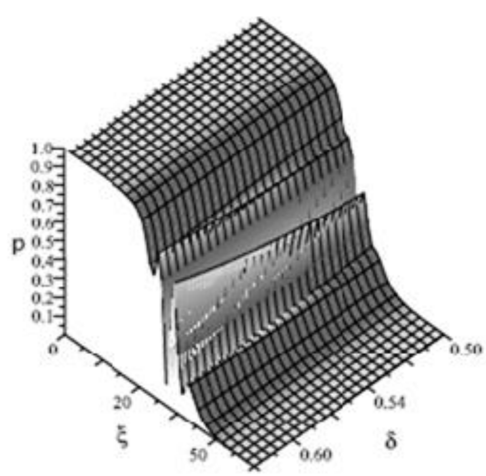

Figure 1: Two dimensional pressure wave propagation in second grade liquid with heat transfer at $\tau=3$ sec.

The pressure wave profile of the shock wave propagation in a pseudo compressible second grade viscoelastic liquid with out bubble-bubble interaction will be discussed here using equation (38). Fig 1 shows the two dimensional shock wave propagation of the bubbly-liquid in ξ and δ directions at $\tau = 3$ sec. Fig 2 shows the two dimensional shock wave propagation in ξ over the range of a specified τ which shows the behaviour of shock wave over time. The figure described wave propagation from $\tau = 0 \cdots 40$ sec, where the oscillation is fully developed.

The dynamics of shock waves propagation in mono-dispersed, pseudo compressible bubbly second grade liquid flow are investigated using the following parameters [25]: $p_0 = 10{,}287.12\text{Pa}$, $\rho_l = 1000\text{kg/m}^3$, $\rho_g = 0.01\text{kg/m}^3$, $\lambda_1 = 1.179\text{MPa}^2$, $\lambda_2 = -6.55$, $R_0 = 1\text{mm}$, $\sigma = 0.0535\text{N/m}$, $\mu = 164\text{mPa.s}$, $c_l = 1452.220795\text{ms}^{-1}$ and $N = 200$, Nu=10, and $\gamma = 1.4$.

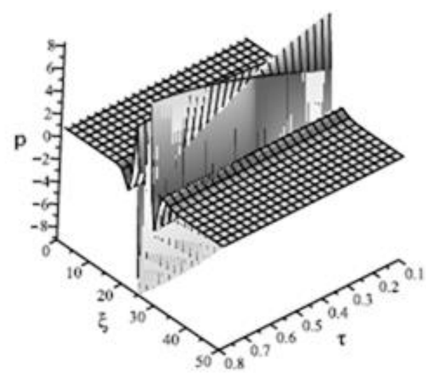

Figure 2: Two dimensional pressure wave propagation in second grade liquid with heat transfer

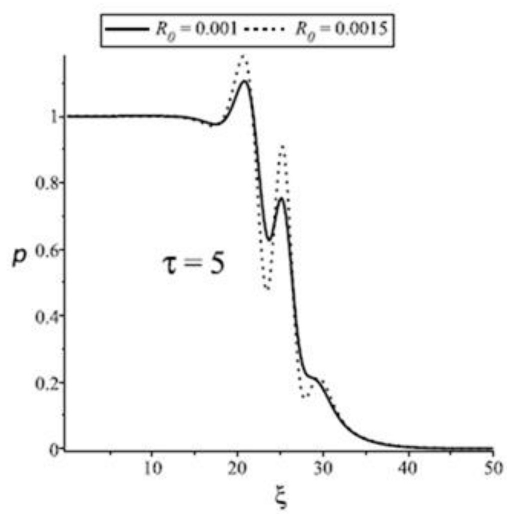

Figure 3: Effect of bubble radious variation on the pressure wave propagation with heat transfer at $\tau = 5$ sec.

Fig. 3 shows the influence of the bubble radius to pressure profile of shock wave propagation in bubbly compressible second grade liquid. The figure indicates that the size of the bubble has no influence on the amplitude of the shock wave as at $\tau = 5$ and $\xi = 0..15$, then from $\xi = 15$ to $\xi = 30$, amplitude of the shock wave of bubbe with higher radius is much more higher that the bubble with smaller radius.

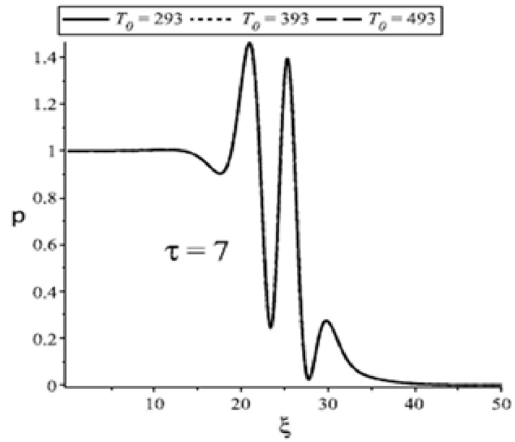

Figure 4: Temperature variation on the wave propagation in second grade liquid $\tau = 7$ sec.

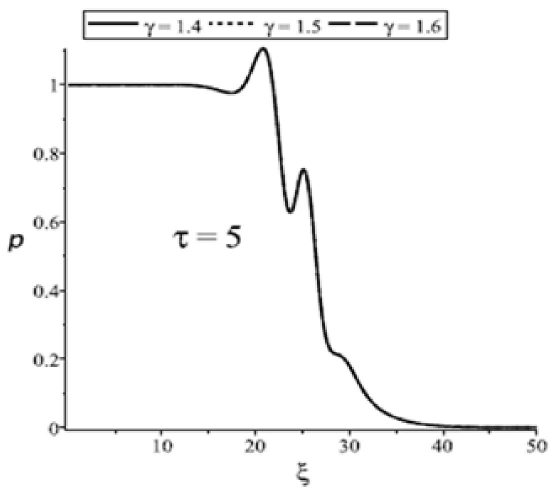

Figure 5: Tolytropic index variation on the wave propagation in second grade liquid $\tau = 5$ sec.

Shock wave as at τ = 5 and ξ = 0..15, then from ξ = 15 to ξ = 30, amplitude of the shock wave of bubbe with higher radius is much higher that the bubble with smaller radius. Fig. 4 and Fig. 5 analysis the effect of the mixture temperature and the polytropic index, which describes the thermal behavior of gas inside the bubbles. The effect of the thermal variation of the polytropic index has less influence on the wave propagation of the pressure profile, at the longer time the oscillation of the wave propagation increases due the increase in the mixture temperature and gas polytropic index, it is observed that the effect of viscosity to pressure profile of shock wave propagation dominate the effect of thermal conductivity. The more viscous is the liquid, the lower the amplitude wave. A highly viscous second grade liquid dissipate faster and vice versa.

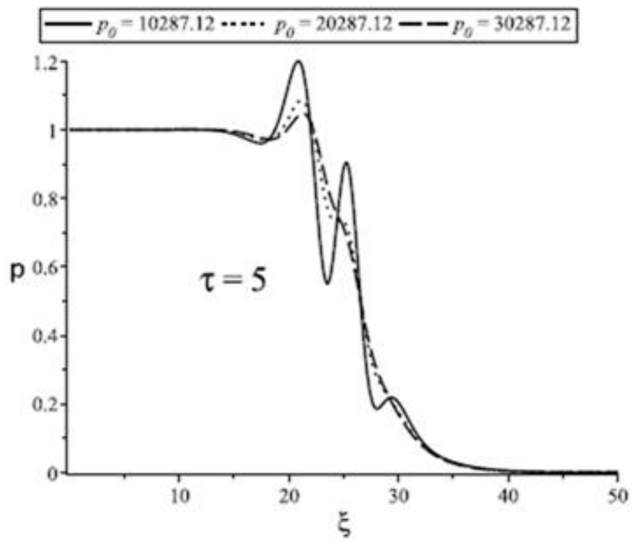

Figure 6: Presure variation on shock wave propagation in second grade liquid $\tau = 5$ sec.

Fig 6 shows the effect of pressure variation to pressure profile of the shock wave propagation. It is observed that change in pressure has a significant effect on the shock wave amplitude. The effect is just the higher the pressure, the higher the shock wave amplitude, and the steeper the wave. These results are in agree with results and analysis in [17].

6. Conclusion

In this paper, the nonlinear wave propagation in bubbly second-grade liquid, with modified Rayleigh-Plesset bubble equation are investigated. The mixture equations are combined with of state for gas equation to study the wave propagation in the bubbly liquid flow. Perturbation method is used in simplifying state variables. A (2+1) KdV-Burgers equation is obtained by considering the pressure wave profile. Adomian decomposition method of solution is adopted to graphically simulate the results. Analysing our simulation with the assumptions that the interaction between bubble-bubble, cluster radius, and number of bubble in a cluster have no effect on the nonlinear wave propagation in compressible second grade liquid flow. Taking into account the heat transfer between the bubble and the liquid, the pressure, initial bubble radius and thermal properties of the gas affect the shock wave propagation in bubble second-grade liquid flow.

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