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Downhill Sombor and Modified Downhill Sombor Indices of Graphs

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Abstract. In this study, we introduce the downhill Sombor and modified downhill Sombor indices and their corresponding exponentials of a graph. Furthermore, we compute these indices for some standard graphs, wheel graphs and honeycomb networks.

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1. Introduction

In this paper, G denotes a finite, simple, connected graph, $V(G)$ and $E(G)$ denote the vertex set and edge set of G. The degree $d_G(u)$ of a vertex u is the number of vertices adjacent to u. A u-v path P in G is a sequence of vertices in G, starting with u and ending at V, such that consecutive vertices in P are adjacent, and no vertex is repeated. A path $\pi = v_1, v_2, ... v_{k+1}$ in G is a downhill path if for every i, $1 \le i \le k$, $d_G(v_i) \ge d_G(v_{i+1})$.

A vertex v is downhill dominated by a vertex u if there exists a downhill path originating from u to v. The downhill neighbourhood of a vertex v is denoted by $N_{dn}(v)$ and defined as: $N_{dn}(v) = \{u: v \text{ downhill dominates } u\}$. The downhill degree $d_{dn}(v)$ of a vertex v is the number of downhill neighbors of v [1].

Recently, some downhill indices were studied in [2, 3, 4].

The Sombor index was introduced in [5] and it is defined as

$$SO(G) = \sum_{uv \in E(G)} \sqrt{d_G(u)^2 + d_G(v)^2}.$$

Motivated by the definition of Sombor index, we introduce the downhill Sombor index of a graph and it is defined as

$$DWSO(G) = \sum_{uv \in E(G)} \sqrt{d_{dn}(u)^2 + d_{dn}(v)^2}.$$

Considering the downhill Sombor index, we introduce the downhill Sombor exponential of a graph G and defined it as

$$DWSO(G,x)=\sum_{uv\in E(G)} x^{\sqrt{d_{dn}(u)^2+d_{dn}(v)^2}}.$$

We define the modified downhill Sombor index of a graph G as

$$^{m}DWSO(G)=\sum_{uv\in E(G)} \frac{1}{\sqrt{d_{dn}(u)^2+d_{dn}(v)^2}}.$$

Considering the modified downhill Sombor index, we introduce the modified downhill Sombor exponential of a graph G and defined it as

$$^{m}DWSO(G,x)=\sum_{uv\in E(G)} x^{\frac{1}{\sqrt{d_{dn}(u)^2+d_{dn}(v)^2}}}.$$

Recently, some Sombor indices were studied in [6, 7, 8].

In this paper, the downhill Sombor index, modified downhill Sombor index and their corresponding exponentials of certain graphs, honeycomb networks, are computed.

2. Results for some standard graphs

Proposition 1. Let G be r-regular with n vertices and $r\geq 2$. Then

$$DWSO(G)=\frac{nr(n-1)}{\sqrt{2}}.$$

Proof: Let G be an r-regular graph with n vertices and $r \geq 2$ and $\frac{nr}{2}$ edges. Then $d_{dn}(v)=n-1$ for every v in G. We obtain

$$DWSO(G)=\sum_{uv\in E(G)} \sqrt{d_{dn}(u)^2+d_{dn}(v)^2}=\frac{nr}{2}\sqrt{(n-1)^2+(n-1)^2}=\frac{nr(n-1)}{\sqrt{2}}.$$

Corollary 1.1. Let C_n be a cycle with $n\geq 3$ vertices. Then

$$DWSO(C_n)=\sqrt{2}n(n-1).$$

Corollary 1.2. Let K_n be a complete graph with $n\geq 3$ vertices. Then

$$DWSO(K_n)=\sqrt{2}n(n-1)^3.$$

Proposition 2. Let P_n be a path with $n\geq 3$ vertices. Then

$$DWSO(G)=\left(\sqrt{2}n-3\sqrt{2}+2\right)(n-1).$$

Proof: Let P_n be a path with $n\geq 3$ vertices. Clearly, P_n has two types of edges based on the downhill degree of end vertices of each edge as follows:

$E_1 = \{uv \in E(G)|\ d_{dn}(u)=0,\ d_{dn}(v) = n-1\}$, $|E_1| = 2$.

$E_2 = \{uv \in E(G)|\ d_{dn}(u)= d_{dn}(v) = n-1\}$, $|E_2| = n-3$.

Then $$DWSO(G)=\sum_{uv\in E(G)} \sqrt{d_{dn}(u)^2+d_{dn}(v)^2}$$

$$=2\sqrt{0^2+(n-1)^2}+(n-3)\sqrt{(n-1)^2+(n-1)^2}=\left(\sqrt{2}n-3\sqrt{2}+2\right)(n-1).$$

Proposition 3. Let $K_{m,n}$ be a complete bipartite graph with $m < n$. Then

$$DWSO(K_{m,n}) = mn^2.$$

Proof. Let $K_{m,n}$ be a complete bipartite graph with $m < n$. There are $m+n$ vertices and mn edges. Clearly, $K_{m,n}$ has one type of edges based on the downhill degree of end vertices of each edge as follows:

$$E_1 = \{uv \in E(K_{m,n}) \mid d_{dn}(u)=0,\ d_{dn}(v) = n\}, \qquad |E_1| = mn.$$

Then $DWSO(K_{m,n}) = \sum_{uv\in E(K_{m,n})} \sqrt{d_{dn}(u)^2 + d_{dn}(v)^2} = mn\sqrt{0^2+n^2} = mn^2.$

3. Results for wheel graphs

Let W_n be a wheel with n+1vertices and $2n$ edges, $n\geq 4$. Then there are two types of edges based on the downhill degree of end vertices of each edge as follows:

$$E_1 = \{uv \in E(W_n) \mid d_{dn}(u) = n,\ d_{dn}(v) = n-1\}, \qquad |E_1| = n.$$
$$E_2 = \{uv \in E(W_n) \mid d_{dn}(u) = d_{dn}(v) = n-1\}, \qquad |E_2| = n.$$

Theorem 1. Let W_n be a wheel with n+1vertices and $2n$ edges, $n\geq 4$. Then

$$DWSO(W_n) = n\sqrt{2n^2-2n+1} + \sqrt{2}n(n-1).$$

Proof. We deduce

$$\begin{aligned} DWSO(W_n) &= \sum_{uv\in E(W_n)} \sqrt{d_{dn}(u)^2 + d_{dn}(v)^2} \\ &= n\sqrt{n^2+(n-1)^2} + n\sqrt{(n-1)^2+(n-1)^2} \\ &= n\sqrt{2n^2-2n+1} + \sqrt{2}n(n-1). \end{aligned}$$

Theorem 2. Let W_n be a wheel with n+1vertices and $2n$ edges, $n\geq 4$. Then

$$DWSO(W_n, x) = nx^{\sqrt{2n^2-2n+1}} + nx^{\sqrt{2}(n-1)}.$$

Proof. We obtain

$$\begin{aligned} DWSO(W_n, x) &= \sum_{uv\in E(G)} x^{\sqrt{d_{dn}(u)^2+d_{dn}(v)^2}} = nx^{\sqrt{n^2+(n-1)^2}} + nx^{\sqrt{(n-1)^2+(n-1)^2}} \\ &= nx^{\sqrt{2n^2-2n+1}} + nx^{\sqrt{2}(n-1)}. \end{aligned}$$

Theorem 3. Let W_n be a wheel with n+1vertices and $2n$ edges , $n\geq 4$. Then

$${}^{m}DWSO(W_n) = \frac{n}{\sqrt{2n^2-2n+1}} + \frac{n}{\sqrt{2}(n-1)}.$$

Proof. We deduce $\quad {}^{m}DWSO(G) = \sum_{uv\in E(G)} \frac{1}{\sqrt{d_{dn}(u)^2 + d_{dn}(v)^2}}$

$$= \frac{n}{\sqrt{n^2+(n-1)^2}} + \frac{n}{\sqrt{(n-1)^2+(n-1)^2}} = \frac{n}{\sqrt{2n^2-2n+1}} + \frac{n}{\sqrt{2}(n-1)}.$$

Theorem 4. Let W_n be a wheel with n+1vertices and 2n edges, n≥4. Then

$$^{m}DWSO(W_n,x)=nx^{\frac{1}{\sqrt{2n^2-2n+1}}}+nx^{\frac{1}{\sqrt{2}(n-1)}}.$$

Proof. We obtain

$$^{m}DWSO(W_n,x)=\sum_{uv\in E(W_n)}x^{\frac{1}{\sqrt{d_{dn}(u)^2+d_{dn}(v)^2}}}=nx^{\frac{1}{\sqrt{n^2+(n-1)^2}}}+nx^{\frac{1}{\sqrt{(n-1)^2+(n-1)^2}}}$$

$$=nx^{\frac{1}{\sqrt{2n^2-2n+1}}}+nx^{\frac{1}{\sqrt{2}(n-1)}}.$$

4. Results for honeycomb networks

If we recursively use hexagonal tiling in a particular pattern, honeycomb networks are formed. These networks are very useful in computer graphics and also in chemistry. A honeycomb network of dimension n is denoted by HC_n where n is the number of hexagons between central and boundary hexagon.

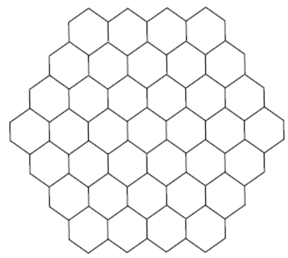

Figure 1: A 4-dimensional honeycomb network

Let H be the graph of honeycomb network $HC_{n,}$ .where $n \geq 3$. By calculation, we obtain that H has $6n^2$ vertices and $9n^2 - 3n$ edges. Then there are four types of edges based on the downhill degree of end vertices of each edge as follows:

$E_1 = \{uv \in E(H) \mid d_{dn}(u) = 1, d_{dn}(v) = 1\}$, $|E_1| = 6$.
$E_2 = \{uv \in E(H) \mid d_{dn}(u) = 1, d_{dn}(v) = 6n^2 - 1\}$, $|E_2| = 12$.
$E_3 = \{uv \in E(H) \mid d_{dn}(u) = 0, d_{dn}(v) = 6n^2 - 1\}$, $|E_2| = 12(n-2)$.

$E_4 = \{uv \in E(H) \mid d_{dn}(u)= d_{dn}(v) = 6n^2 - 1\}$, $|E_4| = 9n^2 - 15n+6$.

Theorem 5. Let H be a honeycomb network with $6n^2$ vertices, n≥4. Then

$$DWSO(H)=6\sqrt{2}+12\sqrt{36n^4-12n^2+2}+12(n-2)(6n^2-1)+\sqrt{2}(9n^2-15n+6)(6n^2-1).$$

Proof. We deduce

$$DWSO(H)=\sum_{uv\in E(H)}\sqrt{d_{dn}(u)^2+d_{dn}(v)^2}$$

$$=6\sqrt{1^2+1^2}+12\sqrt{1^2+(6n^2-1)^2}+12(n-2)\sqrt{0^2+(6n^2-1)^2}$$

$$+\left(9n^2-15n+6\right)\sqrt{\left(6n^2-1\right)^2+\left(6n^2-1\right)^2}$$
$$=6\sqrt{2}+12\sqrt{36n^4-12n^2+2}+12(n-2)\left(6n^2-1\right)+\sqrt{2}\left(9n^2-15n+6\right)\left(6n^2-1\right).$$

Theorem 6. Let H be a honeycomb network with 6 n^2vertices, $n\geq4$. Then
$$DWSO(H,x)=6x^{\sqrt{2}}+12x^{\sqrt{1+\left(6n^2-1\right)^2}}+12(n-2)x^{\left(6n^2-1\right)}+\left(9n^2-15n+6\right)x^{\sqrt{2}\left(6n^2-1\right)}.$$
Proof. We obtain
$$DWSO(H,x)=\sum_{uv\in E(H)}x^{\sqrt{d_{dn}(u)^2+d_{dn}(v)^2}}$$
$$=6x^{\sqrt{1^2+1^2}}+12x^{\sqrt{1^2+\left(6n^2-1\right)^2}}+12(n-2)x^{\sqrt{0^2+\left(6n^2-1\right)^2}}+\left(9n^2-15n+6\right)x^{\sqrt{\left(6n^2-1\right)^2+\left(6n^2-1\right)^2}}$$
$$=6x^{\sqrt{2}}+12x^{\sqrt{1+\left(6n^2-1\right)^2}}+12(n-2)x^{\left(6n^2-1\right)}+\left(9n^2-15n+6\right)x^{\sqrt{2}\left(6n^2-1\right)}.$$

Theorem 7. Let H be a honeycomb network with 6 n^2vertices, $n\geq4$. Then
$$^{m}DWSO(H)=\frac{6}{\sqrt{2}}+\frac{12}{\sqrt{36n^4-12n^2+2}}+\frac{12(n-2)}{6n^2-1}+\frac{9n^2-15n+6}{\sqrt{2}\left(6n^2-1\right)}.$$
Proof. We deduce
$$^{m}DWSO(H)=\sum_{uv\in E(H)}\frac{1}{\sqrt{d_{dn}(u)^2+d_{dn}(v)^2}}$$
$$=\frac{6}{\sqrt{1^2+1^2}}+\frac{12}{\sqrt{1^2+\left(6n^2-1\right)^2}}+\frac{12(n-2)}{\sqrt{0^2+\left(6n^2-1\right)^2}}+\frac{9n^2-15n+6}{\sqrt{\left(6n^2-1\right)^2+\left(6n^2-1\right)^2}}$$
$$=\frac{6}{\sqrt{2}}+\frac{12}{\sqrt{36n^4-12n^2+2}}+\frac{12(n-2)}{6n^2-1}+\frac{9n^2-15n+6}{\sqrt{2}\left(6n^2-1\right)}.$$

Theorem 8. Let H be a honeycomb network with 6 n^2vertices, $n\geq4$. Then
$$^{m}DWSO(H,x)=6x^{\frac{1}{\sqrt{2}}}+12x^{\frac{1}{\sqrt{36n^4-12n^2+2}}}+12(n-2)x^{\frac{1}{\left(6n^2-1\right)}}+\left(9n^2-15n+6\right)x^{\frac{1}{\sqrt{2}\left(6n^2-1\right)}}.$$
Proof. We obtain
$$^{m}DWSO(H,x)=\sum_{uv\in E(H)}x^{\frac{1}{\sqrt{d_{dn}(u)^2+d_{dn}(v)^2}}}$$
$$=6x^{\frac{1}{\sqrt{1^2+1^2}}}+12x^{\frac{1}{\sqrt{1^2+\left(6n^2-1\right)^2}}}+12(n-2)x^{\frac{1}{\sqrt{0^2+\left(6n^2-1\right)^2}}}+\left(9n^2-15n+6\right)x^{\frac{1}{\sqrt{\left(6n^2-1\right)^2+\left(6n^2-1\right)^2}}}$$
$$=6x^{\frac{1}{\sqrt{2}}}+12x^{\frac{1}{\sqrt{36n^4-12n^2+2}}}+12(n-2)x^{\frac{1}{\left(6n^2-1\right)}}+\left(9n^2-15n+6\right)x^{\frac{1}{\sqrt{2}\left(6n^2-1\right)}}.$$

5. Conclusion

In this paper, the downhill Sombor index, modified downhill Sombor index and their corresponding exponentials of certain graphs, honeycomb networks are determined.

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