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# Forgotten Index of *k*-Splitting Graphs

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***Abstract.*** The objective of this paper is to derive explicit formulas for the forgotten index of $k$-splitting of generalized transformation graphs denoted as $spl_k(G^{ab})$. Subsequently, we also derived analogous expressions for the complements of $spl_k(G^{ab})$.

***Keywords:*** Forgotten index, Generalized transformation graphs, k-Splitting of graphs

***AMS Mathematics Subject Classification (2010):*** 05C05, 05C07, 05C35

## 1. Introduction

Throughout this paper, we examine undirected, simple, and finite graphs. Let $G$ be $(n, m)$ −graph defined as a graph characterized by the vertex set $V(G)$ and the edge set $E(G)$, where $|V(G)| = n$ and $|E(G)| = m$, with the elements of $V(G)$ referred to as vertices and the elements of $E(G)$ designated as edges of the graph $G$. Two vertices are considered adjacent in $G$ if they are connected by a common edge, which is described as being incident to those two vertices. The degree of a vertex $u$ in $V(G)$, represented by $d_G(u)$, quantifies the number of edges that are incident to $u$. The complement $\overline{G}$ of graph $G$ constitutes a simple graph that retains the same vertex set as $G$, whereby two vertices $u$ and $v$ are adjacent in $\overline{G}$ if and only if they are non-adjacent in $G$. For additional graph-theoretic terminologies and definitions, we direct the reader to the reference [13].

The vertices within the molecular graph are representative of the atoms constituting the molecule, while the edges signify the covalent bonds that interconnect these atoms. A topological index is defined as a graph invariant that assigns a unique real number to each molecular graph. A plethora of such descriptors has been examined in the realm of theoretical chemistry and has demonstrated practical applications, particularly within the contexts of Quantitative Structure-Activity Relationship (QSAR) and Quantitative Structure-Property Relationship (QSPR)[14, 20]. Topological indices are classified into two principal categories, specifically degree-based indices and distance-based indices. Noteworthy examples of degree-based indices include the first Zagreb index, the second Zagreb index, the forgotten index, the hyper Zagreb index, the Randic index, the harmonic index, the geometric-arithmetic index, and the redefined third Zagreb index, among others. For a comprehensive exploration of degree-based topological indices, we direct the reader to reference[6, 7, 8, 9, 10]. Among the diverse array of degree-based indices, the first and second Zagreb indices have garnered substantial scholarly attention. In the year 1972, Gutman et al. [7] formally introduced the first and second Zagreb indices pertaining to a graph $G$. To date, numerous researchers worldwide are delving into these indices, pushing the boundaries of knowledge to advanced methodologies. Subsequently, in 2008, Došlić [5] articulated the definitions of the first and second Zagreb coindices, which pertain to all non-adjacent pairs of vertices. For further insights regarding the Zagreb indices and coindices, as well as their various applications, refer to[2, 10, 15, 16, 17, 19, 23]. In the year 2015, Basavanagoud et al. [3] introduced novel graph operations referred to as generalized transformations denoted as $G^{ab}$, and derived the formulations for both the first and second Zagreb indices as well as the coindices pertaining to these graphs and their corresponding complements. In 2017, Vaidya et al. [22] proposed a new graph operation termed $k$-splitting of a graph $G$ and conducted an analysis on the energy associated with this operation.

In the current study, we concentrate on deriving explicit formulations for the forgotten index pertaining to the $k$-splitting of generalized transformation graphs $spl_k(G^{ab})$. Subsequently, we also derive comparable expressions for the complements of $spl_k(G^{ab})$.

## 2. Preliminaries

The first Zagreb index $M_1(G)$ [7] and second Zagreb index $M_2(G)$ [7] are defined as

$$M_1(G) = \sum_{u \in V(G)} d_G(u)^2 \quad and \quad M_2(G) = \sum_{uv \in E(G)} d_G(u)\ d_G(v)$$

respectively.

The first Zagreb can also be expressed as [4]

$$M_1(G) = \sum_{uv \in E(G)} [d_G(u) + d_G(v)].$$

Boris Furtula and Ivan Gutman [6] have put forward a degree based topological indices viz., a forgotten topological index which is defined as

$$F(G) = \sum_{u \in V(G)} d_G(u)^3 = \sum_{uv \in E(G)} [d_G(u)^2 + d_G(v)^2].$$

## 3. Generalized transformation graph $G^{ab}$ and the $k$-splitting graph

Let $G$ denote a graph characterized by the vertex set $V(G)$ and the edge set $E(G)$, and let $\alpha$ and $\beta$ represent two elements belonging to the union $V(G) \cup E(G)$. The associativity of the elements $\alpha$ and $\beta$ is defined as $+$ if they exhibit adjacency or incidence within the graph $G$; conversely, it is denoted as $-$ in the absence of such a relationship. Let $ab$ signify a 2-permutation of the set $\{+,-\}$. The elements $\alpha$ and $\beta$ are said to correspond to the initial term $a$ of $ab$ if both entities reside within $V(G)$ or $E(G)$, whereas they correspond to the subsequent term $b$ of $ab$ if one of the entities is contained in $V(G)$ and the other is situated in $E(G)$. The generalized transformation graph $G^{ab}$ is formulated on the vertex set $V(G) \cup E(G)$. A pair of vertices $\alpha$ and $\beta$ in $G^{ab}$ is connected by an edge if and only if their associativity within $G$ aligns with the corresponding term of $ab$. In this context, there exist four distinct graphical transformations of graphs, specifically $G^{++}$, $G^{+-}$, $G^{-+}$, and $G^{--}$, corresponding to the four unique 2-permutations of the set $\{+,-\}$.

In an alternative formulation, the generalized transformation graph $G^{ab}$ constitutes a graph characterized by the vertex set $V(G) \cup E(G)$, where $\alpha,\beta \in V(G^{ab})$ are considered to be adjacent within $G^{ab}$ if and only if the conditions $(i)$ and $(ii)$ are satisfied:

$(i)$ $\alpha,\beta \in V(G^{ab})$, with $\alpha$ and $\beta$ being adjacent in $G$ under the condition that $a=+$, whereas $\alpha$ and $\beta$ are non-adjacent in $G$ when $a=-$.

$(ii)$ If $\alpha \in V(G)$ and $\beta \in E(G)$, then $\alpha$ and $\beta$ are incident in $G$ when $b=+$, while $\alpha$ and $\beta$ are not incident in $G$ if $b=-$.

The vertex $u$ of $G^{ab}$ that corresponds to a vertex $u$ of $G$ is designated as a point vertex. Conversely, the vertex $e$ of $G^{ab}$ that corresponds to an edge $e$ of $G$ is termed a line vertex.

The $k$-splitting of a graph $G$, denoted as $spl_k(G)$, is derived by augmenting each vertex $u$ of $G$ with $k$ additional vertices, labeled as $u'$, $u''$, ..., $u^{(k)}$, in such a manner that $u^{(i)}$, where $1 \le i \le k$, is adjacent to every vertex that shares adjacency with $u$ in the original graph $G$.

Now we present the main results of our work through following sections.

## 4. Results

### 4.1. Forgotten Index of $spl_k(G^{++})$

Let $G$ be a $(n,m)$-graph and $spl_k(G^{++})$ represents $k$-splitting of $G^{++}$.

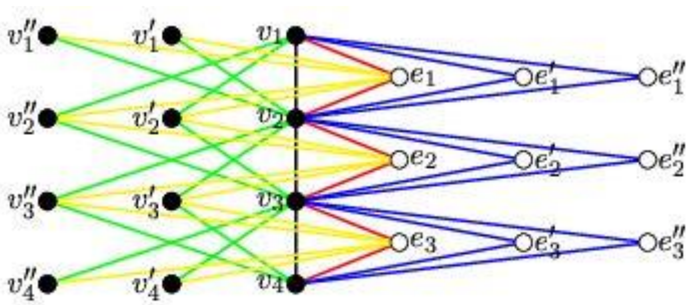


**Figure 1:** $spl_k(P_4^{++})$: $2-splitting\ of\ P_4^{++}$

**Proposition 1.** *Let $G$ be a $(n,m)$-graph. Then*

$(i) d_{spl_k(G^{++})}(v) = 2(k+1)d_G(v)$ *where* $v \in V(G)$

$(ii) d_{spl_k(G^{++})}(e) = 2(k+1)$ *where* $e \in E(G)$

$(iii) d_{spl_k(G^{++})}(v') = 2d_G(v)$ *where* $v' \in G^{++}$ *due to vertex* $v$ *in* $G$

$(iv) d_{spl_k(G^{++})}(e') = 2$ where $e'$ is vertex in $k-$splitting of $G^{++}$ due to edge $e$ in $G$.

**Proposition 2.** *Let $G$ be a $(n,m)$-graph. Then order and size of $spl_k(G^{++})$ are* $(n+m)(k+1)$ and $3m(2k+1)$.

**Theorem 1.** *Let $G$ be a $(n,m)$-graph. Then*

$$F\big(spl_k(G^{++})\big) = 8(k+1)^2F(G) + [4+4k^2(k+1)+4k(k+1)^2+4k]F(G) +8m(k+1)^2+8mk+8mk(k+1)$$

**Proof:** Partitioning edge set of $spl_k(G^{++})$ as follows:

$$E(spl_k(G^{++})) = E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5$$

where,

$E_1 = \{uv:\ uv \in E(G)\}$

$E_2 = \{ue: vertex\ u\ in\ G\ is\ incident\ to\ edge\ e\ in\ G\}$

$E_3 = \{uv': vertex\ u\ in\ G\ is\ adjacent\ to\ vertex\ v' \in G^{++} due\ to\ vertex\ v\ inG\}$

$E_4 = \{ue': vertex\ u\ in\ G\ is\ incident\ to\ vertex\ e' \in G^{++} due\ to\ edge\ e\ in\ G\}$

$E_5 = \{u'e: vertex\ u' \in G^{++} due\ to\ vertex\ u\ in\ G\ incident\ to\ edge\ e\ in\ G\}$

Clearly $|E_1| = m, |E_2| = 2m, |E_3| = 2mk, |E_4| = 2mk$ and $|E_5| = 2mk$

Consider,

$$F\big(spl_k(G^{++})\big) = \sum_{uv\in\left(E(spl_k(G^{++}))\right)} \left[d_{spl_k(G^{++})}(u)^2 + d_{spl_k(G^{++})}(v)^2\right]$$

$$= \sum_{uv\in E_1} \left[d_{spl_k(G^{++})}(u)^2 + d_{spl_k(G^{++})}(v)^2\right]$$

$$+ \sum_{ue\in E_2} \left[d_{spl_k(G^{++})}(u)^2 + d_{spl_k(G^{++})}(e)^2\right]$$

$$+ \sum_{uv'\in E_3} \left[d_{spl_k(G^{++})}(u)^2 + d_{spl_k(G^{++})}(v')^2\right]$$

$$+ \sum_{ue'\in E_4} \left[d_{spl_k(G^{++})}(u)^2 + d_{spl_k(G^{++})}(e')^2\right]$$

$$+ \sum_{u'e\in E_5} \left[d_{spl_k(G^{++})}(u')^2 + d_{spl_k(G^{++})}(e)^2\right]$$

$$F\big(spl_k(G^{++})\big) = \sum_{uv\in E_1} [4(k+1)^2 d_G(u)^2 + 4(k+1)^2 d_G(v)^2]$$

$$+ \sum_{uv\in E_2} [4(k+1)^2 d_G(u)^2 + 4(k+1)^2]$$

$$+\sum_{uv\in E_3}[4(k+1)^2 d_G(u)^2+4d_G(v)^2]+\sum_{uv\in E_4}[4(k+1)^2 d_G(u)^2+4]$$

$$+\sum_{ue\in E_5}[4d_G(u)^2+4(k+1)^2]$$

$$=4(k+1)^2\sum_{uv\in E_1}[d_G(u)^2+d_G(v)^2]\ +4(k+1)^2\sum_{uv\in E_2}d_G(u)^2$$

$$+4(k+1)^2\sum_{uv\in E_2}1+\sum_{uv\in E_3}[4(k^2+2k+1)d_G(u)^2+4d_G(v)^2]$$

$$+4(k+1)^2\sum_{uv\in E_4}d_G(u)^2\ +4\sum_{uv\in E_4}1+4\sum_{uv\in E_5}d_G(u)^2+4(k+1)^2\sum_{uv\in E_5}1$$

$$=4(k+1)^2\sum_{uv\in E_(G)}[d_G(u)^2+d_G(v)^2]+4(k+1)^2\sum_{u\in V(G)}d_G(u)^2\cdot d_G(u)$$

$$+4(k+1)^2\sum_{uv\in E_2}1+4\sum_{uv\in E(G)}[d_G(u)^2+d_G(v)^2]$$

$$+4k(k+2)(k)\sum_{u\in V(G)}d_G(u)^2\cdot d_G(u)+4k(k+1)^2\sum_{u\in V(G)}d_G(u)^2\cdot d_G(u)$$

$$+4\sum_{uv\in E_4}1+4k\sum_{u\in V(G)}d_G(u)^2\cdot d_G(u)+4(k+1)^2\sum_{uv\in E_5}1$$

$$=4(k+1)^2F(G)+4(k+1)^2F(G)+8m(k+1)^2+4F(G)$$
$$+4k^2(k+1)F(G)+4k(k+1)^2F(G)+8mk+4kF(G)+8mk(k+1)$$
$$=8(k+1)^2F(G)+[4+4k^2(k+1)+4k(k+1)^2+4k]F(G)$$
$$+8m(k+1)^2+8mk+8mk(k+1)$$

### 4.2. Forgotten Index of $spl_k(G^{+-})$

Let $G$ be a $(n,m)$-graph and $spl_k(G^{+-})$ represent $k$-splitting of $G^{+-}$.

**Proposition 3.** *Let $G$ be a $(n,m)$-graph. Then*

$(i)d_{spl_k(G^{+-})}(v)=m(k+1)$ where $v\in V(G)$

$(ii)d_{spl_k(G^{+-})}(e)=(n-2)(k+1)$ where $e\in E(G$

$(iii)d_{spl_k(G^{+-})}(v')=m$ where $v'\in G^{+-}$ due to vertex v in G

$(iv)d_{spl_k(G^{+-})}(e')=(n-2)$ where $e'\in k-$ splitting of $G^{+-}$ due to edge e in G.

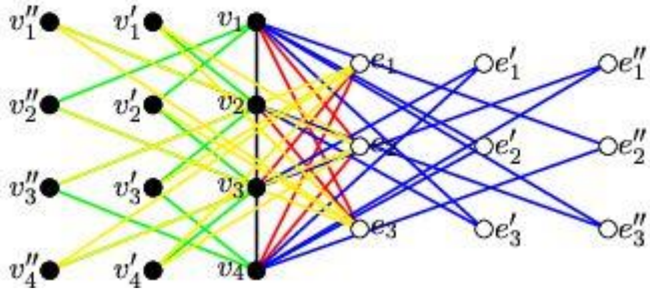


**Figure 2:** $spl_k(P_4^{+-})$:2−$splitting of$ $P_4^{+-}$

**Proposition 4.** *Let* $G$ *be a* $(n,m)$*-graph. Then order and size of* $spl_k(G^{+-})$ *are* $(n+m)(k+1)$ *and* $m(n-1)(2k+1)$.

**Theorem 2.** *Let* $G$ *be a* $(n,m)$*-graph. Then*

$$F\big(spl_k(G^{+-})\big) = 2m^3[(k+1)^2 + k((k+1)^2+1)] + m(n-2)[(m^2+(n-2)^2)(k+1) + k([m^2(k+1)^2+(n-2)^2] + [m^2+(n-2)^2(k+1)^2])]$$

**Proof:** Partitioning edge set of $spl_k(G^{+-})$ as follows:

$$E(spl_k(G^{+-})) = E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5$$

where

$E_1 = \{uv : uv \in E(G)\}$
$E_2 = \{ue$: vertex u in G is not incident to edge e in G$\}$
$E_3 = \{uv'$: vertex u in G is adjacent to vertex v′ ∈ $G^{+-}$due to vertex v in G$\}$
$E_4 = \{ue'$: vertex u in G is incident to vertex e′ ∈ $G^{+-}$due to edge e in G$\}$
$E_5 = \{u'e$: vertexu′ ∈ $G^{+-}$duetovertexu ∈ GnotincidenttoedgeeinG$\}$

Clearly,
$|E_1| = m, |E_2| = m(n-2), |E_3| = 2mk, |E_4| = mk(n-2)$ and $|E_5| = mk(n-2)$.

Consider,

$$F\big(spl_k(G^{+-})\big) = \sum_{uv\in\left(E\left(spl_k(G^{+-})\right)\right)} \left[d_{spl_k(G^{+-})}(u)^2 + d_{spl_k(G^{+-})}(v)^2\right]$$

$$= \sum_{uv\in E_1} \left[d_{spl_k(G^{\pm})}(u)^2 + d_{spl_k(G^{\pm})}(v)^2\right]$$

$$+ \sum_{ue\in E_2} \left[d_{spl_k(G^{+-})}(u)^2 + d_{spl_k(G^{+-})}(e)^2\right]$$

$$+ \sum_{uv'\in E_3} \left[d_{spl_k(G^{\pm})}(u)^2 + d_{spl_k(G^{\pm})}(v')^2\right]$$

$$+\sum_{ue'\in E_4}\left[d_{spl_k(G^{+-})}(u)^2+d_{spl_k(G^{+-})}(e')^2\right]$$
$$+\sum_{u'e\in E_5}\left[d_{spl_k(G^{+-})}(u')^2+d_{spl_k(G^{+-})}(e)^2\right]$$

$$=\sum_{uv\in E_1}[m^2(k+1)^2+m^2(k+1)^2]$$
$$+\sum_{ue\in E_2}[m^2(k+1)^2+(n-2)^2(k+1)^2]$$

$$+\sum_{uv'\in E_3}[m^2(k+1)^2+m^2]+\sum_{ue'\in E_4}[m^2(k+1)^2+(n-2)^2]$$
$$+\sum_{u'e\in E_5}[m^2+(n-2)^2(k+1)^2]$$

$$=2m^2(k+1)^2\sum_{uv\in E_1}1+(m^2+(n-2)^2)(k+1)\sum_{ue\in E_2}1$$
$$+m^2[(k+1)^2+1]\sum_{uv'\in E_3}1+[m^2(k+1)^2+(n-2)^2]\sum_{ue'\in E_4}1$$

$$+[m^2+(n-2)^2(k+1)^2]\sum_{u'e\in E_5}1$$

$$=2m^3(k+1)^2+m(n-2)(m^2+(n-2)^2)(k+1)+2km^3[(k+1)^2+1]$$
$$+mk(n-2)[m^2(k+1)^2+(n-2)^2]\ +mk(n-2)[m^2+(n-2)^2(k+1)^2]$$
$$=2m^3[(k+1)^2+k((k+1)^2+1)]+m(n-2)[(m^2+(n-2)^2)(k+1)$$
$$+k([m^2(k+1)^2+(n-2)^2]+[m^2+(n-2)^2(k+1)^2])]$$

### 4.3. Forgotten Index of $spl_k(G^{-+})$

Let $G$ be a $(n,m)$-graph and $spl_k(G^{-+})$ represents $k$-splitting of $G^{-+}$.

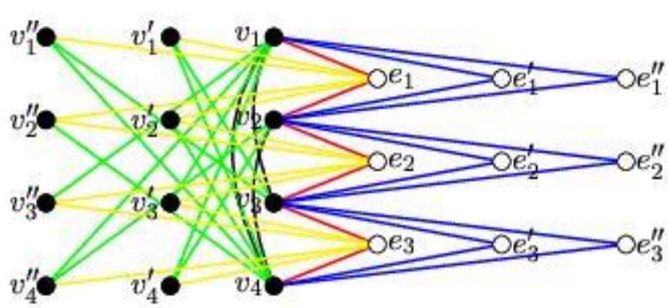


**Figure 3:** $spl_k(P_4^{-+})$:$2-splitting\ of\ P_4^{-+}$

**Proposition 5.** *Let $G$ be a $(n,m)$-graph. Then*

$$(i)d_{spl_k(G^{-+})}(v)=(n-1)(k+1)\quad where\ v\in V(G)$$
$$(ii)d_{spl_k(G^{-+})}(e)=2(k+1)\quad where\ e\in E(G)$$
$$(iii)d_{spl_k(G^{-+})}(v')=(n-1)\quad where\ v'\in G^{-+}\ due\ to\ vertex\ v\ in\ G$$
$$(iv)d_{spl_k(G^{-+})}(e')=2\quad where\ e'\in G^{-+} due\ to\ edge\ e\ in\ G$$

**Proposition 6.** *Let $G$ be a $(n,m)$-graph. Then order and size of $spl_k(G^{-+})$ are* $(n+m)(k+1)$ and $\frac{1}{2}[n(n-1)+2m](2k+1)$.

**Theorem 3.** *Let $G$ be a $(n,m)$-graph. Then*

$$F\big(spl_k(G^{-+})\big)=[2(n-1)^2(k+1)^2+(n-1)^2[(k+1)^2+1]\left(\frac{n(n-1)}{2}-m\right)$$
$$+2m\{(k+1)^2[(n-1)^2+4]+k([(n-1)^2(k+1)^2+4]$$
$$+[(n-1)^2+4(k+1)^2])\}$$

**Proof:** Partitioning edge set of $spl_k(G^{-+})$ as follows:

$$E(spl_k(G^{-+}))=E_1\cup E_2\cup E_3\cup E_4\cup E_5$$

where

$E_1=\{uv: uv\notin E(G)\}$
$E_2=\{ue: vertex\ u\ in\ G\ is\ incident\ to\ edge\ e\ in\ G\}$
$E_3=\{uv': vertex\ u\in G\ is\ not\ adjacent\ to\ vertex\ v'\in G^{-+} due\ to\ vertex\ v\ in\ G\}$
$E_4=\{ue'$: vertex u ∈ G is incident to vertex e′ ∈ $G^{-+}$due to edge e in G}
$E_5=\{u'e$: vertex u′ ∈ $G^{-+}$due to vertex u ∈ G is not incident to edge e in G}

Clearly,

$|E_1|=\frac{n(n-1)}{2}-m, |E_2|=2m, |E_3|=2k\left(\frac{n(n-1)}{2}-m\right),$
$|E_4|=2mk$ and $|E_5|=2mk.$

Consider,

$$F\big(spl_k(G^{-+})\big)=\sum_{uv\in\left(E\left(spl_k(G^{-+})\right)\right)}\left[d_{spl_k(G^{-+})}(u)^2+d_{spl_k(G^{-+})}(v)^2\right]$$
$$=\sum_{uv\in E_1}\left[d_{spl_k(G^{\mp})}(u)^2+d_{spl_k(G^{\mp})}(v)^2\right]$$
$$+\sum_{ue\in E_2}\left[d_{spl_k(G^{-+})}(u)^2+d_{spl_k(G^{-+})}(e)^2\right]$$
$$+\sum_{uv'\in E_3}\left[d_{spl_k(G^{\mp})}(u)^2+d_{spl_k(G^{\mp})}(v')^2\right]$$
$$+\sum_{ue\prime\in E_4}\left[d_{spl_k(G^{-+})}(u)^2+d_{spl_k(G^{-+})}(e')^2\right]$$

$$+\sum_{u'e\in E_5}\left[d_{spl_k(G^{-+})}(u')^2+d_{spl_k(G^{-+})}(e)^2\right]$$

$$F\big(spl_k(G^{-+})\big)=\sum_{uv\in E_1}[(n-1)^2(k+1)^2+(n-1)^2(k+1)^2]$$
$$+\sum_{ue\in E_2}[(n-1)^2(k+1)^2+4(k+1)^2]$$
$$+\sum_{uv'\in E_3}[(n-1)^2(k+1)^2+(n-1)^2]$$
$$+\sum_{ue'\in E_4}[(n-1)^2(k+1)^2+4]+\sum_{u'e\in E_5}[(n-1)^2+4(k+1)^2]$$
$$=2(n-1)^2(k+1)^2\sum_{uv\in E_1}1+(k+1)^2[(n-1)^2+4]\sum_{ue\in E_2}1$$
$$+(n-1)^2[(k+1)^2+1]\sum_{uv'\in E_3}1+[(n-1)^2(k+1)^2+4]\sum_{ue'\in E_4}1$$
$$+[(n-1)^2+4(k+1)^2]\textstyle\sum_{u'e\in E_5}1$$
$$=2(n-1)^2(k+1)^2\left(\frac{n(n-1)}{2}-m\right)+2m(k+1)^2[(n-1)^2+4]$$
$$+(n-1)^2[(k+1)^2+1]\left(\frac{n(n-1)}{2}-m\right)+2mk[(n-1)^2(k+1)^2+4]$$
$$+2mk[(n-1)^2+4(k+1)^2]$$
$$=[2(n-1)^2(k+1)^2+(n-1)^2[(k+1)^2+1]\left(\frac{n(n-1)}{2}-m\right)$$
$$+2m\{(k+1)^2[(n-1)^2+4]+k([(n-1)^2(k+1)^2+4]$$
$$+[(n-1)^2+4(k+1)^2])\}$$

### 4.4. Forgotten Index of $spl_k(G^{--})$

Let $G$ be a $(n,m)$-graph and $spl_k(G^{--})$ represents $k$-splitting of $G^{--}$.

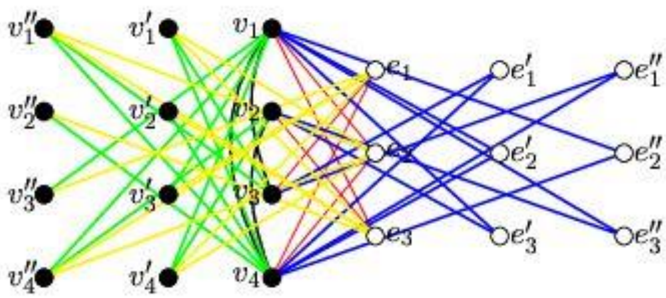


**Figure 4:** $spl_k(P_4^{--})$:2$-splitting\ of\ P_4^{-}$

**Proposition 5.** *Let $G$ be a $(n,m)$-graph. Then*

$(i)d_{spl_k(G^{--})}(v)=\big(n+m-1-2d_G(v)\big)(k+1)$ where v ∈ V(G)

$(ii) d_{spl_k(G^{--})}(e) = (n-2)(k+1)$ where $e \in E(G)$

$(iii) d_{spl_k(G^{--})}(v') = \big(n+m-1-2d_G(v)\big)$ where $v' \in G^{--}$ due to vertex v in G

$(iv) d_{spl_k(G^{--})}(e') = (n-2)$ where $e' \in G^{--}$ due to edge e inG

**Proposition 6.** *Let $G$ be a $(n,m)$-graph. Then order and size of $spl_k(G^{--})$ are*

$$(n+m)(k+1) \text{ and } \frac{1}{2}[n(n-1)+2m(n-3)](2k+1).$$

**Theorem 4.** *Let $G$ be a $(n,m)$-graph. Then*

$$F\big(spl_k(G^{--})\big) = [12(k+1)^2 + 12 - 12(n+m-1) - 8(k+1)^2(n+m-1)$$
$$-4(k+1)^2(n+m-1)M_1(G) + [(n+m-1)^2$$
$$+(n-2)^2(k+1)^2]m(n-2) + [(n+m-1)^2(k+1)^2 + (n-2)^2$$
$$+(n+m-1)^2 + (n-2)^2]mk(n-2)$$

**Proof:** Partitioning edge set of $spl_k(G^{--})$ as follows:

$$E(spl_k(G^{--})) = E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5$$

where

$E_1 = \{uv : uv \notin E(G)\}$

$E_2 = \{ue$: vertex u in G is not incident to edge e in G$\}$

$E_3 = \{uv'$: vertex $u \in G$ is not adjacent to vertex $v' \in G^{--}$ due to vertex v in G$\}$

$E_4 = \{ue'$: vertex $u \in G$ is not incident to vertex $e' \in G^{--}$ due to edge e in G$\}$

$E_5 = \{u'e$: vertexu$' \in G^{--}$duetovertexu $\in$ GisnotincidenttoedgeeinG$\}$

Clearly,

$$|E_1| = \frac{n(n-1)}{2} - m, |E_2| = m(n-2), |E_3| = 2k\left(\frac{n(n-1)}{2} - m\right),$$
$$|E_4| = mk(n-2) \text{ and } |E_5| = mk(n-2).$$

Consider,

$$F\big(spl_k(G^{--})\big) = \sum_{uv \in \left(E(spl_k(G^{--}))\right)} \left[d_{spl_k(G^{--})}(u)^2 + d_{spl_k(G^{--})}(v)^2\right]$$

$$= \sum_{uv \in E_1} \left[d_{spl_k(G^{--})}(u)^2 + d_{spl_k(G^{--})}(v)^2\right]$$

$$+ \sum_{ue \in E_2} \left[d_{spl_k(G^{--})}(u)^2 + d_{spl_k(G^{--})}(e)^2\right]$$

$$+ \sum_{uv' \in E_3} \left[d_{spl_k(G^{--})}(u)^2 + d_{spl_k(G^{--})}(v')^2\right]$$

$$+ \sum_{ue' \in E_4} \left[d_{spl_k(G^{--})}(u)^2 + d_{spl_k(G^{--})}(e')^2\right]$$

$$+ \sum_{u'e \in E_5} \left[d_{spl_k(G^{--})}(u')^2 + d_{spl_k(G^{--})}(e)^2\right]$$

$$F\big(spl_k(G^{--})\big) = \sum_{uv\in E_1} [(n+m-1-2d_G(u))^2(k+1)^2 + (n+m-1-2d_G(v))^2(k+1)^2]$$
$$+\sum_{ue\in E_2} [(n+m-1-2d_G(u))^2 + (n-2)^2(k+1)^2]$$
$$+\sum_{uv'\in E_3} [(n+m-1-2d_G(u))^2(k+1)^2 + (n+m-1-2d_G(v))^2]$$
$$+\sum_{ue'\in E_4} [(n+m-1-2d_G(u))^2(k+1)^2 + (n-2)^2]$$
$$+\sum_{u'e\in E_5} [(n+m-1-2d_G(u))^2 + (n-2)^2]$$
$$= (k+1)^2\sum_{uv\in E_1} [(n+m-1-2d_G(u))^2 + (n+m-1-2d_G(v))^2]$$
$$+\sum_{ue\in E_2} [(n+m-1-2d_G(u))^2 + (n-2)^2(k+1)^2]$$
$$+\sum_{uv'\in E_3} [(n+m-1-2d_G(u))^2(k+1)^2 + (n+m-1-2d_G(v))^2]$$
$$+\sum_{ue'\in E_4} [(n+m-1-2d_G(u))^2(k+1)^2 + (n-2)^2]$$
$$+\sum_{u'e\in E_5} [(n+m-1-2d_G(u))^2 + (n-2)^2]$$

$$F\big(spl_k(G^{--})\big) = (k+1)^2\sum_{uv\in E_1} [(n+m-1)^2 + 4d_G(u)^2 - 4(n+m-1)d_G(u)$$
$$+(n+m-1)^2 + 4d_G(v)^2 - 4(n+m-1)d_G(v)]$$
$$+\sum_{ue\in E_2} [(n+m-1)^2 + 4d_G(u)^2 - 4(n+m-1)d_G(u) + (n-2)^2(k+1)^2]$$
$$+\sum_{uv'\in E_3} [(n+m-1)^2 + 4d_G(u)^2 - 4(n+m-1)d_G(u)(k+1)^2$$
$$+(n+m-1)^2 + 4d_G(v)^2 - 4(n+m-1)d_G(v)]$$
$$+\sum_{ue'\in E_4} [[(n+m-1)^2 + 4d_G(u)^2 - 4(n+m-1)d_G(u)](k+1)^2 + (n-2)^2]$$
$$+\sum_{u'e\in E_5} [(n+m-1)^2 + 4d_G(u)^2 - 4(n+m-1)d_G(u) + (n-2)^2]$$

$$F(spl_k(G^{--})) = 2(k+1)^2(n+m-1)^2\sum_{uv\notin E}1 + 4(k+1)^2\sum_{uv\notin E}[d_G(u)^2+d_G(v)^2]$$
$$-4(k+1)^2(n+m-1)\sum_{uv\notin E}[d_G(u)+d_G(v)]$$
$$+(n+m-1)^2\sum_{ue\in E_2}1 + 4\sum_{ue\in E_2}d_G(u)^2 - 4(n+m-1)\sum_{ue\in E_2}d_G(u)$$
$$+(n-2)^2(k+1)^2\sum_{ue\in E_2}1 + (n+m-1)^2[(k+1)^2+1]\sum_{uv'\in E_3}1$$
$$+4(k+1)^2\sum_{uv'\in E_3}d_G(u)^2 - 4(k+1)^2(n+m-1)\sum_{uv'\in E_3}d_G(u)$$
$$+4\sum_{uv'\in E_3}d_G(v)^2 - 4(n+m-1)\sum_{uv'\in E_3}d_G(v)]$$
$$+(n+m-1)^2(k+1)^2\sum_{ue'\in E_4}1 + 4(k+1)^2\sum_{ue'\in E_4}d_G(u)^2$$
$$-4(k+1)^2(n+m-1)\sum_{ue'\in E_4}d_G(u) + (n-2)^2\sum_{ue'\in E_4}1$$
$$+(n+m-1)^2\sum_{u'e\in E_5}1 + 4\sum_{u'e\in E_5}d_G(u)^2$$
$$-4(n+m-1)\sum_{u'e\in E_5}d_G(u) + (n-2)^2\sum_{u'e\in E_5}1$$

$$F(spl_k(G^{--})) = 2(k+1)^2(n+m-1)^2\left[\frac{n(n-1)}{2}-m\right] + 4(k+1)^2\overline{F}(G)$$
$$-4(k+1)^2(n+m-1)\overline{M}_1(G) + (n+m-1)^2m(n-2) + 4F(G)$$
$$-4(n+m-1)\sum_{u\in V(G)}d_G(u)^2 + (n-2)^2(k+1)^2m(n-2)$$
$$+(n+m-1)^2[(k+1)^2+1]2k\left[\frac{n(n-1)}{2}-m\right] + 4(k+1)^2F(G)$$
$$-4(k+1)^2(n+m-1)\sum_{u\in V(G)}d_G(u)^2 + 4F(G)$$
$$-4(n+m-1)\sum_{v\in V(G)}d_G(v)^2 + (n+m-1)^2(k+1)^2mk(n-2)$$
$$+4(k+1)^2F(G) - 4(k+1)^2(n+m-1)\sum_{u\in V(G)}d_G(u)^2$$
$$+(n-2)^2mk(n-2) + (n+m-1)^2mk(n-2) + 4F(G)$$
$$-4(n+m-1)\sum_{u\in V(G)}d_G(u)^2 + (n-2)^2mk(n-2)$$

$$
\begin{aligned}
F\big(spl_k(G^{--})\big) = {} & 2(k+1)^2(n+m-1)^2\left[\frac{n(n-1)}{2}-m\right]+4(k+1)^2F(G) \\
& -4(k+1)^2(n+m-1)M_1(G)+(n+m-1)^2m(n-2)+4F(G) \\
& -4(n+m-1)F(G)+(n-2)^2(k+1)^2m(n-2) \\
& +(n+m-1)^2[(k+1)^2+1]2k\left[\frac{n(n-1)}{2}-m\right]+4(k+1)^2F(G) \\
& -4(k+1)^2(n+m-1)F(G)+4F(G)-4(n+m-1)F(G) \\
& +(n+m-1)^2(k+1)^2mk(n-2)+4(k+1)^2F(G) \\
& -4(k+1)^2(n+m-1)F(G)+(n-2)^2mk(n-2) \\
& +(n+m-1)^2mk(n-2)+4F(G) \\
& -4(n+m-1)F(G)+(n-2)^2mk(n-2) \\
F\big(spl_k(G^{--})\big) = {} & [2(k+1)^2(n+m-1)^2+(n+m-1)^2[(k+1)^2 \\
& +1]2k]\left[\frac{n(n-1)}{2}-m\right] \\
& +[12(k+1)^2+12-12(n+m-1)-8(k+1)^2(n+m-1) \\
& -4(k+1)^2(n+m-1)M_1(G)+[(n+m-1)^2 \\
& +(n-2)^2(k+1)^2]m(n-2)+[(n+m-1)^2(k+1)^2 \\
& +(n-2)^2+(n+m-1)^2+(n-2)^2]mk(n-2)
\end{aligned}
$$

## 5. Conclusion

In this paper, we derived explicit formulations for the forgotten index pertaining to the $k$-splitting of generalized transformation graphs $spl_k\big(G^{ab}\big)$. Subsequently, we also derived comparable expressions for the complements of $spl_k\big(G^{ab}\big)$.

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