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Hamacher Operations on Pythagorean Fuzzy Matrices and Fermatean Fuzzy Matrices

Debasish Mahata

Department of Applied Mathematics
Vidyasagar University, Midnapore-721102, India.
e-mail: idebasishmahata@gmail.com

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Abstract. This study aims to extend Pythagorean and Fermatean fuzzy matrices within the framework of Hamacher operations. This paper introduces the concepts of Pythagorean fuzzy matrices and Hamacher operations for Fermatean fuzzy matrices and discusses several desirable properties of these operations, including commutativity, idempotency, and monotonicity. Additionally, we prove De Morganâ€™s laws over complements for these operations. The study also explores scalar multiplication and exponentiation operations for Fermatean fuzzy matrices, examining their algebraic properties. Finally, some necessity properties and possibility operators for Fermatean fuzzy matrices are established.

Keywords: Fuzzy Matrix; Pythagorean Fuzzy Matrix; Farmatean Fuzzy Matrix; Hamacher Sum and Product

AMS Mathematics Subject Classification (2010): 94D05, 08A72

1. Introduction

In data analysis, fuzzy matrices play a crucial role in clustering, classification, and data mining. Fuzzy clustering algorithms, such as fuzzy c-means, use fuzzy matrices to allow data points to belong to multiple clusters with varying degrees of membership [1]. This approach is particularly useful in handling overlapping clusters and improving clustering quality.

In classification, fuzzy matrices aid in developing fuzzy rule-based systems, which are more flexible and capable of handling uncertainties compared to classical methods. Fuzzy rule-based systems use fuzzy matrices to represent the rules and membership functions, allowing for more nuanced decision-making. For instance, in medical diagnosis, fuzzy rule-based systems can use fuzzy matrices to represent the relationships between symptoms and diseases, providing more accurate diagnoses even when the input data is imprecise or incomplete [11].

In data mining, fuzzy matrices are used to discover patterns and relationships in large datasets. For instance, fuzzy association rule mining, which uses fuzzy matrices to represent the relationships between items, can identify frequent patterns in transactional data. This approach is particularly useful in market basket analysis, where the goal is to identify sets of items that are frequently purchased together [6].

Despite their numerous advantages, fuzzy matrices face several challenges. One of the main challenges is computational complexity. Operations on fuzzy matrices, such as multiplication and inversion, can be computationally intensive, especially for large matrices. Researchers are continually developing more efficient algorithms to address these challenges, but computational efficiency remains a significant concern.

Another challenge is the interpretability of fuzzy matrices. While fuzzy matrices provide a powerful tool for modeling uncertainty and imprecision, the resulting models can be difficult to interpret. This is particularly problematic in applications such as decision-making, where the interpretability of the model is crucial for gaining user trust and acceptance. Researchers are exploring various approaches to enhance the interpretability of fuzzy matrices, such as simplifying the structure of fuzzy matrices and developing visualization techniques [30].

Looking ahead, the future of fuzzy matrices looks promising, with several exciting research directions. One area of future research is the development of more efficient computational methods for fuzzy matrices. Advances in parallel computing, such as the use of graphics processing units (GPUs) and distributed computing frameworks, hold great potential for improving the computational efficiency of fuzzy matrix operations [11]. Researchers are also exploring the integration of fuzzy matrices with emerging technologies such as artificial intelligence, big data, and the Internet of Things (IoT) [29]. For instance, in the context of big data, fuzzy matrices can be used to handle the uncertainty and imprecision inherent in large, complex datasets.

The notion of an intuitionistic fuzzy matrix (IFM) was first introduced independently by Khan et al.[19] and Im et al. as a generalization of Thomason's fuzzy matrix. In an IFM, each element is represented by an ordered pair $\langle u_{ij}, u'_{ij} \rangle$ where $u_{ij}, u'_{ij} \in [0,1]$ and $0 \leq u_{ij} + u'_{ij} \leq 1$. Khan and Pal established fundamental operations and relations for IFMs, such as maxmin, minmax, complement, algebraic sum, and algebraic product, demonstrating equality between IFMs. Mondal and Pal[7] explored similarity relations, invertibility conditions, and eigenvalues of IFMs. Zhang and Xu developed into intuitionistic fuzzy value and IFMs, introducing intuitionistic fuzzy similarity relations and applying them in clustering analysis. Emam and Fndh [5] introduced various types of IFMs and devised a method to derive an idempotent IFM from any given one through minmax composition. Muthuraji et al.[10] developed a decomposition technique for intuitionistic fuzzy matrices.

Yager[28] introduced the concept of Pythagorean fuzzy sets (PFS) and formulated aggregation operations for them. Subsequently, Zhang and Xu [31] explored various binary operations on PFS and proposed a decision-making algorithm based on this concept. Utilizing the framework of PFS, Pythagorean fuzzy matrices (PFM) were introduced, and their algebraic operations were defined by Silambarasan and Sriram [25]. Demonstrating further advancements, it was shown that the set of all Pythagorean fuzzy matrices constitutes a commutative monoid concerning algebraic sum and algebraic product[25].

Additionally, the development of Hamacher operations on Pythagorean fuzzy matrices and an investigation into their algebraic properties were carried out [25]. In 2020, scalar multiplication and exponentiation operations for Pythagorean fuzzy matrices were established, accompanied by an exploration of their desirable properties. Senapati and Yager (2020) introduced Fermatean FS (FFS), emphasizing its ability to address greater levels of uncertainty compared to IFS and PFS. This implies that IFS and PFS are specific cases of FFS, thus enabling FFSs to manage heightened uncertainty levels. Building on the theory of FFS, Fermatean fuzzy matrices were introduced, and their algebraic operations were defined by Silambarasan [26].

Further developments included the establishment of scalar multiplication and exponentiation operations for Fermatean fuzzy matrices, alongside an investigation into their algebraic properties. The paper also presents the development of Hamacher operations for Fermatean fuzzy matrices and the proof of their algebraic properties.

The Pythagorean fuzzy matrix (PFM) has emerged as a valuable tool for representing uncertainty in multi-criteria decision-making problems. It incorporates both membership and non-membership degrees, ensuring that the sum of their squares remains equal to or less than 1. Compared to the Intuitionistic Fuzzy Matrix (IFM), the PFM offers greater versatility. There are instances where the PFM can resolve issues that the IFM cannot address.

For example, if a decision maker provides membership and non-membership degrees of 0.7 and 0.6 respectively, it is only compatible with the PFM. Essentially, all intuitionistic fuzzy degrees fall within the spectrum of Pythagorean fuzzy degrees, underscoring the PFM's superior capability in managing uncertain problems.

Figure (0) illustrates the distinctions among IFM, PFM. Any Fuzzy Matrix (IFM) $< u_{ij}, u'_{ij} >$, that is an Intuitionistic Fuzzy Matrix (IFM), is also a Pthyagorean Fuzzy Matrix (PFM) and Fermatean Fuzzy Matrix (FFM).

For any two fuzzy matrices U and V, with elements ranging between 0 and 1, the hierarchical relation holds: $u_{ij}^3 \leq u_{ij}^2 \leq u_{ij}$ and $u'^3_{ij} \leq u'^2_{ij} \leq u'_{ij}$. Consequently, $u_{ij} + u'_{ij} \leq 1$ leads to $u_{ij}^2 + u'^2_{ij} \leq 1$, which further implies $u_{ij}^3 + u'^3_{ij} \leq 1$.

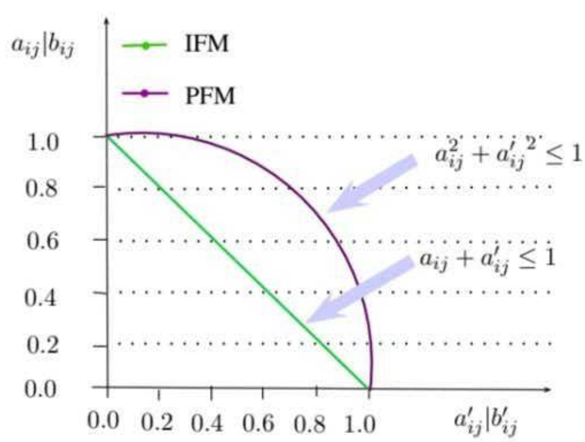

Figure 1: Comparison of spaces of the PFMs and IFMs

Consider the point $(0.9,0.6)$. Observing that $(0.9)^3 + (0.6)^3 \leq 1$ confirms its classification as an FFM.

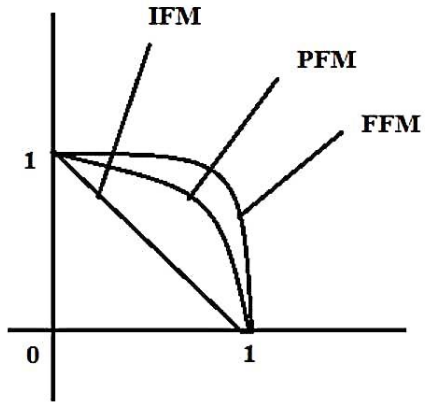

Figure 2: Comparison of Spaces of the PFMs and IFMs and FFMs

However, given that $(0.9)2 + (0.6)2 = 0.81 + 0.36 = 1.17 \geq 1$ and $0.9 + 0.6 \geq 1$, it is evident that $(0.9,0.6)$ neither qualifies as a PFM nor an IFM.

The matrix

$$U = \begin{bmatrix} (0.9,0.6) & (0.2,0.4) \\ (0.3,0.4) & (0.4,0.4) \end{bmatrix}$$

does not meet the criteria of being an IFM and a PFM, yet it satisfies the conditions to be considered as an FFM.

2. Literature review

In 1977, Thomason introduced fuzzy matrix (FM) [27]. After that many works have been published on FMs. Like fuzzy matrices, a lot of researchers worked on intuitionistic fuzzy matrices (IFMs) and published many papers. In 2001, Pal [12] introduced intuitionistic fuzzy determinant (IFD). Motivated by the concept of IFSs and IFD, Pal et al. [13] defined IFM and presented many properties.

Several works on fuzzy and related matrices have been done by many researchers. Two new operators are defined on fuzzy matrices and presented several results [22], similarity relations, invertibility and eigenvalues are investigated for IFM [7] and for bipolar fuzzy matrix [8], inverse of IFMs is studied in [20, 21], triangular fuzzy matrices discussed in [23], circulant triangular fuzzy number matrices are presented in [2], complex nilpotent matrices defined in [3], norm [16], interval-valued fuzzy matrices studied in [24], rank of interval-valued fuzzy matrices is discussed in [9], picture fuzzy matrix is defined in [4], bipolar fuzzy matrices investigated in [17]. New type of fuzzy matrices are introduced whose rows and columns are uncertain, see for the fuzzy matrices [15], for intuitionistic fuzzy matrices [18], for interval-valued fuzzy matrices [14].

To the best of our knowledge, no work have been done on Fermatean fuzzy matrices.

3. Preliminaries

An intuitionistic fuzzy matrix (IFM)is a matrix of pairs $U = (< u_{ij}, u'_{ij} >)$ of a non-negative real numbers $u_{ij}, u'_{ij} \in [0,1]$ satisfying

$$0 \leq (u_{ij} + u'_{ij}) \leq 1, \ \ for\ all\ \ i,j, \tag{1}$$

where u_{ij} and u'_{ij} are the membership and non-membership value of the ijth element.

By changing the restriction of Eqn. (1), different type of FMs can be defined. The general form of the condition (1) is defined below.

$$0 \leq (u_{ij}^{p} + u'_{ij})^{p} \leq 1, \ \ for \ all \ \ i, j, \tag{2}$$

where p is an integer.

If we consider $p = 2$, a new type of fuzzy matrix is generated, known as a Pythagorean fuzzy matrix (PFM), which is defined independently below.

A PFM is a pairs $U = (< u_{ij}, u'_{ij} >)$ of a non-negative real numbers $u_{ij}, u'_{ij} \in [0,1]$ satisfying condition $0 \leq (u_{ij})^2 + (u'_{ij})^2 \leq 1 \quad \forall i, j$, where u_{ij} and u'_{ij} are the membership and non-membership value of the ijth element.

Similarly, if we considered $p = 3$, we obtained another FM known as Fermatean fuzzy matrix (FFM), defined below.

A FFM)is a pairs $U = (< u_{ij}, u'_{ij} >)$ of a non-negative real numbers $u_{ij}, u'_{ij} \in [0,1]$ satisfying condition $0 \leq (u_{ij})^3 + (u'_{ij})^3 \leq 1 \quad \forall i, j$, where u_{ij} and u'_{ij} are the membership and non-membership value of the ijth element.

For other values of p, different types of fuzzy matrices can be defined. However, to the best of my knowledge, no such fuzzy matrices have been defined for higher values of p.

During last few decades, some operators based on t-norms and t-conorms have been defined. Among them, Hamacher operator is one of the well studied operator.

3.1. Hamacher Operations

Hamacher operations includes the Hamacher sum and product, which are examples of t-norms and t-conorms respectively.They are defined as follows:-

$$T(u, v) = u \odot v = \frac{(uv)}{(\gamma+(1-\gamma)(u+v-uv))}$$

and

$$T^*(u, v) = u \oplus v = \frac{(u+v-uv)-(1-\gamma)uv}{1-(1-\gamma)uv}$$

Especially,when $\gamma = 1$, then Hamacher t-norm and t-conorm reduces to

$$T(u, v) = u \odot v = uv$$

and

$$T^*(u, v) = u \oplus v = u + v - uv$$

They are the algebric t-norm and t-conorm respectively.When $\gamma = 2$ Hamacher t-norm and t-conorm reduces to

$$T(u, v) = u \odot v = \frac{uv}{2+(1-2)(u+v-uv)} = \frac{uv}{1+(1-u)(1-b)}$$

and

$$T^*(u, v) = u \oplus v = \frac{u+v}{1+uv}.$$

$T(u, v)$ and $T^*(u, v)$ are called Einstein t-norm and t-conorm respectively.

The combination of the Hamacher operator with PFMs and FFMs yields many interesting results. Some of them are presented in this paper.

4. Hamacher operations on Pythagorean fuzzy matrices

The basic operation on any type of FMs are sum and product. These two operations are defined below.

Definition 1. *Let* $U = (< u_{ij}, u'_{ij} >)$ *and* $V = (< v_{ij}, v'_{ij} >)$ *be any two PFMs of same size, then*

(i) The Hamacher sum between U and V is defined as,

$$U \oplus_H V = (w_{ij}); \forall i, j$$

$$\text{where, } w_{ij} = \begin{cases} < 1,0 >, \text{if} < \mathrm{u_{ij}}, \mathrm{u'_{ij}} >=< 1,0 > \text{and} < \mathrm{v_{ij}}, \mathrm{v'_{ij}} >=< 1,0 > \\ \left\langle \sqrt{\frac{u_{ij}^2+v_{ij}^2-2u_{ij}^2v_{ij}^2}{1-u_{ij}^2v_{ij}^2}}, \sqrt{\frac{u'^2_{ij}v'^2_{ij}}{u'^2_{ij}+v'^2_{ij}-u'^2_{ij}v'^2_{ij}}} \right\rangle, \text{Otherwise} \end{cases} \tag{3}$$

(ii) The Hamacher product between U and V is defined as,

$$U \odot_H V = (d_{ij}); \forall i, j$$

$$\text{where, } d_{ij} = \begin{cases} < 0,1 >, \text{if} < \mathrm{u_{ij}}, \mathrm{u'_{ij}} >=< 0,1 > \text{and} < \mathrm{v_{ij}}, \mathrm{v'_{ij}} >=< 0,1 > \\ \left\langle \sqrt{\frac{u_{ij}^2v_{ij}^2}{u_{ij}^2+v_{ij}^2-u_{ij}^2v_{ij}^2}}, \sqrt{\frac{u'^2_{ij}+v'^2_{ij}-2u'^2_{ij}v'^2_{ij}}{1-u'^2_{ij}v'^2_{ij}}} \right\rangle, \text{Otherwise} \end{cases} \tag{4}$$

The relations between Hamacher sum and Hamacher product is established in the following theorem.

Theorem 1. *Let* U, V *be any two PFMs of the same size, then* $U \odot_H V \leq U \oplus_H V$.

Proof: we know that for any real numbers $u, v \in [0,1]$, the following inequality holds:

$$\sqrt{\frac{u^2v^2}{u^2+v^2-u^2v^2}} \leq \sqrt{\frac{u^2+v^2-2u^2v^2}{1-u^2v^2}} \tag{5}$$

Using the above inequality (5), we have

$$\sqrt{\frac{u_{ij}^2v_{ij}^2}{u_{ij}^2+v_{ij}^2-u_{ij}^2v_{ij}^2}} \leq \sqrt{\frac{u_{ij}^2+v_{ij}^2-2u_{ij}^2v_{ij}^2}{1-u_{ij}^2v_{ij}^2}}$$

and

$$\sqrt{\frac{u'^2_{ij}v'^2_{ij}}{u'^2_{ij}+v'^2_{ij}-u'^2_{ij}v'^2_{ij}}} \leq \sqrt{\frac{u'^2_{ij}+v'^2_{ij}-2u'^2_{ij}v'^2_{ij}}{1-u'^2_{ij}v'^2_{ij}}}.$$

Now, by definition of Hamacher sum and product, we get $U \odot_H V \leq U \oplus_H V$.

Theorem 2. *For any PFM* U, *(i)* $U \oplus_H U \geq U$, *(ii)*$U \odot_H U \leq U$.

Proof :

(i)

$$U \oplus_H U = \langle \sqrt{\frac{2u_{ij}^2-2u_{ij}^4}{1-u_{ij}^4}}, \sqrt{\frac{u'^4_{ij}}{2u'^2_{ij}-u'^4_{ij}}}\rangle$$
$$= \langle \sqrt{\frac{2u_{ij}^2}{1+u_{ij}^2}}, \sqrt{\frac{u'^2_{ij}}{2-u'^2_{ij}}}\rangle$$
$$\geq < u_{ij}, u'_{ij} > \geq U$$

(ii) Similarly, $U \odot_H U \leq U$ can be proved.

Theorem 3. *If* U, V *are any two PFMs of same size and* $U \leq V$, *then* $U \odot_H W \leq V \odot_H W$.

Proof: We know that for any three real numbers $u, v, w \in [0,1]$ and if $u \leq v$ then

$$\sqrt{\frac{u^2w^2}{u^2+w^2-u^2w^2}} \leq \sqrt{\frac{v^2w^2}{v^2+w^2-v^2w^2}} \tag{6}$$

and

$$\sqrt{\frac{u^2+w^2-2u^2w^2}{1-u^2w^2}} \leq \sqrt{\frac{v^2+w^2-2v^2w^2}{1-v^2w^2}} \tag{7}$$

Using inequality (6), we have

$$\sqrt{\frac{u_{ij}^2w^2ij}{u_{ij}^2+w_{ij}^2-u_{ij}^2w_{ij}^2}} \leq \sqrt{\frac{v_{ij}^2w_{ij}^2}{v_{ij}^2+w_{ij}^2-v_{ij}^2w_{ij}^2}}$$

This implies, $\sqrt{\frac{v'^2_{ij}w'^2_{ij}}{v'^2_{ij}+w'^2_{ij}-v'^2_{ij}w'^2_{ij}}} \geq \sqrt{\frac{u'^2_{ij}w'^2_{ij}}{u'^2_{ij}+w'^2_{ij}-u'^2_{ij}w'^2_{ij}}}$

Hence, we get $U \odot_H W \leq V \odot_H W$.

Theorem 4. *If* U, V *be any two PFMs of the same size and* $U \leq V$, $U \oplus_H W \leq V \oplus_H W$.

Proof : Using inequality (7) from Theorem (3), we have

$$\sqrt{\frac{u_{ij}^2+w_{ij}^2-2u_{ij}^2w^2ij}{1-u_{ij}^2w_{ij}^2}} \leq \sqrt{\frac{v_{ij}^2+w_{ij}^2-2v_{ij}^2w_{ij}^2}{1-v_{ij}^2w_{ij}^2}}$$

This implies

$$\sqrt{\frac{v'^2_{ij}+w'^2_{ij}-2v'^2_{ij}w'^2_{ij}}{1-v'^2_{ij}w'^2_{ij}}} \geq \sqrt{\frac{u'^2_{ij}+w'^2_{ij}-2u'^2_{ij}w'^2_{ij}}{1-u'^2_{ij}w'^2_{ij}}}$$

Finally, we get $U \oplus_H W \leq V \oplus_H W$.

Theorem 5. *If* U, V *be any two PFMs of the same size, then*

(i) $(U \cap V) \oplus_H (U \cup V) = U \oplus_H V$,

(ii) $(U \cap V) \odot_H (U \cup V) = U \odot_H V$.

Proof: (i)

$$(U \cap V) \oplus_H (U \cup V) = (< min(u_{ij}, v_{ij}), max(u'_{ij}, v'_{ij}) > \oplus_H$$
$$< max(u_{ij}, v_{ij}), min(u'_{ij}, v'_{ij}) >)$$

$=$

$$\langle \sqrt{\frac{min(u_{ij}^2,v_{ij}^2)+max(u_{ij}^2,v_{ij}^2)-2min(u_{ij}^2,v_{ij}^2)max(u_{ij}^2,v_{ij}^2)}{1-min(u_{ij}^2,v_{ij}^2)max(u_{ij}^2,v_{ij}^2)}}, \sqrt{\frac{max(u'^2_{ij},v'^2_{ij})min(u'^2_{ij},v'^2_{ij})}{max(u'^2_{ij},v'^2_{ij})+min(u'^2_{ij},v'^2_{ij})-max(u'^2_{ij},v'^2_{ij})min(u'^2_{ij},v'^2_{ij})}}\rangle$$

$$= \langle \sqrt{\frac{u_{ij}^2+v_{ij}^2-2u_{ij}^2v_{ij}^2}{1-u_{ij}^2v_{ij}^2}}, \sqrt{\frac{u'^2_{ij}v'^2_{ij}}{u'^2_{ij}+v'^2_{ij}-u'^2_{ij}v'^2_{ij}}}\rangle$$

$= U \oplus_H V.$

(ii) $(U \cap V) \odot_H (U \cup V)$

$$= (< min(u_{ij}, v_{ij}), max(u'_{ij}, v'_{ij}) > \odot_H < max(u_{ij}, v_{ij}), min(u'_{ij}, v'_{ij}) >)$$

$$= \langle \sqrt{\frac{\min(u_{ij}^2, v_{ij}^2)\max(u_{ij}^2, v_{ij}^2)}{\min(u_{ij}^2, v_{ij}^2) + \max(u_{ij}^2, v_{ij}^2) - \min(u_{ij}^2, v_{ij}^2)\max(u_{ij}^2, v_{ij}^2)}},$$

$$\sqrt{\frac{max(u'^2_{ij}, v'^2_{ij}) + min(u'^2_{ij}, v'^2_{ij}) - 2max(u'^2_{ij}, v'^2_{ij})min(u'^2_{ij}, v'^2_{ij})}{1 - max(u'^2_{ij}, v'^2_{ij})min(u'^2_{ij}, v'^2_{ij})}}\rangle$$

$$= \langle \sqrt{\frac{u_{ij}^2v_{ij}^2}{u_{ij}^2 + v_{ij}^2 - u_{ij}^2v_{ij}^2}}, \sqrt{\frac{u'^2_{ij} + v'^2_{ij} - 2u'^2_{ij}v'^2_{ij}}{1 - u'^2_{ij}v'^2_{ij}}}\rangle$$

$= U \odot_H V.$

Theorem 6. *If* U, V *be any two PFMs of the same size then*

(i) $(U \oplus_H V)^c = (U)^c \odot_H (V)^c$,

(ii) $(U \odot_H V)^c = (U)^c \oplus_H (V)^c$,

(iii) $(U \oplus_H V)^c \leq (U)^c \odot_H (V)^c$,

(iv) $(U \odot_H V)^c \geq (U)^c \odot_H (V)^c$, where c denotes the complement.

Proof:

(i)

$$(U)^c \odot_H (V)^c = \langle \sqrt{\frac{u'^2_{ij}v'^2_{ij}}{u'^2_{ij}+v'^2_{ij}-u'^2_{ij}v'^2_{ij}}}, \sqrt{\frac{u_{ij}^2+v_{ij}^2-2u_{ij}^2v_{ij}^2}{1-u_{ij}^2v_{ij}^2}}\rangle = (U \oplus_H V)^c$$

(ii)

$$(U)^c \oplus_H (V)^c = \langle \sqrt{\frac{u'^2_{ij}+v'^2_{ij}-2u'^2_{ij}v'^2_{ij}}{1-u'^2_{ij}v'^2_{ij}}}, \sqrt{\frac{u_{ij}^2v_{ij}^2}{u_{ij}^2+v_{ij}^2-u_{ij}^2v_{ij}^2}}\rangle = (U \odot_H V)^c.$$

(iii) We know that,

$$(U \oplus_H V)^c = \langle \sqrt{\frac{u'^2_{ij}v'^2_{ij}}{u'^2_{ij}+v'^2_{ij}-u'^2_{ij}v'^2_{ij}}}, \sqrt{\frac{u_{ij}^2+v_{ij}^2-2u_{ij}^2v_{ij}^2}{1-u_{ij}^2v_{ij}^2}}\rangle$$

and

$$(U)^c \oplus_H (V)^c = \langle \sqrt{\frac{u'^2_{ij}+v'^2_{ij}-2u'^2_{ij}v'^2_{ij}}{1-u'^2_{ij}v'^2_{ij}}}, \sqrt{\frac{u^2_{ij}v^2_{ij}}{u^2_{ij}+v^2_{ij}-u^2_{ij}v^2_{ij}}} \rangle$$

Now using inequality (5), we get

$$\sqrt{\frac{u'^2_{ij}v'^2_{ij}}{u'^2_{ij}+v'^2_{ij}-u'^2_{ij}v'^2_{ij}}} \leq \sqrt{\frac{u'^2_{ij}+v'^2_{ij}-2u'^2_{ij}v'^2_{ij}}{1-u'^2_{ij}v'^2_{ij}}}$$

and

$$\sqrt{\frac{u^2_{ij}+v^2_{ij}-2u^2_{ij}v^2_{ij}}{1-u^2_{ij}v^2_{ij}}} \geq \sqrt{\frac{u^2_{ij}v^2_{ij}}{u^2_{ij}+v^2_{ij}-u^2_{ij}v^2_{ij}}}$$

Hence, $(U \oplus_H V)^c \leq (U)^c \odot_H (V)^c$.

(iv) Similarly, we can prove $(U \odot_H V)^c \geq (U)^c \odot_H (V)^c$.

5. Hamacher operations on Fermatean fuzzy matrices

In this section, the Hamachar operator is defined and investigated for FFMs.

Definition 2. *Let* $U = (< u_{ij}, u'_{ij} >)$ *and* $V = (< v_{ij}, v'_{ij} >)$ *be any two FFMs of same size, then*

(i) The Hamacher sum U and V is defined as,

$U \oplus_H V = (\alpha_{ij})$; for all i, j.

$$where, \alpha_{ij} = \begin{cases} < 1,0 >, \text{if} < \mathrm{u_{ij}}, \mathrm{u'_{ij}} > = < 1,0 > \text{ and } < \mathrm{v_{ij}}, \mathrm{v'_{ij}} > = < 1,0 > \\ \langle \sqrt[3]{\frac{u^3_{ij}+v^3_{ij}-2u^3_{ij}v^3_{ij}}{1-u^3_{ij}v^3_{ij}}}, \sqrt[3]{\frac{u'^3_{ij}v'^3_{ij}}{u'^3_{ij}+v'^3_{ij}-u'^3_{ij}v'^3_{ij}}} \rangle, \text{otherwise} \end{cases}$$

(ii) The Hamacher product U and V is defined as,

$$U \odot_H V = (\beta_{ij}); \forall i, j$$

$$\text{where, } \beta_{ij} = \begin{cases} < 0,1 >, \text{if} < \mathrm{u_{ij}}, \mathrm{u'_{ij}} > = < 0,1 > \text{ and } < \mathrm{v_{ij}}, \mathrm{v'_{ij}} > = < 0,1 > \\ \langle \sqrt[3]{\frac{u^3_{ij}v^3_{ij}}{u^3_{ij}+v^3_{ij}-u^3_{ij}v^3_{ij}}}, \sqrt[3]{\frac{u'^3_{ij}+v'^3_{ij}-2u'^3_{ij}v'^3_{ij}}{1-u'^3_{ij}v'^3_{ij}}} \rangle, \text{otherwise} \end{cases}$$

Lemma 1. *For any real numbers* $u, v \in [0,1]$, *the following inequality holds*

$$\sqrt[3]{\frac{u^3v^3}{u^3+v^3-u^3v^3}} \leq \sqrt[3]{\frac{u^3+v^3-2u^3v^3}{1-u^3v^3}}.$$

Proof: Let $\frac{uv}{u+v-uv} \leq \frac{u+v-2uv}{1-uv}$.

We know that

$$(u+v)^2 \geq 4uv \tag{8}$$

so, $u+v-uv \leq 1-(1-u) \leq 1$
or, $1+3(u+v-uv) \leq 4$
or, $(1+3(u+v-uv))uv \leq 4uv$ (9)

From the equations (8) and (9), we get

$(1+3(u+v-uv))uv \leq 4uv \leq (u+v)^2$
or, $\leq (u+v)^2-(1+3(u+v-uv))uv$
or, $0 \leq (u+v)^2+3(u+v-uv)-uv$
or, $uv \leq (u+v)^2+3u^2v^2-3uv(u+v)$
or, $uv-u^2v^2 \leq (u+v-2uv)(u+v-uv)$
or, $\frac{uv}{u+v-uv} \leq \frac{u+v-2uv}{1-uv}$
or, $\sqrt[3]{\frac{u^3v^3}{u^3+v^3-u^3v^3}} \leq \sqrt[3]{\frac{u^3+v^3-2u^3v^3}{1-u^3v^3}}$

Now, a relation between Hamacher sum and Hamacher product on FFMs is established below.

Theorem 7. *If* U, V *be any two FFMs of the same size, then* $U \odot_H V \leq U \oplus_H V$.
Proof: Using the previous lemmas (1), (2)

$$\sqrt[3]{\frac{u_{ij}^3v_{ij}^3}{u_{ij}^3+v_{ij}^3-u_{ij}^3v_{ij}^2}} \leq \sqrt[3]{\frac{u_{ij}^3+v_{ij}^3-2u_{ij}^3v_{ij}^3}{1-u_{ij}^3v_{ij}^2}}$$

and $\sqrt[3]{\frac{u'^3_{ij}v'^3_{ij}}{u'^3_{ij}+v'^3_{ij}-u'^3_{ij}v'^3_{ij}}} \leq \sqrt[3]{\frac{u'^3_{ij}+v'^3_{ij}-2u'^3_{ij}v'^3_{ij}}{1-u'^3_{ij}v'^3_{ij}}}$.

Now, by definition of Hamacher sum and product on FFMs, $U \odot_H V \leq U \oplus_H V$.

Theorem 8. *For any FFMs* U;

(i) $U \oplus_H U \geq U$,
(ii) $U \odot_H U \leq U$.

Proof:
(i)

$$U \oplus_H U = \left\langle \sqrt[3]{\frac{u_{ij}^3+u_{ij}^3-2u_{ij}^3u_{ij}^3}{1-u_{ij}^3u_{ij}^2}}, \sqrt[3]{\frac{u'^3_{ij}u'^3_{ij}}{u'^3_{ij}+u'^3_{ij}-u'^3_{ij}u'^3_{ij}}} \right\rangle$$
$$= \left\langle \sqrt[3]{\frac{2u_{ij}^3-2u_{ij}^6}{1-u_{ij}^6}}, \sqrt[3]{\frac{u'^6_{ij}}{2u'^3_{ij}-u'^6_{ij}}} \right\rangle$$
$$= \left\langle \sqrt[3]{\frac{2u_{ij}^3}{1+u_{ij}^3}}, \sqrt[3]{\frac{u'^3_{ij}}{2-u'^3_{ij}}} \right\rangle$$
$$\geq \langle u_{ij}^3, u'^3_{ij} \rangle$$
$$\geq \langle u_{ij}, u'_{ij} \rangle \geq U.$$

(ii) Similarly, $U \odot_H U \leq U$ can be proved.

Lemma 2. *For any three real numbers* $u, v, w \in [0,1]$, *if* $u \le v$ *then*

(i) $\sqrt[3]{\frac{u^3w^3}{u^3+w^3-u^3w^3}} \le \sqrt[3]{\frac{v^3w^3}{v^3+w^3-v^3w^3}}$,

(ii) $\sqrt[3]{\frac{u^3+w^3-2u^3w^3}{1-u^3w^3}} \le \sqrt[3]{\frac{v^3+w^3-2v^3w^3}{1-v^3w^3}}$.

Proof: Let $\frac{uw}{u+w-uw} \le \frac{vw}{v+w-vw}$.

We know that

$$u \le v \ \ implies \ \ uw^2 \le vw^2$$
or, $uw^2 + uvw(1-w) \le vw^2 + uvw(1-w)$
or, $uw^2 + uvw - uvw^2 \le vw^2 + uvw - uvw^2$
or, $uw(w+v-vw) \le vw(w+u-uw)$
or, $\frac{uw}{u+w-uw} \le \frac{vw}{v+w-vw}$
or, $\sqrt[3]{\frac{u^3w^3}{u^3+w^3-u^3w^3}} \le \sqrt[3]{\frac{v^3w^3}{v^3+w^3-v^3w^3}}$

(ii) Let $\frac{u+w-2uw}{1-uw} \le \frac{v+w-2vw}{1-vw}$.

Since, we know that

$$u \le v \Longrightarrow u(1-w)^2 \le v(1-w)^2$$
or, $u(1-2w+w^2) \le v(1-2w+w^2)$
or, $u(1-2w+w^2) \le v(1-2w+w^2)$
or, $u - 2uw + uw^2 \le v - 2vw + vw^2$
or, $u - 2uw + uw^2 + (w - uvw + 2uvw^2) \le v - 2vw + vw^2 + (w - uvw + 2uvw^2)$
or, $u + w - 2uw - vw(u + w - 2uw) \le v + w - 2vw - uw(v + w - 2vw)$
or, $(u + w - 2uw)(1 - vw) \le (v + w - 2vw)(1 - uw)$
or, $\frac{u+w-2uw}{1-uw} \le \frac{v+w-2vw}{1-vw}$
or, $\sqrt[3]{\frac{u^3+w^3-2u^3w^3}{1-u^3w^3}} \le \sqrt[3]{\frac{v^3+w^3-2v^3w^3}{1-v^3w^3}}$

Theorem 9. *If* U, V, W *be any three FFMs of the same size and* $U \le V$, $U \odot_H W \le V \odot_H W$.

Proof: Using Lemma (2)(i),

$$\sqrt[3]{\frac{u_{ij}^3 w^3 ij}{u_{ij}^3+w_{ij}^3-u_{ij}^3 w_{ij}^3}} \le \sqrt[3]{\frac{v_{ij}^3 w_{ij}^3}{v_{ij}^3+w_{ij}^3-v_{ij}^3 w_{ij}^3}}$$

$$\Longrightarrow \sqrt[3]{\frac{v\prime_{ij}^3 w\prime_{ij}^3}{v\prime_{ij}^3+w\prime_{ij}^3-v\prime_{ij}^3 w\prime_{ij}^3}} \ge \sqrt[3]{\frac{u\prime_{ij}^3 w\prime_{ij}^3}{u\prime_{ij}^3+w\prime_{ij}^3-u\prime_{ij}^3 w\prime_{ij}^3}}.$$

Thus, $U \odot_H W \le V \odot_H W$.

Theorem 10. *If* U, V, W *be any three FFMs of the same size and* $U \leq V$, $U \oplus_H W \leq V \oplus_H W$.
Proof: Let $u_{ij} \leq v_{ij}$ and $u'_{ij} \leq v'_{ij}$, for all i, j.
Using Lemma (2)(ii)

$$\sqrt[3]{\frac{u_{ij}^3 + w_{ij}^3 - 2u_{ij}^3 w^3 ij}{1 - u_{ij}^3 w_{ij}^3}} \leq \sqrt[3]{\frac{v_{ij}^3 + w_{ij}^3 - 2v_{ij}^3 w_{ij}^3}{1 - v_{ij}^3 w_{ij}^3}}$$

and $\sqrt[3]{\frac{v'^3_{ij}+w'^3_{ij}-2v'^3_{ij}w'^3_{ij}}{1-v'^3_{ij}w'^3_{ij}}} \geq \sqrt[3]{\frac{u'^3_{ij}+w'^3_{ij}-2u'^3_{ij}w'^3_{ij}}{1-u'^3_{ij}w'^3_{ij}}}$.

Hence, we get $U \oplus_H W \leq V \oplus_H W$.

Theorem 11. *If* U, V *be any two FFMs of the same size, then*
(i) $(U \cap V) \oplus_H (U \cup V) = U \oplus_H V$,
(ii) $(U \cap V) \odot_H (U \cup V) = U \odot_H V$.
Proof: (i) $(U \cap V) \oplus_H (U \cup V)$

$$= (< min(u_{ij}, v_{ij}), max(u'_{ij}, v'_{ij}) > \oplus_H < max(u_{ij}, v_{ij}), min(u'_{ij}, v'_{ij}) >)$$

$$= \langle \sqrt[3]{\frac{min(u_{ij}^3, v_{ij}^3) + max(u_{ij}^3, v_{ij}^3) - 2min(u_{ij}^3, v_{ij}^3) max(u_{ij}^3, v_{ij}^3)}{1 - min(u_{ij}^3, v_{ij}^3) max(u_{ij}^3, v_{ij}^3)}},$$

$$\sqrt[3]{\frac{max(u'^3_{ij}, v'^3_{ij}) min(u'^3_{ij}, v'^3_{ij})}{max(u'^3_{ij}, v'^3_{ij}) + min(u'^2_{ij}, v'^3_{ij}) - max(u'^3_{ij}, v'^3_{ij}) min(u'^3_{ij}, v'^3_{ij})}} \rangle$$

$$= \langle \sqrt[3]{\frac{u_{ij}^3 + v_{ij}^3 - 2u_{ij}^3 v_{ij}^3}{1 - u_{ij}^3 v_{ij}^3}}, \sqrt[3]{\frac{u'^3_{ij} v'^3_{ij}}{u'^3_{ij} + v'^3_{ij} - u'^3_{ij} v'^3_{ij}}} \rangle$$

$$= U \oplus_H V.$$

(ii) $(U \cap V) \odot_H (U \cup V)$

$$= (< min(u_{ij}, v_{ij}), max(u'_{ij}, v'_{ij}) > \odot_H < max(u_{ij}, v_{ij}), min(u'_{ij}, v'_{ij}) >)$$

$$= \langle \sqrt[3]{\frac{min(u_{ij}^3, v_{ij}^3) max(u_{ij}^2, v_{ij}^3)}{min(u_{ij}^3, v_{ij}^3) + max(u_{ij}^3, v_{ij}^3) - min(u_{ij}^3, v_{ij}^3) max(u_{ij}^3, v_{ij}^3)}},$$

$$\sqrt[3]{\frac{max(u'^3_{ij}, v'^3_{ij}) + min(u'^3_{ij}, v'^3_{ij}) - 2max(u'^3_{ij}, v'^2_{ij}) min(u'^3_{ij}, v'^3_{ij})}{1 - max(u'^3_{ij}, v'^3_{ij}) min(u'^3_{ij}, v'^3_{ij})}} \rangle$$

$$= \langle \sqrt[3]{\frac{u_{ij}^3 v_{ij}^3}{u_{ij}^3 + v_{ij}^3 - u_{ij}^3 v_{ij}^3}}, \sqrt[3]{\frac{u'^3_{ij} + v'^3_{ij} - 2u'^3_{ij} v'^3_{ij}}{1 - u'^3_{ij} v'^3_{ij}}} \rangle$$

$$= U \odot_H V.$$

Theorem 12. *If* U, V *be any two FFMs of the same size then*
(i) $(U \oplus_H V)^c = (U)^c \odot_H (V)^c$,
(ii) $(U \odot_H V)^c = (U)^c \oplus_H (V)^c$,

(iii) $(U \oplus_H V)^c \leq (U)^c \odot_H (V)^c$,
(iv) $(U \odot_H V)^c \geq (U)^c \odot_H (V)^c$, where c denotes complement.

Proof: (i)

$$(U)^c \odot_H (V)^c = \langle \sqrt[3]{\frac{u'^3_{ij}v'^3_{ij}}{u'^3_{ij}+v'^3_{ij}-u'^3_{ij}v'^3_{ij}}}, \sqrt[3]{\frac{u^3_{ij}+v^3_{ij}-2u^3_{ij}v^3_{ij}}{1-u^3_{ij}v^3_{ij}}} \rangle = (U \oplus_H V)^c.$$

(ii)

$$(U)^c \oplus_H (V)^c = \langle \sqrt[3]{\frac{u'^3_{ij}+v'^3_{ij}-2u'^3_{ij}v'^3_{ij}}{1-u'^3_{ij}v'^3_{ij}}}, \sqrt[3]{\frac{u^3_{ij}v^3_{ij}}{u^3_{ij}+v^3_{ij}-u^3_{ij}v^3_{ij}}} \rangle$$

$$= (U \odot_H V)^c$$

(iii) We know that,

$$(A \oplus_H V)^c = \langle \sqrt[3]{\frac{u'^3_{ij}v'^3_{ij}}{u'^3_{ij}+v'^3_{ij}-u'^3_{ij}v'^3_{ij}}}, \sqrt[3]{\frac{u^3_{ij}+v^3_{ij}-2u^3_{ij}v^3_{ij}}{1-u^3_{ij}v^3_{ij}}} \rangle$$

and

$$(U)^c \oplus_H (V)^c = \langle \sqrt{\frac{u'^3_{ij}+v'^3_{ij}-2u'^3_{ij}v'^3_{ij}}{1-u'^3_{ij}v'^3_{ij}}}, \sqrt[3]{\frac{u^3_{ij}v^3_{ij}}{u^3_{ij}+v^3_{ij}-u^3_{ij}v^3_{ij}}} \rangle.$$

Using Lemma 2(i) and 2(ii),we get

$$\sqrt[3]{\frac{u'^3_{ij}v'^3_{ij}}{u'^3_{ij}+v'^3_{ij}-u'^3_{ij}v'^3_{ij}}} \leq \sqrt[3]{\frac{u'^3_{ij}+v'^3_{ij}-2u'^3_{ij}v'^3_{ij}}{1-u'^3_{ij}v'^3_{ij}}}$$

and

$$\sqrt[3]{\frac{u^3_{ij}+v^3_{ij}-2u^3_{ij}v^3_{ij}}{1-u^3_{ij}v^3_{ij}}} \geq \sqrt[3]{\frac{u^3_{ij}v^3_{ij}}{u^3_{ij}+v^3_{ij}-u^3_{ij}v^3_{ij}}}.$$

Hence, $(U \oplus_H V)^c \leq (U)^c \odot_H (V)^c$.
(iv) Similarly, we can prove $(U \odot_H V)^c \geq (U)^c \odot_H (V)^c$.

6. Hamacher scalar multiplication and exponential operations on Fermatean fuzzy matrices

We defined the following operations over Hamacher operations on FFMs. In this section, we form Hamacher scalar multiplication and Hamacher exponentiation operations on FFM U and investigate their algebraic properties.

Theorem 13. *If n is any positive integer and U is FFM, then the Hamacher scalar multiplication operation $(\cdot_H)$ is*

$$n \cdot_H U = (U \oplus_H \ldots . \oplus_H U) = \langle \sqrt[3]{\frac{nu^3_{ij}}{1+(n-1)u^3_{ij}}}, \sqrt[3]{\frac{u'^3_{ij}}{n-(n-1)u'^3_{ij}}} \rangle \tag{10}$$

where $(U \oplus_H \ldots . \oplus_H U)$ represents the n times Hamacher scalar multiplication of U.
Proof: The expression of Eq. (10) is denoted by $P(n)$.
Using mathematical induction, we prove Eq. (10), which holds for any positive integer n.

$$U \cdot_H U = \langle \sqrt[3]{\frac{u_{ij}^3+u_{ij}^3-2u_{ij}^3u_{ij}^3}{1-u_{ij}^3u_{ij}^2}}, \sqrt[3]{\frac{u'^3_{ij}u'^3_{ij}}{u'^3_{ij}+u'^3_{ij}-u'^3_{ij}u'^3_{ij}}} \rangle$$

$$= \langle \sqrt[3]{\frac{2u_{ij}^3-2u_{ij}^6}{1-u_{ij}^6}}, \sqrt[3]{\frac{u'^6_{ij}}{2u'^3_{ij}-u'^6_{ij}}} \rangle$$

$$= \langle \sqrt[3]{\frac{2u_{ij}^3}{1+u_{ij}^3}}, \sqrt[3]{\frac{u'^3_{ij}}{2-u'^3_{ij}}} \rangle$$

$$2 \cdot_H U = \langle \sqrt[3]{\frac{2u_{ij}^3}{1+(2-1)u_{ij}^3}}, \sqrt[3]{\frac{u'^3_{ij}}{2-(2-1)u'^3_{ij}}} \rangle$$

$$n \cdot_H U = \langle \sqrt[3]{\frac{nu_{ij}^3}{1+(n-1)u_{ij}^3}}, \sqrt[3]{\frac{u'^3_{ij}}{n-(n-1)u'^3_{ij}}} \rangle.$$

$P(n)$ holds.

Suppose that Eq. (10) holds for $n = m$, then

$$m \cdot_H U = \langle \sqrt[3]{\frac{mu_{ij}^3}{1+(m-1)u_{ij}^3}}, \sqrt[3]{\frac{u'^3_{ij}}{m-(m-1)u'^3_{ij}}} \rangle.$$

So,

$$(m+1) \cdot_H U = ((m \cdot_H U) \cdot_H U)$$

$$= \langle \sqrt[3]{\frac{u_{ij}^3(m+1)(1-u_{ij}^3)}{(1+mu_{ij}^3)(1-u_{ij}^3)}}, \sqrt[3]{\frac{(u'^3_{ij})^2}{(m+1-mu'^3_{ij})u'^3_{ij}}} \rangle$$

$$= \langle \sqrt[3]{\frac{u_{ij}^3(m+1)}{(1+mu_{ij}^3)}}, \sqrt[3]{\frac{u'^3_{ij}}{(m+1-mu'^3_{ij})}} \rangle$$

$$= \langle \sqrt[3]{\frac{(m+1)u_{ij}^3}{1+((m+1)-1)u_{ij}^3}}, \sqrt[3]{\frac{u'^3_{ij}}{(m+1)-((m+1)-1)u'^3_{ij}}} \rangle.$$

When $n = m + 1$

$$n \cdot_H U = (U \oplus_H \ldots \oplus_H U) = \langle \sqrt[3]{\frac{nu_{ij}^3}{1+(n-1)u_{ij}^3}}, \sqrt[3]{\frac{u'^3_{ij}}{n-(n-1)u'^3_{ij}}} \rangle \text{ also holds.}$$

Thus by induction hypothesis, $P(n)$ holds for any positive integer n.

Theorem 14. *If n is any positive integer and U is FFM, then the Hamacher exponentiation opertion $(\wedge_H)$ is*

$$U^{\wedge_H n} = (U \odot_H \ldots \odot_H U) = \langle \sqrt[3]{\frac{u_{ij}^3}{n-(n-1)u_{ij}^3}}, \sqrt[3]{\frac{nu'^3_{ij}}{1+(n-1)u'^3_{ij}}} \rangle \tag{11}$$

where $(U \odot_H \ldots \odot_H U)$ represents the n times Hamacher exponentiation of U. The expression of Eq. (11) is denoted by $P'(n)$.

Proof: Using mathematical induction, we can prove the equation (11), which holds for any positive integer n.

$$U^{\wedge_H 1} = \langle \sqrt[3]{\frac{u_{ij}^3 u_{ij}^3}{u_{ij}^3 + u_{ij}^3 - u_{ij}^3 u_{ij}^3}}, \sqrt[3]{\frac{u'^3_{ij} + u'^3_{ij} - 2u'^3_{ij} u'^3_{ij}}{1 - u'^3_{ij} u'^2_{ij}}} \rangle$$

$$= \langle \sqrt[3]{\frac{u_{ij}^6}{2u_{ij}^3 - u_{ij}^6}}, \sqrt[3]{\frac{2u'^3_{ij} - 2u'^6_{ij}}{1 - u'^6_{ij}}} \rangle$$

$$= \langle \sqrt[3]{\frac{u_{ij}^3}{2 - u_{ij}^3}}, \sqrt[3]{\frac{2u'^3_{ij}}{1 + u'^3_{ij}}} \rangle$$

$$U^{\wedge_H 2} = \langle \sqrt[3]{\frac{u_{ij}^3}{2 - (2-1)u_{ij}^3}}, \sqrt[3]{\frac{2u'^3_{ij}}{1 + (2-1)u'^3_{ij}}} \rangle$$

$$U^{\wedge_H n} = \langle \sqrt[3]{\frac{u_{ij}^3}{n - (n-1)u_{ij}^3}}, \sqrt[3]{\frac{nu'^3_{ij}}{1 + (n-1)u'^3_{ij}}}. \rangle$$

Thus, $P'(n)$ holds.

Suppose that Eq. (11) holds for $n = m$, then

$$U^{\wedge_H m} = (U \odot_H \ldots \odot_H U) = \langle \sqrt[3]{\frac{u_{ij}^3}{m - (m-1)u_{ij}^3}}, \sqrt[3]{\frac{mu'^3_{ij}}{1 + (m-1)u'^3_{ij}}} \rangle \tag{12}$$

When $n = m + 1$,

$$U^{\wedge_H (m+1)} = \langle \sqrt[3]{\frac{u_{ij}^3}{(m+1) - ((m+1)-1)u_{ij}^3}}, \sqrt[3]{\frac{(m+1)u'^3_{ij}}{1 + ((m+1)-1)u'^3_{ij}}} \rangle$$

$$U^{\wedge_H n} = (U \odot_H \ldots \odot_H U) = \langle \sqrt[3]{\frac{u_{ij}^3}{n - (n-1)u_{ij}^3}}, \sqrt[3]{\frac{nu'^3_{ij}}{1 + (n-1)u'^3_{ij}}} \rangle$$

also holds.

Use the induction hypothesis, $P'(n)$ holds for any positive integer n.

Note that $n \cdot_H U$ and $U^{\wedge_H n}$ are also FFMs.

Theorem 15. *For any FFM U, and any positive integer n, $n \cdot_H U$ and $U^{\wedge_H n}$ are also FFMs.*

Proof: $0 \le u_{ij}^3 \le 1;\ 0 \le u'^3_{ij} \le 1;\ \ 0 \le u_{ij}^3 + u'^3_{ij} \le 1 \le 2;$

$n > 1, (n-1)u_{ij}^3 > -1, \text{i.e.}\ 1 + (n-1)u_{ij}^3 > 0; n - (n-1)u'^3_{ij} = u'^3_{ij} + n(1 - u'^3_{ij})$
$\ge u'^3_{ij}.$

Then we have, $\sqrt[3]{\frac{nu_{ij}^3}{1 + (n-1)u_{ij}^3}} \ge 0 \quad \sqrt[3]{\frac{u'^3_{ij}}{n - (n-1)u'^3_{ij}}} \ge 0.$

We consider that

$$1+(n-1)u_{ij}^3 = nu_{ij}^3 + 1 - u_{ij}^3 \geq nu_{ij}^3$$

and

$$n-(n-1)u'^3_{ij} = u'^3_{ij} + n(1-u'^3_{ij}) \geq u'^3_{ij}$$

Thus,

$$\sqrt[3]{\frac{nu_{ij}^3}{1+(n-1)u_{ij}^3}} \leq 1, \sqrt[3]{\frac{u'^3_{ij}}{n-(n-1)u'^3_{ij}}} \leq 1.$$

$$u_{ij}^3 + u'^3_{ij} \leq 1 \Rightarrow 0 \leq u'^3_{ij} \leq 1 - u_{ij}^3$$

$$\Rightarrow \sqrt[3]{\frac{nu_{ij}^3}{1+(n-1)u_{ij}^3}} + \sqrt[3]{\frac{u'^3_{ij}}{n-(n-1)u'^3_{ij}}}$$

$$= \sqrt[3]{\frac{nu_{ij}^3}{1+(n-1)u_{ij}^3}} + \sqrt[3]{\frac{1}{\frac{n}{u'^3_{ij}}-(n-1)}}$$

$$\leq \sqrt[3]{\frac{nu_{ij}^3}{1+(n-1)u_{ij}^3}} + \sqrt[3]{\frac{1}{\frac{n}{1-u_{ij}^3}-(n-1)}}; \because 0 \leq u'^3_{ij} \leq 1 - u_{ij}^3$$

Therefore,

$$0 \leq \sqrt[3]{\frac{nu_{ij}^3}{1+(n-1)u_{ij}^3}} \leq 1, 0 \leq \sqrt[3]{\frac{u'^3_{ij}}{n-(n-1)u'^3_{ij}}} \leq 1$$

$$\Rightarrow \sqrt[3]{\frac{nu_{ij}^3}{1+(n-1)u_{ij}^3}} + \sqrt[3]{\frac{u'^3_{ij}}{n-(n-1)u'^3_{ij}}} \leq 1$$

In the same way, we get

$$0 \leq \sqrt[3]{\frac{u_{ij}^3}{n-(n-1)u_{ij}^3}} \leq 1, 0 \leq \sqrt[3]{\frac{nu'^3_{ij}}{1+(n-1)u'^3_{ij}}} \leq 1$$

$$\Rightarrow \sqrt[3]{\frac{u_{ij}^3}{n-(n-1)u_{ij}^3}} + \sqrt[3]{\frac{nu'^3_{ij}}{1+(n-1)u'^3_{ij}}} \leq 1.$$

Hence, $n \cdot_H U$ and $U^{\wedge_H n}$ are also FFMs.

Theorem 16. *Let* U, V *any two FFMs of the same size and for any integer* n*.*

(i) $n \cdot_H (U \cap V) = (n \cdot_H U) \cap (n \cdot_H V)$,
(ii) $n \cdot_H (U \cup V) = (n \cdot_H U) \cup (n \cdot_H V)$,
(iii) $(U \cap V)^{\wedge_H n} = (U)^{\wedge_H n} \cap (V)^{\wedge_H n}$,
(iv) $(U \cup V)^{\wedge_H n} = (U)^{\wedge_H n} \cup (V)^{\wedge_H n}$.

Proof: (i) We know that

$$(U \cap V) = \langle min(u_{ij}, v_{ij}), max(u'_{ij}, v'_{ij})\rangle.$$

Then

$n \cdot_H (U \cap V) = \langle w_{ij}, w'_{ij}\rangle,\ n \cdot_H U = \langle d_{ij}, d'_{ij}\rangle, n \cdot_H V = \langle e_{ij}, e'_{ij}\rangle$

where,

$$w_{ij} = \sqrt[3]{\frac{n(min(u_{ij}^3, v_{ij}^3))}{1+(n-1)(min(u_{ij}^3, v_{ij}^3))}}$$

$$w_{ij} = min\langle \sqrt[3]{\frac{n(u_{ij}^3)}{1+(n-1)(u_{ij}^3)}}, \sqrt[3]{\frac{n(v_{ij}^3)}{1+(n-1)v_{ij}^3}}\rangle$$

$$w_{ij} = min\langle d_{ij}, e_{ij}\rangle \tag{13}$$

and

$$w'_{ij} = \sqrt[3]{\frac{max(u'^3_{ij}, v'^3_{ij}))}{n-(n-1)(min(u'^3_{ij}, v'^3_{ij}))}}$$

$$w'_{ij} = max\langle \sqrt[3]{\frac{(u'^3_{ij})}{n-(n-1)(u'^3_{ij})}}, \sqrt[3]{\frac{(v'^3_{ij})}{n-(n-1)v'^3_{ij}}}\rangle$$

$$w'_{ij} = max\langle d'_{ij}, e'_{ij}\rangle \tag{14}$$

Comparing Eqs. (13) and (14), we have

$$(n \cdot_H U) \cap (n \cdot_H V) = \langle min(d_{ij}, d'_{ij}), max(e_{ij}, e'_{ij})\rangle$$

$$(n \cdot_H U) \cap (n \cdot_H V)$$
$$= min\langle \sqrt[3]{\frac{nu_{ij}^3}{1+(n-1)u_{ij}^3}}, \sqrt[3]{\frac{u'^3_{ij}}{n-(n-1)u'^3_{ij}}}\rangle, max\langle \sqrt[3]{\frac{nv_{ij}^3}{1+(n-1)v_{ij}^3}}, \sqrt[3]{\frac{v'^3_{ij}}{n-(n-1)v'^3_{ij}}}\rangle.$$

Thus, we get $n \cdot_H (U \cap V) = (n \cdot_H U) \cap (n \cdot_H V)$.

(ii) Prof is similar to above.

(iii) We know that

$$(U \cap V) = \langle min(u_{ij}, v_{ij}), max(u'_{ij}, v'_{ij})\rangle$$

then $(U \cap V)^{\wedge_H n} = \langle w_{ij}, w'_{ij}\rangle, (U)^{\wedge_H n} = \langle d_{ij}, d'_{ij}\rangle, (V)^{\wedge_H n} = \langle e_{ij}, e'_{ij}\rangle$

where,

$$w_{ij} = \sqrt[3]{\frac{(min(u_{ij}^3, v_{ij}^3))}{n-(n-1)(min(u_{ij}^3, v_{ij}^3))}}$$

$$w_{ij} = min\langle \sqrt[3]{\frac{(u_{ij}^3)}{n-(n-1)(u_{ij}^3)}}, \sqrt[3]{\frac{n(v_{ij}^3)}{n-(n-1)v_{ij}^3}}\rangle$$

$$w_{ij} = min\langle d_{ij}, e_{ij}\rangle. \tag{15}$$

and

$$w'_{ij} = \sqrt[3]{\frac{n(max(u'^3_{ij}, v'^3_{ij})}{1+(n-1)(max(u'^3_{ij}, v'^3_{ij}))}}$$

$$w'_{ij} = max\langle \sqrt[3]{\frac{n(u'^3_{ij})}{1+(n-1)(u'^3_{ij})}}, \sqrt[3]{\frac{n(v'^3_{ij})}{1+(n-1)v'^3_{ij}}} \rangle$$

$$w'_{ij} = max\langle d'_{ij}, e'_{ij} \rangle \tag{16}$$

Comparing Eqs. (15) and (16), we have

$$(n \cdot_H U) \cap (U)^{\wedge_H n} \cap (V)^{\wedge_H n} = \langle min(d_{ij}, d'_{ij}), max(e_{ij}, e'_{ij}) \rangle$$

So, $(n \cdot_H U) \cap (U)^{\wedge_H n} \cap (V)^{\wedge_H n}$

$$= min\langle \sqrt[3]{\frac{u^3_{ij}}{n-(n-1)u^3_{ij}}}, \sqrt[3]{\frac{nu'^3_{ij}}{1+(n-1)u'^3_{ij}}} \rangle, \max\langle \sqrt[3]{\frac{v^3_{ij}}{n-(n-1)v^3_{ij}}}, \sqrt[3]{\frac{nv'^3_{ij}}{1+(n-1)v'^3_{ij}}} \rangle$$

(iv) Similarly, we can prove $(U \cup V)^{\wedge_H n} = (U)^{\wedge_H n} \cup (V)^{\wedge_H n}$.

7. Necessity and possibility operators on FFSs

Definition 3. *For any FFM U, the necessity $(\square)$ and the possibility $(\diamond)$ operators are defined as follows:*

$$\square\, U = \langle u_{ij}, \sqrt[3]{1-u^3_{ij}} \rangle$$

and

$$\diamond\, U = \langle \sqrt[3]{1-u'^3_{ij}}, u'_{ij} \rangle$$

Theorem 17. *For U, V be any two FFMs of the same size, then*

(i) $\square\,(U \oplus_H V) = \square\, U \oplus_H \square\, V$,

(ii) $\diamond\,(U \oplus_H V) = \diamond\, U \oplus_H \diamond\, V$.

Proof: (i)

$$\square\,(U \oplus_H V) = \langle \sqrt[3]{\frac{u^3_{ij}+v^3_{ij}-2u^3_{ij}v^3_{ij}}{1-u^3_{ij}v^3_{ij}}}, \sqrt[3]{1-\frac{u^3_{ij}+v^3_{ij}-2u^3_{ij}v^3_{ij}}{1-u^3_{ij}v^3_{ij}}} \rangle$$

$$= \langle \sqrt[3]{\frac{u^3_{ij}+v^3_{ij}-2u^3_{ij}v^3_{ij}}{1-u^3_{ij}v^3_{ij}}}, \sqrt[3]{\frac{1-u^3_{ij}v^3_{ij}-u^3_{ij}-v^3_{ij}+2u^3_{ij}v^3_{ij}}{1-u^3_{ij}v^3_{ij}}} \rangle$$

$$= \langle \sqrt[3]{\frac{u^3_{ij}+v^3_{ij}-2u^3_{ij}v^3_{ij}}{1-u^3_{ij}v^3_{ij}}}, \sqrt[3]{\frac{(1-u^3_{ij})(1-v^3_{ij})}{1-u^3_{ij}v^3_{ij}}} \rangle \tag{17}$$

Now,

$$\square\, U \oplus_H \square\, V = \langle u_{ij}, \sqrt[3]{1-u^3_{ij}} \rangle \oplus_H \langle v_{ij}, \sqrt[3]{1-v^3_{ij}} \rangle$$

$$= \langle \sqrt[3]{\frac{u^3_{ij}+v^3_{ij}-2u^3_{ij}v^3_{ij}}{1-u^3_{ij}v^3_{ij}}}, \sqrt[3]{\frac{(1-u^3_{ij})(1-v^3_{ij})}{1-u^3_{ij}v^3_{ij}}} \rangle \tag{18}$$

From Eqs. (17) and (18), we have $\Box\,(U \oplus_H V) = \Box\, U \oplus_H \Box\, V$.

(ii)

$$\diamond\,(U \oplus_H V) = \left\langle \sqrt[3]{1 - \frac{u'^3_{ij} v'^3_{ij}}{u'^3_{ij} + v'^3_{ij} - u'^3_{ij} v'^3_{ij}}}, \sqrt[3]{\frac{u'^3_{ij} v'^3_{ij}}{u'^3_{ij} + v'^3_{ij} - u'^3_{ij} v'^3_{ij}}} \right\rangle$$

$$= \left\langle \sqrt[3]{\frac{u'^3_{ij} + v'^3_{ij} - 2u'^3_{ij} v'^3_{ij}}{u'^3_{ij} + v'^3_{ij} - u'^3_{ij} v'^3_{ij}}}, \sqrt[3]{\frac{u'^3_{ij} v'^3_{ij}}{u'^3_{ij} + v'^3_{ij} - u'^3_{ij} v'^3_{ij}}} \right\rangle$$

$$= \diamond\, U \oplus_H \diamond\, V.$$

Theorem 18. *For U, V be any two FFMs of the same size, then*

(i) $\Box\,(U \odot_H V) = \Box\, U \odot_H \Box\, V$,

(ii) $\diamond\,(U \odot_H V) = \diamond\, U \odot_H \diamond\, V$.

Proof: (i)

$$\Box\,(U \odot_H V) = \left\langle \sqrt[3]{\frac{u^3_{ij} v^3_{ij}}{u^3_{ij} + v^3_{ij} - u^3_{ij} v^3_{ij}}}\ \sqrt[3]{1 - \frac{u^3_{ij} v^3_{ij}}{u^3_{ij} + v^3_{ij} - u^3_{ij} v^3_{ij}}} \right\rangle$$

$$= \left\langle \sqrt[3]{\frac{u^3_{ij} v^3_{ij}}{u^3_{ij} + v^3_{ij} - u^3_{ij} v^3_{ij}}}, \sqrt[3]{\frac{u^3_{ij} + v^3_{ij} - 2u^3_{ij} v^3_{ij}}{u^3_{ij} + v^3_{ij} - u^3_{ij} v^3_{ij}}} \right\rangle$$

$$= \Box\, U \odot_H \Box\, V.$$

(ii)

$$\diamond\,(U \odot_H V) = \left\langle \sqrt[3]{1 - \frac{u'^3_{ij} + v'^3_{ij} - 2u'^3_{ij} v'^3}{1 - u'^3_{ij} v'^3_{ij}}}, \sqrt[3]{\frac{u'^3_{ij} + v'^3_{ij} - 2u'^3_{ij} v'^3_{ij}}{1 - u'^3_{ij} v'^3_{ij}}} \right\rangle$$

$$= \left\langle \sqrt[3]{\frac{(1 - u'^3_{ij})(1 - v'^3_{ij})}{1 - u'^3_{ij} v'^3_{ij}}}, \sqrt[3]{\frac{u'^3_{ij} + v'^3_{ij} - 2u'^3_{ij} v'^3_{ij}}{1 - u'^3_{ij} v'^3_{ij}}} \right\rangle$$

$$= \diamond\, U \odot_H \diamond\, V.$$

Theorem 19. *For U, V be any two FFMs of the same size, then*

(i) $(\Box\,(U^c \oplus_H V^c))^c = \diamond\, U \odot_H \diamond\, V$,

(ii) $(\Box\,(U^c \odot_H V^c))^c = \diamond\, U \oplus_H \diamond\, V$.

Proof: (i)

$$\Box\,(U^c \oplus_H V^c) = \left\langle \sqrt[3]{\frac{u'^3_{ij} + v'^3_{ij} - 2u'^3_{ij} v'^3_{ij}}{1 - u'^3_{ij} v'^3_{ij}}}, \sqrt[3]{1 - \frac{u'^3_{ij} + v'^3_{ij} - 2u'^3_{ij} v'^3}{1 - u'^3_{ij} v'^3_{ij}}} \right\rangle$$

$$= \left\langle \sqrt[3]{\frac{u'^3_{ij} + v'^3_{ij} - 2u'^3_{ij} v'^3_{ij}}{1 - u'^3_{ij} v'^3_{ij}}}, \sqrt[3]{\frac{(1 - u'^3_{ij})(1 - v'^3_{ij})}{1 - u'^3_{ij} v'^3_{ij}}} \right\rangle.$$

Now,

$$(\Box\,(U^c \oplus_H V^c))^c = \left\langle \sqrt[3]{\frac{(1 - u'^3_{ij})(1 - v'^3_{ij})}{1 - u'^3_{ij} v'^3_{ij}}}, \sqrt[3]{\frac{u'^3_{ij} + v'^3_{ij} - 2u'^3_{ij} v'^3_{ij}}{1 - u'^3_{ij} v'^3_{ij}}} \right\rangle$$

Again,

$$\diamond U \odot_H \diamond V = \left\langle \sqrt[3]{\frac{(1-u'^3_{ij})(1-v'^3_{ij})}{1-u'^3_{ij}v'^3_{ij}}}, \sqrt[3]{\frac{u'^3_{ij}+v'^3_{ij}-2u'^3_{ij}v'^3_{ij}}{1-u'^3_{ij}v'^3_{ij}}} \right\rangle$$

Therefore, we get $(\square\, (U^c \oplus_H V^c))^c = \diamond U \odot_H \diamond V$.

Theorem 20. *For* U, V *be any two FFMs of the same size, then*

(i) $(\diamond\, (U^c \oplus_H V^c))^c = \square U \odot_H \square V$,

(ii) $(\diamond\, (U^c \odot_H V^c))^c = \square U \oplus_H \square V$.

Proof: (i)

$$(U^c \odot_H V^c) = \left\langle \sqrt[3]{1 - \frac{u^3_{ij}v^3_{ij}}{u^3_{ij}+v^3_{ij}-u^3_{ij}v^3_{ij}}}, \sqrt[3]{\frac{u^3_{ij}v^3_{ij}}{u^3_{ij}+v^3_{ij}-u^3_{ij}v^3_{ij}}} \right\rangle$$

$$= \left\langle \sqrt[3]{\frac{u^3_{ij}+v^3_{ij}-2u^3_{ij}v^3_{ij}}{u^3_{ij}+v^3_{ij}-u^3_{ij}v^3_{ij}}}, \sqrt[3]{\frac{u^3_{ij}v^3_{ij}}{u^3_{ij}+v^3_{ij}-u^3_{ij}v^3_{ij}}} \right\rangle.$$

Now,

$$(\diamond\, (U^c \oplus_H V^c))^c = \left\langle \sqrt[3]{\frac{u^3_{ij}v^3_{ij}}{u^3_{ij}+v^3_{ij}-u^3_{ij}v^3_{ij}}}, \sqrt[3]{\frac{u^3_{ij}+v^3_{ij}-2u^3_{ij}v^3_{ij}}{u^3_{ij}+v^3_{ij}-u^3_{ij}v^3_{ij}}} \right\rangle$$

Again,

$$\square U \odot_H \square V = \left\langle \sqrt[3]{\frac{u^3_{ij}v^3_{ij}}{u^3_{ij}+v^3_{ij}-u^3_{ij}v^3_{ij}}}, \sqrt[3]{\frac{u^3_{ij}+v^3_{ij}-2u^3_{ij}v^3_{ij}}{u^3_{ij}+v^3_{ij}-u^3_{ij}v^3_{ij}}} \right\rangle$$

Therefore, $(\diamond\, (U^c \oplus_H V^c))^c = \square U \odot_H \square V$.

(ii) Prof is similar.

Theorem 21. *For any FFM and for any positive integer* n,

(i) $\square\, (n \cdot_H U) = n \cdot_H (\square\, U)$,

(ii) $\diamond\, (n \cdot_H U) = n \cdot_H (\diamond\, U)$,

(iii) $\square\, U^{\wedge_H n} = (\square\, U)^{\wedge_H n}$,

(iv) $\diamond\, U^{\wedge_H n} = (\diamond\, U)^{\wedge_H n}$.

Proof: (i)

$$\square\, (n \cdot_H U) = \left\langle \sqrt[3]{\frac{nu^3_{ij}}{1+(n-1)u^3_{ij}}}, \sqrt[3]{1 - \frac{nu^3_{ij}}{1+(n-1)u^3_{ij}}} \right\rangle$$

$$= \left\langle \sqrt[3]{\frac{nu^3_{ij}}{1+(n-1)u^3_{ij}}}, \sqrt[3]{\frac{1+(n-1)u^3_{ij}-nu^3_{ij}}{1+(n-1)u^3_{ij}}} \right\rangle$$

$$= \left\langle \sqrt[3]{\frac{nu^3_{ij}}{1+(n-1)u^3_{ij}}}, \sqrt[3]{\frac{1+nu^3_{ij}-u^3_{ij}-nu^3_{ij}}{1+(n-1)u^3_{ij}}} \right\rangle$$

$$\Box\,(n\cdot_H U)=\langle\sqrt[3]{\frac{nu_{ij}^3}{1+(n-1)u_{ij}^3}},\sqrt[3]{\frac{1-u_{ij}^3}{1+(n-1)u_{ij}^3}}\rangle. \tag{19}$$

Again,

$$n\cdot_H(\Box\,U)=\langle\sqrt[3]{\frac{nu_{ij}^3}{1+(n-1)u_{ij}^3}},\sqrt[3]{\frac{1-u_{ij}^3}{n-(n-1)(1-u_{ij}^3)}}\rangle$$

$$=\langle\sqrt[3]{\frac{nu_{ij}^3}{1+(n-1)u_{ij}^3}},\sqrt[3]{\frac{1-u_{ij}^3}{n-n+nu_{ij}^3+1-u_{ij}^3}}\rangle$$

$$n\cdot_H(\Box\,U)=\langle\sqrt[3]{\frac{nu_{ij}^3}{1+(n-1)u_{ij}^3}},\sqrt[3]{\frac{1-u_{ij}^3}{1+(n-1)u_{ij}^3}}\rangle. \tag{20}$$

$\Box\,(n\cdot_H U)=n\cdot_H(\Box\,U)$.

Other proofs are similar.

8. Conclusion

In this paper, Hamacher operations for PFMs were formulated based on the principles of PFMs. Their algebraic properties were explored, and it was demonstrated that the set of all PFMs, under Hamacher addition and multiplication, constitutes a commutative monoid. The algebraic structure of PFMs under Hamacher operations was examined, providing deep insights into their practical applications. Additionally, the validity of De Morgan laws was confirmed.

Hamacher operations were also extended to the Fermatean fuzzy framework in this research. Hamacher operations for FFMs were developed and analyzed, and their algebraic properties were examined. It was shown that the set of FFMs, under Hamacher addition and multiplication, also forms a commutative monoid. The algebraic structure of Fermatean fuzzy matrices under Hamacher operations was studied, offering significant insights into their practical applications. Furthermore, the validity of De Morgan laws was confirmed. Scalar multiplication and exponentiation operations on FFMs were introduced, and their algebraic properties were examined. Finally, several properties of necessity and possibility operators applied to FFMs were verified.

It should be noted that the Hamacher operations for FFMs presented here hold potential for future applications in the aggregation of Fermatean fuzzy information.

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