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Hybrid Heston-SABR Model: A Comparative Study of Monte-Carlo and Finite Difference Numerical Methods

Mwenya Joel*[1], Elias R Offen[2], Emmanuel E Sinkwembe[3] **and** ***Silas S Mirau[4]***

[1,4]Department of Applied and Computational Mathematics,
Nelson Mandela African Institute of Science and Technology,
P.O. Box 447, Arusha, Tanzania.
[1]E-mail: mwenyaj@yahoo.com
[2]Department of Mathematics, University of Botswana, Plot 4775 Village,
Gaborone, Botswana.
[2]E-mail: offen@ub.ac.bw
[3]Department of Mathematics, University of Dar Es Salaam, P.O. Box 35091,
Dar Es Salaam, Tanzania.
[3]E-mail: sinkwembe2001@gmail.com
[4]E-mail: silas.mirau@authornm-aist.ac.tz
*Corresponding author

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Abstract. Numerical techniques play a crucial role in the derivative pricing of options, particularly when no closed-form analytical formula exists. This study aims to compare two prominent numerical methods: Monte Carlo simulation and finite difference methods, as they are applied to the hybrid Heston-SABR model. The hybrid Heston-SABR model, provides a sophisticated basis for modeling the dynamics of financial derivatives. In this paper, we explore these two primary numerical methods commonly employed by financial experts to determine option prices. We evaluate the performance, accuracy, and computational efficiency of Monte Carlo simulation and finite difference methods in solving the partial differential equations (PDEs) and pricing options under the stated model. We assess the convergence of both methods for valuing European options within the hybrid Heston-SABR framework.

We observed that when pricing European options under this model, both approaches converge faster, are more accurate, and are unconditionally stable. We have also established that the numerical method's accuracy and stability are affected by the maturity time. Further, we have determined that changing the maturity time T affects the trade-off between numerical accuracy and computing efficiency in pricing European call options for the two numerical approaches under this hybrid model.

Keywords: Derivative pricing; Finite Difference methods; Monte Carlo Simulation; Numerical methods; Hybrid Heston-SABR model, Stochastic volatility.

AMS Mathematics Subject Classification (2020): 65C05, 65M06, 91G20 and 91G60.

1. Introduction

Numerical methods are very powerful in approximating the values of equations, integrals and differential equations where analytical solutions fail. Two examples of numerical methods for solving differential equations are the Monte Carlo methods and the Finite difference methods. These two are some of the methods used in finance.

Boyle et al. [1] was a pioneering researcher who introduced the Monte Carlo method to the realm of finance. This numerical technique is widely employed and proves valuable in situations where no closed-form solution exists. Particularly adept at valuing both standard and contingent options, it leverages risk assessment results. In an environment where risk is neutral, a sampling technique is used to calculate the expected payoff. The key steps involved in applying this method are as follows: Simulate the trajectory of the underlying asset under risk-neutral conditions for the desired time horizon, and discount the payoff associated with each simulated path. Reiterate this process for a substantial quantity of simulated sample routes and then determine the option's value by taking the mean of the discounted cash flows for the entire sample pathways.

In their seminal work [2] were pioneers in utilizing finite difference methods to price options that lacked closed-form solutions. They focused on valuing American options on stocks with discrete dividend payments. By approximating the differential equation across the integration area using a system of algebraic equations, the finite difference method also tackled the Black-Scholes partial differential equation, as elaborated by [3]. The three branches of finite difference methods of relevance include: Explicit Finite Difference which is simple and easy to implement, but may suffer from stability issues, Implicit Finite Difference which is stable but requires solving nonlinear equations and the Crank-Nicolson Scheme which is a compromise between explicit and implicit methods, and in our working we implement this type of finite difference method.

The hybrid Heston-SABR model is an interesting area of research that combines elements from both models. The study conducted by [4] explores the intricate connection between the Heston and SABR models. It meticulously examines how an expansion of the implied volatility within the Heston model is formulated to align with the implied volatility patterns observed in the SABR model. The paper authored by [5] examines proficient stochastic volatility models, notably exploring ZABR (Zero-Alpha Beta Rho) models. It underscores the significance of comprehending the dynamic interactions among diverse stochastic volatility models and their practical implications. In their paper [6] introduces an additional component to the Heston-Hull-White (HHW) hybrid model. The extended model combines the Heston stochastic volatility model with the Hull-White stochastic interest rate model, with the goal of capturing the relationship between the underlying asset price and the interest rate while maintaining analytical tractability. The authors have effectively obtained an analytical expression for the characteristic function of the underlying asset price within this context. Leveraging this expression, they offer a closed-form European option pricing formula based on the two-factor HHW hybrid model.

To underscore the significance of incorporating correlation, empirical analyses are conducted to compare the model's performance with that of the traditional HHW model, particularly using European options linked to the SP 500 index.

Although not directly pertaining to the Hybrid Heston-SABR model, there exist supplementary investigations on interconnected subjects, these include [7-12]. These have given us the impetus to have an insight and give a comparative study of the Monte Carlo and the Finite difference numerical methods on the hybrid Heston-SABR model.

2. Materials and methods (Some mathematical tools).

As given by [13], in this paper, we examine the following model, described by

$$\begin{cases} d\Pi(t) = r\Pi(t)dt + \sqrt{\eta(t)}\Pi^{\beta}(t)dW^{\Pi}(t) \\ d\eta(t) = \kappa(\theta \quad \eta(t))dt + \alpha\sqrt{\eta(t)}dW^{\eta}(t), \\ \qquad \text{where } r \in R, \beta \in [0,1) \text{ and } \kappa, \theta, \alpha, \eta(0) > 0 \end{cases} \tag{1}$$

for $0 < t < T$, the given maturity time of the option. $\Pi(t)$ and $\eta(t)$ are random variables that represent the asset price and its variance at time t, respectively. The parameters κ, θ, α, β are real numbers, $dW^{\Pi}(t)$ and $dW^{\eta}(t)$ are Wiener processes, with $\rho \in [-1,1)$.

To obtain the partial differential equation (PDE) corresponding to the Hybrid Heston-SABR model, as specified in equation (1), we will use the Itô's lemma to express the dynamics of a function of the state variables and then equating the resulting expression to zero. The function we are interested in is the option price, which depends on both the stock price, $\Pi(t)$, and the stochastic volatility, $\eta(t)$. Let us define the option price as $\Lambda(t,\Pi,\eta)$, in which the stock price is Π, the time is t, and the stochastic volatility is η. Therefore;

$$d\Lambda = \frac{\partial \Lambda}{\partial t}dt + \frac{\partial \Lambda}{\partial \Pi}d\Pi + \frac{\partial \Lambda}{\partial \eta}d\eta$$
$$+\frac{1}{2}\frac{\partial^2 \Lambda}{\partial \Pi^2}\left(d\Pi^2\right) + \frac{1}{2}\frac{\partial^2 \Lambda}{\partial \eta^2}\left(d\eta^2\right) + \frac{\partial^2 \Lambda}{\partial \Pi \partial \eta}\left(d\Pi d\eta\right) \quad . \tag{2}$$

Now, we need to calculate the changes in $d\Pi$ and $d\eta$ using the given SDEs in equation (1), thus, substituting these into the expression for $d\Lambda$, we get:

$$d\Lambda = \frac{\partial \Lambda}{\partial t}dt + \frac{\partial \Lambda}{\partial \Pi}\left[r\Pi dt + \sqrt{\eta}\Pi^{\beta} dW^{\Pi}\right]$$

$$+\frac{\partial \Lambda}{\partial \eta}\left[\kappa(\theta - \eta)dt + \alpha\sqrt{\eta}dW^{\eta}\right] + \frac{1}{2}\frac{\partial^2 \Lambda}{\partial \Lambda^2}\left[\sqrt{\eta}\Pi^{\beta} dW^{\Pi}\right]^2 \tag{3}$$

$$+\frac{1}{2}\frac{\partial^2 \Lambda}{\partial \eta^2}\left[\alpha^2 \eta \Pi^{\beta} dW^{\eta}\right]^2 + \frac{\partial^2 \Lambda}{\partial \Pi \partial \eta}\left[\sqrt{\eta}\Pi^{\beta} dW^{\Pi}\right]\left[\alpha\sqrt{\eta}\Pi^{\beta} dW^{\eta}\right] \quad .$$

Let us simplify the expression by collecting terms and recognizing that dW^{Π} and dW^{η} are stochastic differentials with zero mean and variance dt :

$$d\Lambda = \left[\frac{\partial \Lambda}{\partial t} + r\Pi\frac{\partial \Lambda}{\partial \Pi} + \kappa(\theta - \eta)\frac{\partial \Pi}{\partial \eta}\right)dt$$

$$+\left[\frac{1}{2}\alpha^2\eta\frac{\partial^2 \Lambda}{\partial \eta} + \frac{1}{2}\Pi^{2\beta}\eta\frac{\partial^2 \Lambda}{\partial \Pi^2}\right)dt \tag{4}$$

$$+\sqrt{\eta}\Pi^{\beta}\frac{\partial \Lambda}{\partial \Pi}dW^{\Pi} + \alpha\sqrt{\eta}\frac{\partial \Lambda}{\partial \eta}dW^{\eta} \quad .$$

Then, to obtain the partial differential equation (PDE) for the option price $\Lambda(t, \Pi, \eta)$, we set the coefficient of dt in the above expression to zero:

$$\frac{\partial \Lambda}{\partial t} + r\Pi\frac{\partial \Lambda}{\partial \Pi} + \kappa(\theta - \eta)\frac{\partial \Lambda}{\partial \eta} + \frac{1}{2}\alpha^2\eta\frac{\partial^2 \Lambda}{\partial \eta} + \frac{1}{2}\Pi^{2\beta}\eta\frac{\partial^2 \Lambda}{\partial \Pi^2}. \tag{5}$$

This is the PDE for the option price in the Hybrid Heston-SABR model. It's a parabolic PDE that depends on the stock price Π, the stochastic volatility η, and the parameters r, β, κ, θ and α. Solving this PDE requires numerical methods and we will use the Monte Carlo and finite difference methods.

3. Pricing European option for hybrid Heston-SABR model

We employ the Monte Carlo and Finite difference numerical methods to price European options for the hybrid Heston-SABR model. This model is complex, and the use of these numerical methods can help manage and solve the intricacies involved. Specifically, for the Monte Carlo simulations because of it is flexibility, accuracy in high dimensions and easiness to implement. For the finite difference methods (FDM), it provides a deterministic solution to the PDEs, which can be more stable and accurate for the given problem, it handles boundary conditions well, which is crucial for accurately pricing options and other derivatives. FDM discretizes the problem on a grid, allowing for precise control over the numerical solution and error estimation.

3.1. Monte-Carlo methods

To use Monte Carlo methods to find option prices the hybrid Heston-SABR given by equation (1) with its partial differential equation (5), we need to simulate the stochastic processes involved, the stock price (Π), and the stochastic volatility (η). Then, we can estimate the option price by averaging the payoffs of the simulated paths. The procedure we followed when using this method are; we firstly initialize the parameters of the model, then simulate the paths for both the stock and the volatility, using Euler-Maruyama scheme. This is done by initializing arrays Π_{paths} and η_{paths} with zeros to store the simulated paths. Set the initial values for stock price (Π_0) and volatility (η_0). For each path we generate random increments dW^{Π} and dW^{η} using normal distributions. Then, update the stock price path using the stochastic differential equation (SDE):

$$\Pi_{i,j+1} = \Pi_{i,j} + r \quad \Pi_{i,j} \quad dt + \sqrt{max(\eta_{i,j}, 0)} \Pi_{i,j}^{\beta} dW_s \, . \tag{6}$$

Update the volatility path using another SDE:

$$\eta_{i,j+1} = \eta_{i,j} + \kappa \quad \left(\theta - max\left(\eta_{i,j}, 0\right)\right) \quad dt + \alpha \quad \sqrt{max(\eta_{i,j}, 0)} dW_v \tag{7}$$

Next, we compute the option payoffs based on the simulated stock price paths and a given strike price (K). Which returns the maximum of zero and the disparity between the ultimate stock price and the strike price can be represented as:

$$\text{Payoff} = max\left(\Pi_{\text{final}} - K, 0\right)$$

Finally, we estimate the option price by simulating stock price and volatility paths using simulated paths to calculate option payoffs. Taking the average of the option payoffs and discounting it to present value:

$$\text{Option Price} = \frac{1}{\text{num\_paths}} \sum_{i=1}^{\text{num\_paths}} \text{Payoff}_i \quad e^{\,rT} \, . \tag{8}$$

3.2. Finite difference methods

This method entails discretizing both the partial differential equation and the boundary conditions, accomplished by employing either a forward or a backward difference approximation. The PDE given in (5) is discretized in both the time and space variables. Let us denote the grid points in time as t_i (where $t_i = \Delta\Pi$) and in space as Π_j and η_k (where, $\Pi_j = j\Delta\Pi$ and $\eta_k = k\Delta\eta$).

$$\frac{\partial \Lambda}{\partial t} \approx \frac{\Lambda_{i+1,j,k} - \Lambda_{i,j,k}}{\Delta t}$$

$$\frac{\partial \Lambda}{\partial \Pi} \approx \frac{\left(\Lambda_{i,j+1,k} - \Lambda_{i,j,k}\right)}{\Delta \Pi}$$

$$\frac{\partial \Lambda}{\partial \eta} \approx \frac{\left(\Lambda_{i,j,k+1} - \Lambda_{i,j,k}\right)}{\Delta \eta}$$

$$\frac{\partial^2 \Lambda}{\partial \Pi^2} \approx \frac{\left(\Lambda_{i,j+1,k} - 2\Lambda_{i,j,k} + \Lambda_{i,j-1,k}\right)}{\Delta \Pi^2}$$

$$\frac{\partial^2 \Lambda}{\partial \eta^2} \approx \frac{\left(\Lambda_{i,j,k+1} - 2\Lambda_{i,j,k} + \Lambda_{i,j,k-1}\right)}{\Delta \eta^2}.$$

Substituting these finite difference approximations into the PDE (5), we get

$$\begin{gathered}\frac{\Lambda_{i+1,j,k} - \Lambda_{i,j,k}}{\Delta t} + r\Pi_j \frac{\left(\Lambda_{i,j+1,k} - \Lambda_{i,j,k}\right)}{\Delta \Pi} + \kappa(\theta \quad \eta)\frac{\left(\Lambda_{i,j,k+1} - \Lambda_{i,j,k}\right)}{\Delta \eta} \\ \frac{1}{2}\alpha\eta_k \frac{\left(\Lambda_{i,j,k+1} - 2\Lambda_{i,j,k} + V_{i,j,k \quad 1}\right)}{\Delta \eta^2} + \frac{1}{2}\Pi_j^{2\beta}\eta_k \frac{\left(\Lambda_{i,j+1,k} - 2\Lambda_{i,j,k} + \Lambda_{i,j \quad 1,k}\right)}{\Delta \Pi^2} \quad .\end{gathered} \tag{9}$$

Rearranging for $\Lambda_{i+1,j,k}$

$$\begin{gathered}\Lambda_{i+1,j,k} = \Lambda_{i,j,k} + \Delta t\left[r\Pi_j \frac{\left(\Lambda_{i,j+1,k} - \Lambda_{i,j,k}\right)}{\Delta \Pi} + \kappa(\theta \quad \eta)\frac{\left(\Lambda_{i,j,k+1} - \Lambda_{i,j,k}\right)}{\Delta \eta}\right] \\ + \Delta t\left[\frac{1}{2}\alpha\eta_k \frac{\left(\Lambda_{i,j,k+1} - 2\Lambda_{i,j,k} + \Lambda_{i,j,k-1}\right)}{\Delta \eta^2} + \frac{1}{2}\Pi_j^{2\beta}\eta_k \frac{\left(\Lambda_{i,j+1,k} \quad 2\Lambda_{i,j,k} + \Lambda_{i,j-1,k}\right)}{\Delta \Pi^2}\right] \quad .\end{gathered} \tag{10}$$

This equation (10) allows us to update the value of Λ at each coordinate point (i,j,k) based on the values at the neighbouring coordinate points and the known parameters.

3.3. Crack-Nicolson finite difference method

The Crack-Nicolson method combines a backward difference for the time derivative and central differences for the space derivatives. Using this approach to numerically solve the equation provided in (5), we get:

$$\frac{\Lambda_{i,j}^{k+1}-\Lambda_{i,j}^{k}}{\Delta t}=\frac{r\Pi_i\Lambda_{i+1,j}^{k+1}-\Lambda_{i-1,j}^{k+1}}{\Delta\Pi}+\kappa\left(\theta-\eta_j\right)\frac{\Lambda_{i,j+1}^{k+1}-\Lambda_{i,j-1}^{k+1}}{2\Delta\eta}$$

$$+\frac{1}{2}\alpha^2\eta_j\left(\frac{\Lambda_{i+1,j}^{k+1}-2\Lambda_{i,j}^{k+1}+\Lambda_{i-1,j}^{k+1}}{\Delta\Pi^2}\right)+\frac{1}{2}\Pi_i^{2\beta}\eta_j\left(\frac{\Lambda_{i,j+1}^{k+1}-2\Lambda_{i,j}^{k+1}+\Lambda_{i,j+1}^{k+1}}{\Delta\eta^2}\right)\quad . \tag{11}$$

Rearrange the equation to solve for $\Lambda_{i,j}^{k+1}$, we get

$$\Lambda_{i,j}^{k+1}=\Lambda_{i,j}^{k}+\Delta t\left[\frac{r\Pi_i\Lambda_{i+1,j}^{k+1}-\Lambda_{i-1,j}^{k+1}}{\Delta\Pi}+\kappa\left(\theta-\eta_j\right)\frac{\Lambda_{i,j+1}^{k+1}-\Lambda_{i,j-1}^{k+1}}{2\Delta\eta}\right)$$

$$+\Delta t\left[\frac{1}{2}\alpha^2\eta_j\left(\frac{\Lambda_{i+1,j}^{k+1}-2V_{i,j}^{k+1}+\Lambda_{i-1,j}^{k+1}}{\Delta\Pi^2}\right)\right) \tag{12}$$

$$+\Delta t\left[\frac{1}{2}\Pi_i^{2\beta}\eta_j\left(\frac{\Lambda_{i,j+1}^{k+1}-2\Lambda_{i,j}^{k+1}+\Lambda_{i,j+1}^{k+1}}{\Delta\eta^2}\right)\right)\quad .$$

3.4. Stability analysis

Since the Crank-Nicolson approach takes into account the average of the option price values at time steps k and $k+1$, it is an implicit finite difference scheme. The time step size Δt needs to meet the following requirements in order to be stable:

$$\frac{\Delta t}{(\Delta\Pi)^2}\leq\frac{1}{2}\quad\text{(for spatial discretization)}\ .$$

$$\frac{\Delta t}{(\Delta\eta)^2}\leq\frac{1}{2}\quad\text{(for volatility discretization)}\ .$$

In terms of both space and time, the Crank-Nicolson approach is second-order accurate, and it reduces numerical oscillations compared to explicit methods. Errors in initial conditions or model parameters will propagate over time. Accurate initialization and calibration of model parameters are crucial for reliable pricing. Consistency, stability, and convergence are the three fundamental elements that characterize a numerical method. A fundamental principle in the realm of numerical techniques for partial differential equations (PDEs) is the Lax equivalence theorem. This theorem, established as a classical result, articulates that a consistent approximation of a linear PDE converges if and only if it demonstrates stability [14]. We have assessed the given discretization scheme in by analyzing the truncation error, stability properties, and convergence behavior. Thus, confirming consistency, stability, and convergence, we can conclude that the method provides an accurate numerical solution.

Consistency means that the finite difference method approximates the correct PDE as the grid spacing approaches zero. To verify consistency, we compare the finite difference equation (FDE) with the PDE and examine the difference (local truncation error) as $\Delta\Pi$ and $\Delta\eta$ tend to zero. Let's analyze the terms in the FDE:

- The first term involves the price gradient: $\dfrac{r\Pi_i \Lambda_{i+1,j}^{k+1} - \Lambda_{i-1,j}^{k+1}}{\Delta\Pi}$.

- The second term involves the volatility gradient: $\kappa(\theta-\eta_j)\dfrac{\Lambda_{i,j+1}^{k+1} - \Lambda_{i,j-1}^{k+1}}{2\Delta\eta}$.

- The third term is the price-related second derivative:

$$\frac{1}{2}\alpha^2\eta_j\left(\frac{\Lambda_{i+1,j}^{k+1} \quad 2\Lambda_{i,j}^{k+1} + \Lambda_{i-1,j}^{k+1}}{\Delta\Pi^2}\right).$$

- The fourth term involves the volatility-related second derivative:

$$\frac{1}{2}\Pi_i^{2\beta}\eta_j\left(\frac{\Lambda_{i,j+1}^{k+1} - 2\Lambda_{i,j}^{k+1} + \Lambda_{i,j+1}^{k+1}}{\Delta\eta^2}\right).$$

The difference between the PDE and FDE approaches zero as $\Delta\Pi$ and $\Delta\eta$ decrease, then the FDE is consistent with the PDE. Thus, FDE is consistent because it approximates the correct PDE as the grid spacing becomes finer. Consistency is a necessary condition for convergence, which ensures that the numerical solution approaches the true solution of the PDE.

The stability of a finite difference scheme depends on its ability to maintain bounded solutions over time. Let's examine the finite difference strategy for the PDE's stability:

$$\begin{aligned}\Lambda_{i,j}^{k+1} = \Lambda_{i,j}^{k} &+ \Delta t\left[\frac{r\Pi_i \Lambda_{i+1,j}^{k+1} - \Lambda_{i-1,j}^{k+1}}{\Delta\Pi} + \kappa(\theta-\eta_i)\frac{\Lambda_{i+1,j}^{k+1} - \Lambda_{i,j-1}^{k+1}}{2\Delta\eta}\right] \\ &+ \Delta t\left[\frac{1}{2}\alpha^2\eta_i\left(\frac{\Lambda_{i+1,j}^{k+1} - 2\Lambda_{i,j}^{k+1} + \Lambda_{i-1,j}^{k+1}}{\Delta\Pi^2}\right)\right] + \Delta t\left[\frac{1}{2}\Pi_i^{2\beta}\eta_j\left(\frac{\Lambda_{i+1,j}^{k+1} - 2\Lambda_{i,j}^{k+1} + \Lambda_{i,j+1}^{k+1}}{\Delta\eta^2}\right)\right].\end{aligned} \tag{13}$$

To assess stability, we perform a von Neumann stability analysis by assuming a solution of the form

$$\Lambda_{i,j}^{k} = A^{k} e^{i(\xi i \Delta x + v j \Delta \eta)}.$$

Substitute this into the finite difference scheme and analyze the amplification factor (A). The scheme is stable if $\left(|\mathrm{A}| \leq 1\right)$ for all (ξ, v).

Let's assume a solution of the form $\Lambda_{i,j,k} = A_k e^{i\left(\xi_i \Delta\Pi + v_j \Delta\eta\right)}$, where A is the amplification factor, $\Delta\Pi$ and $\Delta\eta$ are the spatial step sizes, ξ and v are dimensionless wavenumbers. Substituting this solution into the finite difference scheme given in equation (13), we get:

$$
\begin{aligned}
& A_{k+1} e^{i\left(\xi_i \Delta\Pi + v_j \Delta\eta\right)} \\
& = A_k e^{i\left(\xi_i \Delta\Pi + v_j \Delta\eta\right)} + \Delta t\Big[r\Pi_i A_{k+1} e^{i\left(\xi_{i+1} \Delta\Pi + v_j \Delta\eta\right)} \quad A_{k+1} e^{i\left(\xi_{i\ \ 1} \Delta x + v_j \Delta\eta\right)} \Delta\Pi \Big) \\
& + \kappa\left(\theta \quad \eta_j\right) A_{k+1} e^{i\left(\xi_i \Delta\Pi + v_{j+1} \Delta\eta\right)} \quad A_{k+1} e^{i\left(\xi_i \Delta\Pi + v_{j-1} \Delta\eta\right)} 2\Delta\eta \\
& + \frac{1}{2}\alpha^2 \eta_j \left(A_{k+1} e^{i\left(\xi_{i+1} \Delta\Pi + v_j \Delta\eta\right)} \quad 2A_{k+1} e^{i\left(\xi_i \Delta\Pi + v_j \Delta\eta\right)} + A_{k+1} e^{i\left(\xi_{i-1} \Delta\Pi + v_j \Delta\eta\right)} \right) \Delta\Pi^2 \Big) \\
& + \frac{1}{2}\Pi_i^2 \beta \eta_j \left(A_{k+1} e^{i\left(\xi_i \Delta\Pi + v_{j+1} \Delta\eta\right)} \quad 2A_{k+1} e^{i\left(\xi_i \Delta\Pi + v_j \Delta\eta\right)} + A_{k+1} e^{i\left(\xi_i \Delta\Pi + v_{j+1} \Delta\eta\right)} \right) \Big) \quad .
\end{aligned}
\tag{14}
$$

Now, we simplify and factor out $A_k e^{i\left(\xi_i \Delta\Pi + v_j \Delta\eta\right)}$ to obtain:

$$A = 1 + \Delta t \Phi(\xi, v)$$

where $\Phi(\xi, v)$ represents the stability function. For the scheme to be stable, $|\mathrm{A}| \leq 1$ for all ξ and v. This is equivalent to $|\Phi(\xi, v)| \leq 0$ for all ξ and v. The stability of the scheme depends on the specific values of Δt, $\Delta\Pi$, $\Delta\eta$, r, Π_i, η_j, α, β, κ, θ, and the ranges of ξ and v. The stability conditions for this scheme involved examining the behavior of $\Phi(\xi, v)$ over the entire range of ξ and v.

4. Numerical results and discussion

We evaluate the effectiveness of both numerical methods for pricing a European option within the framework of the hybrid Heston-SABR model alongside different maturity time $(T = 1,2,3)$ for the following parameters: $S_0 = 100$, $r = 0.05$, $\kappa = 1.0$, $\theta = 0.04$, $\alpha = 0.2$, $\beta = 0.5$, $v = 0.1$. We display the numerical examples of these results in Tables 1, 2, 3, for different strike prices at the aforementioned maturity times respectively. In Figure 3(a), we plot both Monte Carlo and Crank-Nicolson option prices for different maturities on the same graph, allowing for easy comparison between the two methods. Figure 3(b) displays the one for Monte Carlo option prices and Figure 3(c) for Crank-Nicolson option prices, each plot displaying the prices for different maturities, using the hybrid Heston-SABR model.

Table 1: Tabular Comparison of Numerical Methods for a European Option

Number	Strike Price	Monte Carlo	Crack-Nicolson
1	20	10.693431126863498	10.693431127382029
2	30	19.865005233288283	19.865005234301258
3	40	30.564984661341530	30.564984662927714
4	50	41.264852997083830	41.264852999241704
5	60	50.436115038742244	50.436115041398450
6	70	61.135879229734300	61.135879232956235
7	80	71.835611878816760	71.835611882617120
8	90	81.006792851434950	81.006792855728920
9	100	91.706487601289520	91.706487606148410
10	110	102.40616852460067	102.40616853002845
11	120	111.57731480037250	111.57731480629903
12	130	122.27697708142286	122.27697708792262
13	140	132.97663188669280	132.97663189377400

Notes: $T = 1$, $S_0 = 100$ $r = 0.05$, $\beta = 0.5$, $\theta = 0.04$, $\kappa = 1.0$, $\alpha = 0.2$

Table 2:

Number	Strike Price	Monte Carlo	Crack-Nicolson
1	20	10.890422368603792	10.890422372962922
2	30	20.236841316091052	20.236841324604477
3	40	31.140608838655440	31.140608851997690
4	50	42.044149975402590	42.044149993565020
5	60	51.389932997921690	51.389933020230146
6	70	62.293261903747960	62.293261930861500
7	80	73.196526531366290	73.196526563302830
8	90	82.542144346221780	82.542144382291200
9	100	93.445331739741400	93.445331780609760
10	110	104.34849095624963	104.34849100191924
11	120	113.69403806270275	113.69403811251152
12	130	124.59715928811504	124.59715934273655
13	140	135.50026527856136	135.50026533800880

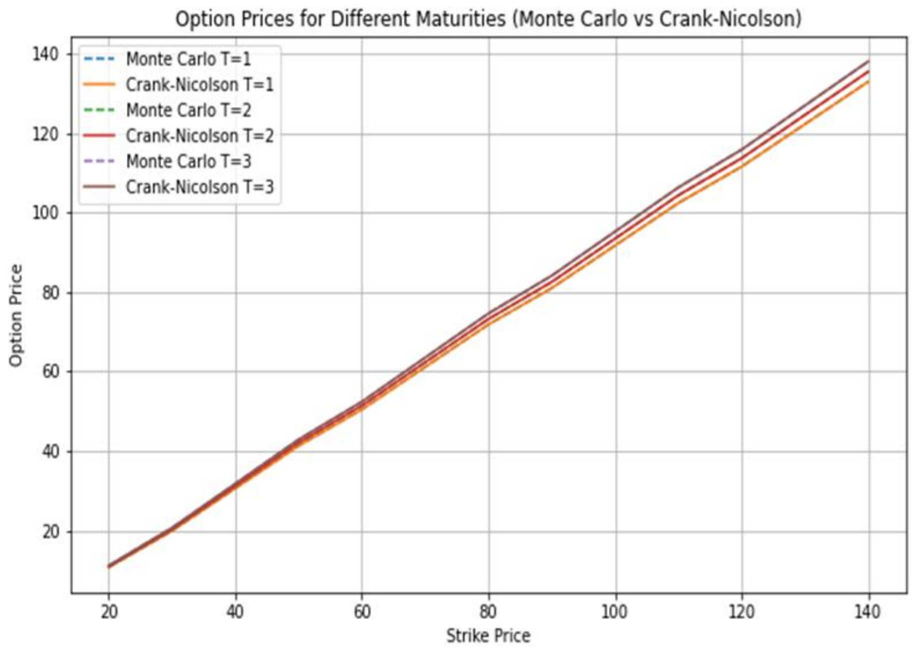
Notes: $T = 2$, $S_0 = 100$ $r = 0.05$, $\beta = 0.5$, $\theta = 0.04$, $\kappa = 1.0$, $\alpha = 0.2$

Table 3:

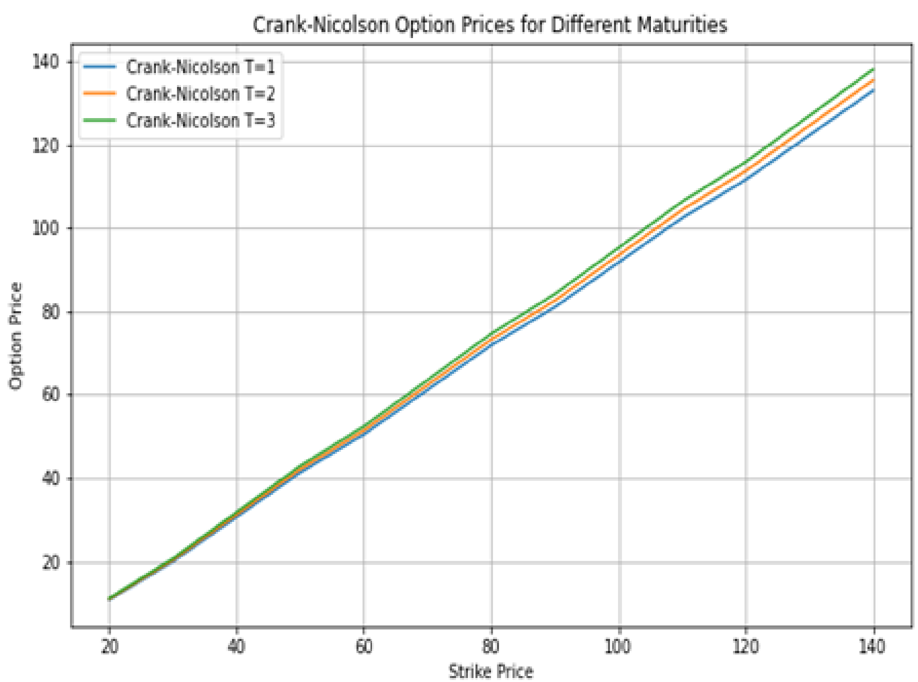
Number	Strike Price	Monte Carlo	Crack-Nicolson
1	20	11.091039185398849	11.091039200466130
2	30	20.615636056679957	20.615636086114012
3	40	31.727072875972485	31.727072922114623
4	50	42.838163695596100	42.838163758415990
5	60	52.361788651919840	52.361788729051360
6	70	63.472555101162230	63.472555194934040
7	80	74.583223307588140	74.583223418027740
8	90	84.106595760875820	84.106595885593720
9	100	95.217145921903440	95.217146063241200
10	110	106.32765301663643	106.32765317460012
11	120	115.85091739789384	115.85091757013700
12	130	126.96136642603899	126.96136661492570
13	140	138.07179216856824	138.07179237410844

Notes: $T = 3$, $S_0 = 100$ $r = 0.05$, $\beta = 0.5$, $\theta = 0.04$, $\kappa = 1.0$, $\alpha = 0.2$

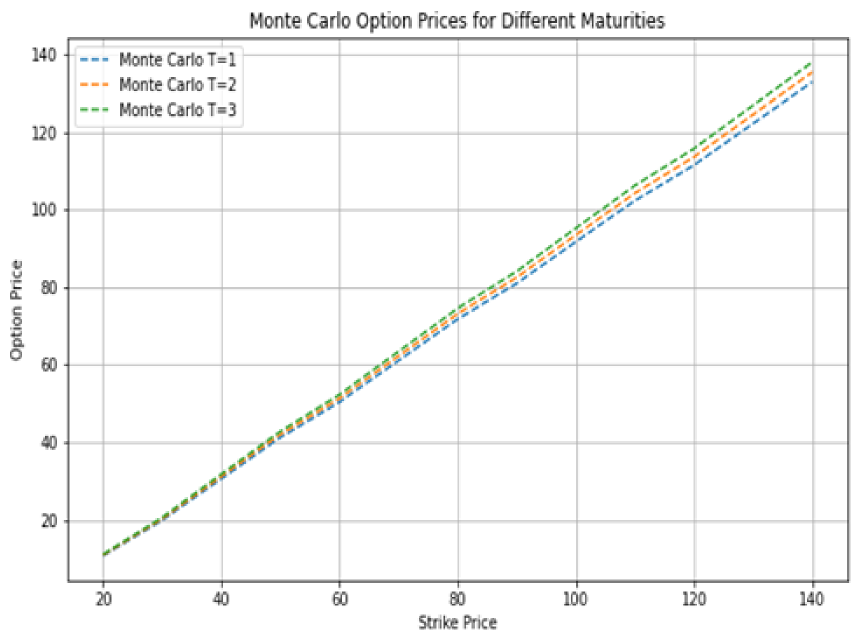
In the provided tables 1, 2, 3, we show that for all the varied maturity durations and strike prices, the two systems perform well and are consistent with small decimal discrepancies.


Option Prices for Different Maturities (Monte Carlo vs Crank-Nicolson)
Monte Carlo T=1
Crank-Nicolson T=1
Monte Carlo T=2
Crank-Nicolson T=2
Monte Carlo T=3
Crank-Nicolson T=3
Option Price
Strike Price
140
120
100
80
60
40
20
20
40
60
80
100
120
140

(a)


Crank-Nicolson Option Prices for Different Maturities
Crank-Nicolson T=1
Crank-Nicolson T=2
Crank-Nicolson T=3
Option Price
Strike Price
140
120
100
80
60
40
20
20
40
60
80
100
120
140

(b)



(c)

Figure 1: (a) Compares option prices using both numerical methods at different maturity times; (b) Shows option prices at different maturity times using the Crank-Nicolson method; (c) Indicates option prices at different maturity times using the Monte Carlo method.

We have also established that adjusting the maturity time T impacts the trade-off between computational efficiency and numerical accuracy in pricing European call options using the Heston-SABR model for the two numerical methods. Figure 3(a), (b) and (c) plots represent a combined graph of the two methods and individual methods respectively, for the visualization of the numerical techniques.

5. Conclusion

Overall, it has been determined that both approaches offer benefits and drawbacks when utilizing them. The Monte Carlo method is easy to use, broadly adaptable, and capable of handling complicated payoffs and models such as the hybrid Heston-SABR model. However, they require a lot of computing power to provide reliable results, and many simulations are needed. Crank-Nicolson finite difference approaches are comparatively stable and effective for pricing vanilla options. In addition, they can be rather challenging to code and need complex techniques to solve huge sparse linear systems of equations. In summary, when pricing European options under the hybrid Heston-SABR model, both approaches converge faster, are more accurate, and are unconditionally stable. The study concludes that both Monte Carlo simulation and finite difference methods are effective for pricing European options under the hybrid Heston-SABR model. Both methods demonstrate fast convergence, high accuracy, and unconditional stability. However, the accuracy and stability of these numerical methods are influenced by the maturity time of the options. Additionally, there is a trade-off between numerical accuracy and computational efficiency when adjusting the maturity time T.

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REFERENCES

1. P. Boyle, M. Broadie and P. Glasserman, Monte Carlo methods for security pricing, *J. Econ. Dyn. Control*, 21(8–9) (1997) 1267–1321, doi: 10.1016/s0165-1889(97)00028-6.
2. M. J. Brennan and E. S. Schwartz, Finite Difference Methods and Jump Processes Arising in the Pricing of Contingent Claims: A Synthesis, *J. Financ. Quant. Anal.*, 12(4) (1977) 659. doi: 10.1017/S0022109000023462.
3. A. Tveito and R. Winther, *Introduction to Partial Differential Equations: A Computational Approach*, Springer, (1998).
4. P. S. Hagan, A. Lesniewski and D. Woodward, Implied Volatility Formulas for Heston Models, 2018 (98) (2018) 44-57. doi: 10.1002/wilm.10723.
5. M. Felpel, J. Kienitz and T. McWalter, Effective stochastic volatility: applications to ZABR-type models, *SSRN Electron. J.*, (2020), doi: 10.2139/ssrn.3518141.
6. S. Huang, X. He and S. Qian, An analytical approximation of European option prices under a hybrid GARCH-Vasicek model with double exponential jump in the bid-ask price economy, *AIMS Math.*, 9(5) (2024) 11833–11850. doi: 10.3934/math.2024579.
7. I. Florescu, M. C. Mariani and I. Sengupta, Option pricing with transaction costs and stochastic volatility, *Electron. J. Differ. Equations*, 165 (2014) 1–19.
8. M. C. Mariani, E. K. Ncheuguim, and I. Sengupta, Option Pricing With Transaction, 1–34.
9. D. Yan, A comprehensive study of option pricing, 106 (2022) 522–524. doi: 10.1017/S0004972722000740.
10. J. Choi and Y. K. Kwok, Simulation schemes for the Heston model with Poisson conditioning, *Eur. J. Oper. Res.*, 314(1) (2024) 363–376. doi: 10.1016/j.ejor.2023.10.048.
11. S. J. A. Malham, J. Shen and A. Wiese, Series expansions and direct inversion for the heston model, *SIAM J. Financ. Math.*, 12(1) (2021) 487–549. doi: 10.1137/19M126791X.
12. S. Foulon and K. J. in 't Hout, Adi Finite Difference Schemes for Option Pricing, *Int. J. Numer. Anal. Model.*, 7(2) (2010) 303–320.
13. A. W. van der Stoep, L. A. Grzelak and C. W. Oosterlee, The Heston stochastic-local volatility model: efficient Monte Carlo simulation, *SSRN Electron. J.*, 17(7) (2014), doi: 10.2139/ssrn.2278122.
14. P. D. Lax and R. D. Richtmyer, Survey of the stability of linear finite difference equations, *Commun. Pure Appl. Math.*, 9(2) (1956) 267–293. doi: 10.1002/cpa.3160090206.