Annals of Pure and Applied Mathematics
Vol. 29, No. 1, 2024, 51-72
ISSN: 2279-087X (P), 2279-0888(online)
Published on 13 February 2024
www.researchmathsci.org
DOI: http://dx.doi.org/10.22457/apam.v29n1a06929

Annals of Pure and Applied Mathematics

Computation of Different Topological Indices of Barbell Graph, Pan Graph and Sun Graph Using M-Polynomial

***Smith K. Phadnis*[1*] and *Mallikarjun Basanna Kattimani*[2]**

[1,2]Department of Mathematics, The Oxford College of Engineering
Bangalore-560068, India. Email: smita.deshpande142009@gmail.com
[*]Corresponding author. Email: mallikarjunbk64@gmail.com

Received 5 January 2024; accepted 12 February 2024

Abstract. Topological indices are numerical parameters used to study the physical and chemical residences of compounds. Degree-based topological indices have been studied extensively and can be correlated with many properties of the compounds. In the factors of degree-based topological indices M-polynomial played an important role. Using M-polynomial we can derive several degree-based topological indices of a graph. In this article, we have obtained 18 different topological indices of the Barbell graph, Pan graph and Sun graph.

Keywords: Graphs, M-polynomial, Degree-based topological index.

AMS Mathematics Subject Classification (2010): 05C31

1. Introduction

All graphs considered here are finite, nontrivial, without isolated vertices, undirected, without loops or multiple edges. Undefined terms or notations in this paper may be found in Harary [10]. Vertex set is denoted by $V(G)$, edge set is denoted by $E(G)$ for a graph G. The degree $d_G(v)$ of a vertex $\in V(G)$ is the number of edges incident to it in G. An isolated vertex or singleton graph is a vertex with degree zero. Let $\{v_1, v_2, \dots v_n\}$ be the vertices of G and let $d_i = d_G(v_i)$. The n-barbell graph denoted by $B(p,n)$is the simple graph obtained by connecting two copies of a complete graph K_n by a bridge. The pan graph denoted by $P_n$is the graph obtained by joining a cycle graph to a singleton graph with a bridge. The n-sun graph denoted by $S(n)$ is a graph with $2n$ vertices consisting of a central complete graph with K_n an outer ring of n vertices, each of which is joined to both endpoints of the closest outer edge of the central core. In graph theory, information about a graph can be encoded in the form of a polynomial and many such polynomials are available in the literature. One of the most commonly used polynomials called M-polynomial denoted by $f(x,y)$ throughout the article, makes use of a degree of vertices of the graph. The illustration of the M-polynomial was done by Deutsch and Klavžar in the year 2015 [8]. We denote the degree of a vertex u by d_u. The degree of an edge is defined to be $d_{uv} = d_u + d_v - 2$. In the year 1947, the first topological index in graphs was introduced by Weiner which is known as the Weiner index [1]. In the year 1972, the first and second Zagreb indices were introduced [13]. In the year 2003, modifications to Zagreb indices were introduced [14]. Many topological indices are studied in the literature. In 1975, the Randic index was introduced [34] and a general Randic index was introduced in the year 1998 [7]. In the same year Atom Bond Connectivity index was introduced [15]. Kulli introduced a series of indices namely Banahatti indices, hyper Banahatti indices and their modified versions [32-34]. Khalaf et al obtained different topological indices using the M-polynomial of Book Graph [3] in the year 2020. In this article, we have obtained the M-polynomial of 3 different graphs and hence various topological indices associated with these graphs. The barbell graph denoted by $B(p,n)$ is a graph with $2n$ vertices and $n(n-1)+1$ edges which is a simple graph obtained by connecting two copies of a complete graph K_n by a bridge. The pan graph denoted by P_n is the graph with $(n+1)$ vertices and $(n+1)$ edges which is obtained by joining a cycle graph to a singleton graph with a bridge. The sun graph denoted by $S(n)$ is a graph with $2n$ vertices and $\frac{n^2+3n}{2}$ edges consisting of a central complete graph with K_n an outer ring of n vertices, each of which is joined to both endpoints of the closest outer edge of the central core. The M-polynomial of a graph is denoted by $f(x,y)$ or $M(G{:}\,x,y)$ and is defined as $f(x,y) = \sum_{\delta \le i \le j \le \Delta} m_{ij}(G)\, x^i y^j$ where δ and Δ are the minimum and maximum of degree of vertices in the graph G and $m_{ij}(G)$ is the number of edges with end vertices u and v and $(d_u, d_v) = (i,j)$. This polynomial was introduced in 2015 by Deutsch and Klavžarin and it is useful in determining many degree-based topological indices. This motivates us to study the M-polynomial of some complete and cycle-related graphs.

Table 1: Definitions of different topological indices

Topological Index	Definition	Formula to derive index by applying M-Polynomial with $x = y = 1$
First Zagreb Index	$M_1(G) = \sum_{uv \in E(G)} (d_u + d_v)$	$(D_x + D_y) f(x,y)$
Second Zagreb Index	$M_2(G) = \sum_{uv \in E(G)} (d_u \,.\, d_v)$	$(D_x \,.\, D_y) f(x,y)$
Modified Second Zagreb Index	$m_{M_2}(G) = \sum_{uv \in E(G)} \frac{1}{d_u \,.\, d_v}$	$(S_x \,.\, S_y) f(x,y)$
General Randic Index	$R_\alpha(G) = \sum_{uv \in E(G)} (d_u . d_v)^\alpha$	$\left(D_x^\alpha . D_y^\alpha\right) f(x,y)$
Inverse Randic Index	$RR_\alpha(G) = \sum_{uv \in E(G)} \left(\frac{1}{d_u \,.\, d_v}\right)^\alpha$	$\left(S_x^\alpha . S_y^\alpha\right) f(x,y)$
Harmonic Index	$H(G) = \sum_{uv \in E(G)} \left(\frac{2}{d_u + d_v}\right)$	$2 S_x J\, f(x,y)$
Symmetric Division Index	$SSD(G) = \sum_{uv \in E(G)} \left(\frac{d_u}{d_v} + \frac{d_v}{d_u}\right)$	$(D_x S_y + S_x D_y) f(x,y)$

Augmented Zagreb Index	$A(G) = \sum_{uv\in E(G)} \left(\frac{d_u \,.\, d_v}{d_u + d_v - 2}\right)^3$	$\left(S_x^3 Q_{-2} J D_x^3 D_y^3\right) f(x,y)$
Inverse Sum Index	$I(G) = \sum_{uv\in E(G)} \left(\frac{d_u \,.\, d_v}{d_u + d_v}\right)$	$(S_x J D_x D_y) f(x,y)$
Atom-bond Connectivity Index	$ABC(G) = \sum_{uv\in E(G)} \sqrt{\frac{d_u + d_v - 2}{d_u \,.\, d_v}}$	$\left(D_x^{1/2} Q_{-2} J S_x^{1/2} S_y^{1/2}\right) f(x,y$
Geometric Arithmetic Index	$GA(G) = \sum_{uv\in E(G)} \left(\frac{2\sqrt{d_u \,.\, d_v}}{d_u + d_v}\right)$	$(2S_x J D_x^{1/2} D_y^{1/2}) f(x,y)$
First K-Banahatti Index	$B_1(G) = \sum_{uv\in E(G)} (d_u + d_{uv})$	$(D_x + D_y + 2D_x Q_{-2} J) f(x,y)$
Second K-Banahatti Index	$B_2(G) = \sum_{uv\in E(G)} (d_u \,.\, d_{uv})$	$(D_x Q_{-2} J (D_x + D_y)) f(x,y)$
First K-hyper Banahatti Index	$HB_1(G) = \sum_{uv\in E(G)} (d_u + d_{uv})^2$	$\left(D_x^2 + D_y^2 + 2D_x^2 Q_{-2} J + 2D_x Q_{-2} J (D_x + D_y)\right) f($
Second K-hyper Banahatti Index	$HB_2(G) = \sum_{uv\in E(G)} (d_u . d_{uv})^2$	$\left(D_x^2 Q_{-2} J \left(D_x^2 + D_y^2\right)\right) f(x,y)$
Modified First K-hyper Banahatti Index	$mB_1(G) = \sum_{uv\in E(G)} \left(\frac{1}{d_u + d_{uv}}\right)$	$\left(S_x Q_{-2} J (L_x + L_y)\right) f(x,y)$
Modified Second K-hyper Banahatti Index	$mB_2(G) = \sum_{uv\in E(G)} \left(\frac{1}{d_u . d_{uv}}\right)$	$\left(S_x Q_{-2} J (S_x + S_y)\right) f(x,y)$
Harmonic K-Banahatti Index	$H_b(G) = \sum_{uv\in E(G)} \left(\frac{2}{d_u + d_{uv}}\right)$	$\left(2S_x Q_{-2} J (L_x + L_y)\right) f(x,y)$

Table 2: Notations used in computing indices

$D_x = x\frac{\partial f}{\partial x}$	$J = f(x,x)$	$D_x^{1/2} = \sqrt{x\frac{\partial f}{\partial x}} \,.\, f(x,y)$
$D_y = y\frac{\partial f}{\partial y}$	$Q_\alpha = x^\alpha f(x,y)$	$D_y^{1/2} = \sqrt{y\frac{\partial f}{\partial y}} \,.\, f(x,y)$
$L_x = f(x^2, y)$	$S_x = \int_0^x \frac{f(t,y)}{t} dt$	$S_x^{1/2} = \sqrt{\int_0^x \frac{f(t,y)}{t}} \,.\, f(x,y)$
$L_y = f(x, y^2)$	$S_y = \int_0^y \frac{f(x,t)}{t} dt$	$S_y^{1/2} = \sqrt{\int_0^y \frac{f(x,t)}{t}} \,.\, f(x,y)$

Different notations used in the formulae are explained in Table 2.

2. Main results

Theorem 2.1. If $B(p,n)$ is the barbell graph, then

$$f(x,y) = (n-1)(n-2)x^{n-1}y^{n-1} + 2(n-1)x^{n-1}y^{n} + x^{n}y^{n}$$

Proof: By definition of the barbell graph and by computation, we find that$B(p,n)$has $2n$ vertices and$n(n-1)+1$ edges. Based on the degrees of end vertices, the edge set of $B(p,n)$ can be tabulated in *Table 3.*

Table 3: Details of the degrees of vertices and number of edges in B(p, n)

(d_u, d_v)	$(n-1, n-1)$	$(n-1, n)$	(n, n)
Total number of edges	$(n-1, n-2)$	$2n-2$	1

The M-polynomial of $B(p,n)$ is given as

$$M(B(p,n): x,y) = f(x,y) = \sum_{\delta \le i \le j \le \Delta} m_{ij}(G)\, x^i y^j$$

$$f(x,y) = (n-1)(n-2)x^{n-1}y^{n-1} + 2(n-1)x^{n-1}y^{n} + x^{n}y^{n}$$

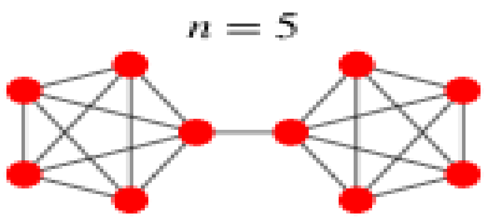

Figure 1: The n-Barbell Graph $B(p,n)$

Theorem 2.2. Topological indices of barbell graph$B(p,n)$ are given by the following.

1. $M_1(B(p,n)) = 2n^3 - 4n^2 + 6n - 2$
2. $M_2(B(p,n)) = n^4 - 3n^3 + 6n^2 - 5n + 2$
3. $m_{M_2}(B(p,n)) = \frac{n^3-n-1}{n^3-n^2}$
4. $R_\alpha(B(p,n)) = (n^2 - 3n + 2)(n-1)^{2\alpha} + 2(n-1)^{\alpha+1}n^\alpha + n^{2\alpha}$
5. $RR_\alpha(B(p,n)) = \frac{n^2-3n+2}{(n-1)^{2\alpha}} + \frac{2n-2}{n^\alpha(n-1)^\alpha} + \frac{1}{n^{2\alpha}}$
6. $H(B(p,n)) = \frac{2n^3-n^2-1}{2n^2-n}$
7. $SSD(B(p,n)) = \frac{2}{n}(n^3 - n^2 + n - 1)$
8. $A(B(p,n)) = \frac{(n-1)^7(n-2)}{(2n-4)^3} + \frac{2n^3(n-1)^4}{(2n-3)^3} + \frac{n^6}{(2n-2)^3}$
9. $I(B(p,n)) = \frac{(n-1)^3(n-2)}{2n-2} + \frac{2n(n-1)^2}{2n-1} + \frac{n^2}{2n}$

10. $ABC(B(p,n)) = (n-2)\sqrt{2n-4} + \frac{2(n-1)}{\sqrt{n(n-1)}}\sqrt{2n-3} + \frac{\sqrt{2n-2}}{n}$
11. $GA(B(p,n)) = \frac{2n^3-7n^2+9n-3+(4n-4)\sqrt{n(n-1)}}{2n-1}$
12. $B_1(B(p,n)) = 6n^3 - 16n^2 + 22n - 10$
13. $B_2(B(p,n)) = 4n^4 - 16n^3 + 32n^2 - 30n + 10$
14. $HB_1(B(p,n)) = 18n^4 - 78n^3 + 164n^2 - 160n + 58$
15. $HB_2(B(p,n)) = 2(n-1)^3(n-2)(2n-4)^2 + 2(n-1)(2n^2 - 2n + 1)(2n-3)^2 + 2n^2(2n-2)^2$
16. $mB_1(B(p,n)) = \frac{2(n-1)(n-2)}{3n-5} + \frac{2(n-1)}{3n-4} + \frac{2(n-1)}{3n-3} + \frac{2}{3n-2}$
17. $mB_2(B(p,n)) = \frac{n(2n-3)(2n-2)+2(2n-1)(2n-2)+2(2n-3)}{n(2n-3)(2n-2)}$
18. $H_b(B(p,n)) = \frac{4(n-1)(n-2)}{3n-5} + \frac{4(n-1)}{3n-4} + \frac{4(n-1)}{3n-3} + \frac{4}{3n-2}$

Proof:

1. First Zagreb Index
   $D_x f(x,y) = (n-1)^2(n-2)x^{n-1}y^{n-1} + 2(n-1)^2x^{n-1}y^n + nx^ny^n$
   $D_y f(x,y) = (n-1)^2(n-2)x^{n-1}y^{n-1} + 2n(n-1)x^{n-1}y^n + nx^ny^n$
   Thus $M_1(B(p,n)) = (D_x + D_y)f(x,y)\ at\ x = 1\ and\ y = 1$ gives
   $M_1(B(p,n)) = 2n^3 - 4n^2 + 6n - 2$
2. Second Zagreb Index
   $D_y f(x,y) = (n-1)^2(n-2)x^{n-1}y^{n-1} + 2n(n-1)x^{n-1}y^n + nx^ny^n$
   $(D_x . D_y)f(x,y) = (n-1)^2(n-2)x^{n-1}y^{n-1} + 2n(n-1)x^{n-1}y^n + nx^ny^n$
   Thus $M_2(B(p,n)) = (D_x . D_y)f(x,y)\ at\ x = 1\ and\ y = 1$ gives
   $M_2(B(p,n)) = n^4 - 3n^3 + 6n^2 - 5n + 2$
3. Modified Second Zagreb Index
   $$S_y(f(x,y)) = (n-2)x^{n-1}y^{n-1} + \frac{2(n-1)x^{n-1}y^{n-1}}{n} + \frac{x^ny^n}{n}$$
   $$(S_x . S_y)f(x,y) = \frac{(n-2)}{(n-1)}x^{n-1}y^{n-1} + 2x^{n-1}y^n + \frac{x^ny^n}{n^2}$$
   Thus $m_{M_2}(B(p,n)) = (S_x . S_y)f(x,y)\ at\ x = 1\ and\ y = 1$ gives
   $$m_{M_2}(B(p,n)) = \frac{n^3 - n - 1}{n^3 - n^2}$$
4. General Randic Index
   $$D_y^\alpha(f(x,y)) = (n-1)(n-2)(n-1)^\alpha x^{n-1}y^{n-1} + 2(n-1)n^\alpha x^{n-1}y^n + n^\alpha x^ny^n$$
   $$(D_x^\alpha . D_y^\alpha)f(x,y) = (n-1)(n-2)(n-1)^{2\alpha}x^{n-1}y^{n-1} + 2(n-1)n^\alpha(n-1)^\alpha x^{n-1}y^n + n^{2\alpha}x^ny^n$$
   Thus $R_\alpha(B(p,n)) = (D_x^\alpha . D_y^\alpha)f(x,y)\ at\ x = 1\ and\ y = 1$ gives
   $R_\alpha(B(p,n)) = (n^2 - 3n + 2)(n-1)^{2\alpha} + 2(n-1)^{\alpha+1}n^\alpha + n^{2\alpha}$

5. Inverse Randic Index

$$S_y^{\alpha}\big(f(x,y)\big) = \frac{(n-1)(n-2)x^{n-1}y^{n-1}}{(n-1)^{\alpha}} + \frac{2(n-1)x^{n-1}y^{n}}{n^{\alpha}} + \frac{x^n y^n}{n^{\alpha}}$$

$$\big(S_x^{\alpha}\,.\,S_y^{\alpha}\big)f(x,y) = \frac{(n-1)(n-2)x^{n-1}y^{n-1}}{(n-1)^{2\alpha}} + \frac{2(n-1)x^{n-1}y^{n}}{n^{\alpha}(n-1)^{\alpha}} + \frac{x^n y^n}{n^{2\alpha}}$$

Thus $RR_{\alpha}(B(p,n)) = \big(S_x^{\alpha}\,.\,S_y^{\alpha}\big)f(x,y)\ at\ x = 1\ and\ y = 1$ gives

$$RR_{\alpha}(B(p,n)) = \frac{n^2-3n+2}{(n-1)^{2\alpha}} + \frac{2n-2}{n^{\alpha}(n-1)^{\alpha}} + \frac{1}{n^{2\alpha}}$$

6. Harmonic Index

$$Jf(x,y) = (n-1)(n-2)x^{2n-2} + 2(n-1)x^{2n-1} + x^{2n}$$

$$2S_x\,Jf(x,y) = (n-2)x^{2n-2} + \frac{4(n-1)x^{2n-1}}{2n-1} + \frac{x^{2n}}{n}$$

Thus $H(B(p,n)) = 2S_x\,Jf(x,y)\ at\ x = 1$ gives

$$H(B(p,n)) = \frac{2n^3-n^2-1}{2n^2-n}$$

7. Symmetric Division Index

$$\big(D_xS_y\big)f(x,y) = (n-1)(n-2)x^{n-1}y^{n-1} + \frac{2(n-1)^2x^{n-1}y^n}{n} + x^ny^n$$

$$\big(S_xD_y\big)f(x,y) = (n-1)(n-2)x^{n-1}y^{n-1} + 2nx^{n-1}y^n + x^ny^n$$

$$\big(D_xS_y + S_xD_y\big)f(x,y)$$
$$= 2(n-1)(n-2)x^{n-1}y^{n-1} + 2\left[\frac{(n-1)^2}{n} + n\right]x^{n-1}y^n + 2x^ny^n$$

Thus $SSD(B(p,n)) = \big(D_xS_y + S_xD_y\big)f(x,y)\ at\ x = y = 1$ gives

$$SSD(B(p,n)) = \frac{2}{n}(n^3-n^2+n-1)$$

8. Augmented Zagreb Index

$$D_y^3\big(f(x,y)\big) = (n-1)(n-2)(n-1)^3x^{n-1}y^{n-1} + 2(n-1)n^3x^{n-1}y^n + n^3x^ny^n$$

$$D_x^3D_y^3\big(f(x,y)\big) = (n-1)^7(n-2)x^{n-1}y^{n-1} + 2(n-1)^4n^3x^{n-1}y^n + n^6x^ny^n$$

$$JD_x^3D_y^3\big(f(x,y)\big) = (n-1)^7(n-2)x^{2n-2} + 2(n-1)^4n^3x^{2n-1} + n^6x^{2n}$$

$$Q_{-2}JD_x^3D_y^3\big(f(x,y)\big) = (n-1)^7(n-2)x^{2n-4} + 2(n-1)^4n^3x^{2n-3} + n^6x^{2n-2}$$

$$\big(S_x^3Q_{-2}JD_x^3D_y^3\big)f(x,y)$$
$$= \frac{(n-1)^7(n-2)x^{2n-4}}{(2n-4)^3} + \frac{2(n-1)^4n^3x^{2n-3}}{(2n-3)^3} + \frac{n^6x^{2n-2}}{(2n-2)^3}$$

Thus $A(B(p,n)) = \big(S_x^3Q_{-2}JD_x^3D_y^3\big)f(x,y)\ at\ x = 1$ gives

$$A(B(p,n)) = \frac{(n-1)^7(n-2)}{(2n-4)^3} + \frac{2n^3(n-1)^4}{(2n-3)^3} + \frac{n^6}{(2n-2)^3}$$

9. Inverse Sum Index

$$\big(D_xD_y\big)f(x,y) = (n-1)^3(n-2)x^{n-1}y^{n-1} + 2n(n-1)^2x^{n-1}y^n + n^2x^ny^n$$

$$(JD_xD_y)f(x,y) = (n-1)^3(n-2)x^{2n-2} + 2n(n-1)^2x^{2n-1} + n^2x^{2n}$$

$$(S_xJD_xD_y)f(x,y) = (n-1)^3(n-2)\frac{x^{2n-2}}{2n-2} + 2n(n-1)^2\frac{x^{2n-1}}{2n-1} + n^2\frac{x^{2n}}{2n}$$

Thus $I(B(p,n)) = (S_xJD_xD_y)f(x,y)\ at\ x = 1$ gives

$$I(B(p,n)) = \frac{(n-1)^3(n-2)}{2n-2} + \frac{2n(n-1)^2}{2n-1} + \frac{n^2}{2n}$$

10. Atom-bond Connectivity Index

$$\left(S_y^{1/2}\right)f(x,y) = \frac{(n-1)(n-2)}{\sqrt{n-1}}x^{n-1}y^{n-1} + \frac{2(n-1)}{\sqrt{n}}x^{n-1}y^n + \frac{x^ny^n}{\sqrt{n}}$$

$$\left(S_x^{1/2}S_y^{1/2}\right)f(x,y) = (n-1)x^{n-1}y^{n-1} + \frac{2(n-1)}{\sqrt{n(n-1)}}x^{n-1}y^n + \frac{x^ny^n}{n}$$

$$\left(JS_x^{1/2}S_y^{1/2}\right)f(x,y) = (n-1)x^{2n-2} + \frac{2(n-1)}{\sqrt{n(n-1)}}x^{2n-1} + \frac{x^{2n}}{n}$$

$$\left(Q_{-2}JS_x^{1/2}S_y^{1/2}\right)f(x,y) = (n-1)x^{2n-4} + \frac{2(n-1)}{\sqrt{n(n-1)}}x^{2n-3} + \frac{x^{2n-2}}{n}$$

$$\left(D_x^{1/2}Q_{-2}JS_x^{1/2}S_y^{1/2}\right)f(x,y)$$
$$= (n-1)\sqrt{2n-4}x^{2n-4} + \frac{2(n-1)}{\sqrt{n(n-1)}}\sqrt{2n-3}\ x^{2n-3}$$
$$+ \frac{\sqrt{2n-2}}{n}x^{2n-2}$$

Thus $ABC(B(p,n)) = \left(D_x^{1/2}Q_{-2}JS_x^{1/2}S_y^{1/2}\right)f(x,y)\ at\ x = 1$ gives

$$ABC(B(p,n)) = (n-2)\sqrt{2n-4} + \frac{2(n-1)}{\sqrt{n(n-1)}}\sqrt{2n-3} + \frac{\sqrt{2n-2}}{n}$$

11. Geometric Arithmetic Index

$$\left(D_y^{1/2}\right)f(x,y) = (n-1)(n-2)\sqrt{(n-1)}x^{n-1}y^{n-1} + 2\sqrt{n}(n-1)x^{n-1}y^n$$
$$+ n^2x^ny^n$$

$$\left(D_x^{1/2}D_y^{1/2}\right)f(x,y)$$
$$= (n-1)^2(n-2)x^{n-1}y^{n-1} + 2(n-1)\sqrt{n(n-1)}x^{n-1}y^n$$
$$+ nx^ny^n$$

$$\left(JD_x^{1/2}D_y^{1/2}\right)f(x,y)$$
$$= (n-1)^2(n-2)x^{2n-2} + 2(n-1)\sqrt{n(n-1)}x^{2n-1} + nx^{2n}$$

$$\left(2S_xJD_x^{1/2}D_y^{1/2}\right)f(x,y)$$
$$= 2(n-1)^2(n-2)\frac{x^{2n-2}}{2(n-1)} + 4(n-1)\sqrt{n(n-1)}\frac{x^{2n-1}}{2n-1}$$
$$+ x^{2n}$$

Thus $GA(B(p,n)) = \left(2S_x J D_x^{1/2} D_y^{1/2}\right) f(x,y)\ at\ x=1$ gives

$$GA(B(p,n)) = \frac{2n^3 - 7n^2 + 9n - 3 + (4n-4)\sqrt{n(n-1)}}{2n-1}$$

12. First K-Banahatti Index

$$(2D_x Q_{-2} J) f(x,y) = 2(n-1)(n-2)(2n-4)x^{2n-4} + 4(n-1)(2n-3)x^{2n-3} + 2(2n-2)x^{2n-2}$$

$$\left(D_x + D_y\right) f(x,y) = 2(n-1)^2(n-2)x^{n-1}y^{n-1} + 2(n-1)(2n-1)(n-1)^2(n-2)x^{n-1}y^n + 2n(n-1)^2(n-2)x^n y^n$$

Thus $B_1(B(p,n)) = \left(D_x + D_y + 2D_x Q_{-2} J\right) f(x,y)\ at\ x=y=1$ gives

$B_1(B(p,n)) = 6n^3 - 16n^2 + 22n - 10$

13. Second K-Banahatti Index

$$\left(Q_{-2} J(D_x + D_y)\right) f(x,y) = 2(n-1)^2(n-2)x^{2n-4} + 2(n-1)(2n-1)x^{2n-3} + 2nx^{2n-2}$$

$$\left(D_x Q_{-2} J(D_x + D_y)\right) f(x,y) = 2(n-1)^2(n-2)(2n-4)x^{2n-4} + 2(n-1)(2n-1)(2n-3)x^{2n-3} + 2n(2n-2)x^{2n-2}$$

Thus $B_2(B(p,n)) = \left(D_x Q_{-2} J(D_x + D_y)\right) f(x,y)\ at\ x=1$ gives

$B_2(B(p,n)) = 4n^4 - 16n^3 + 32n^2 - 30n + 10$

14. First K-hyper Banahatti Index

$$\left(D_x^2 + D_y^2\right) f(x,y) = 2(n-1)^3(n-2)x^{n-1}y^{n-1} + 2(n-1)[n^2 + (n-1)^2]x^{n-1}y^n + 2n^2 x^n y^n$$

$$\left(2{D_x}^2 Q_{-2} J\right) f(x,y) = 2(n-1)(n-2)(2n-4)^2 x^{2n-4} + 4(n-1)(2n-3)^2 x^{2n-3} + 2(2n-2)^2 x^{2n-2}$$

$$\left(2D_x Q_{-2} J(D_x + D_y)\right) f(x,y) = 4(n-1)^2(n-2)(2n-4)x^{2n-4} + 4(n-1)(2n-1)(2n-3)x^{2n-3} + 4n(2n-2)x^{2n-2}$$

Thus $HB_1(B(p,n)) = \left(D_x^2 + D_y^2 + 2{D_x}^2 Q_{-2} J + 2D_x Q_{-2} J\left(D_x + D_y\right)\right) f(x,y)\ at\ x=1$ gives

$HB_1(B(p,n)) = 18n^4 - 78n^3 + 164n^2 - 160n + 58$

15. Second K-hyper Banahatti Index

$$(D_x^2 + D_y^2)f(x,y) = 2(n-1)^3(n-2)x^{n-1}y^{n-1} + 2(n-1)[n^2 + (n-1)^2]x^{n-1}y^n + 2n^2x^ny^n$$

$$\left(J(D_x^2 + D_y^2)\right)f(x,y) = 2(n-1)^3(n-2)x^{2n-2} + 2(n-1)[n^2 + (n-1)^2]x^{2n-1} + 2n^2x^{2n}$$

$$\left(Q_{-2}J(D_x^2 + D_y^2)\right)f(x,y) = 2(n-1)^3(n-2)x^{2n-4} + 2(n-1)[n^2 + (n-1)^2]x^{2n-3} + 2n^2x^{2n-2}$$

$$\left(D_x^2Q_{-2}J(D_x^2 + D_y^2)\right)f(x,y) = 2(n-1)^3(n-2)(2n-4)^2x^{2n-4} + 2(n-1)[n^2 + (n-1)^2](2n-3)^2x^{2n-3} + 2n^2(2n-2)^2x^{2n-2}$$

Thus $HB_2(B(p,n)) = \left(D_x^2Q_{-2}J(D_x^2 + D_y^2)\right)f(x,y)\ at\ x = 1$ gives

$$HB_2(B(p,n)) = 2(n-1)^3(n-2)(2n-4)^2 + 2(n-1)(2n^2-2n+1)(2n-3)^2 + 2n^2(2n-2)^2$$

16. Modified First K-hyper Banahatti Index

$$\left(Q_{-2}J(L_x + L_y)\right)f(x,y) = 2(n-1)(n-2)x^{3n-5} + 2(n-1)x^{3n-4} + 2(n-1)x^{3n-3} + 2x^{3n-2}$$

$$\left(S_xQ_{-2}J(L_x + L_y)\right)f(x,y) = \frac{2(n-1)(n-2)}{3n-5}x^{3n-5} + \frac{2(n-1)}{3n-4}x^{3n-4} + \frac{2(n-1)}{3n-3}x^{3n-3} + \frac{2}{3n-2}x^{3n-2}$$

Thus $mB_1(B(p,n)) = \left(S_xQ_{-2}J(L_x + L_y)\right)f(x,y)\ at\ x = 1$ gives

$$mB_1(B(p,n)) = \frac{2(n-1)(n-2)}{3n-5} + \frac{2(n-1)}{3n-4} + \frac{2(n-1)}{3n-3} + \frac{2}{3n-2}$$

17. Modified Second K-hyper Banahatti Index

$$\left(Q_{-2}J(S_x + S_y)\right)f(x,y) = 2(n-2)x^{2n-4} + 2\left(\frac{2n-1}{n}\right)x^{2n-3} + \frac{2}{n}x^{2n-2}$$

$$\left(S_xQ_{-2}J(S_x + S_y)\right)f(x,y) = \frac{2(n-2)}{2n-4}x^{2n-4} + \frac{2\left(\frac{2n-1}{n}\right)}{2n-3}x^{2n-3} + \frac{2}{n(2n-2)}x^{2n-2}$$

Thus $mB_2(B(p,n)) = \left(S_xQ_{-2}J(S_x + S_y)\right)f(x,y)\ at\ x = 1$ gives

$$mB_2(B(p,n)) = \frac{n(2n-3)(2n-2) + 2(2n-1)(2n-2) + 2(2n-3)}{n(2n-3)(2n-2)}$$

18. Harmonic K-Banahatti Index

$$\left(Q_{-2}J(L_x+L_y)\right)f(x,y) = \frac{2(n-1)(n-2)}{3n-5}x^{3n-5} + \frac{2(n-1)}{3n-4}x^{3n-4} + \frac{2(n-1)}{3n-3}x^{3n-3} + \frac{2}{3n-2}x^{3n-2}$$

$$\left(2S_xQ_{-2}J(L_x+L_y)\right)f(x,y) = \frac{4(n-1)(n-2)}{3n-5}x^{3n-5} + \frac{4(n-1)}{3n-4}x^{3n-4} + \frac{4(n-1)}{3n-3}x^{3n-3} + \frac{4}{3n-2}x^{3n-2}$$

Thus $H_b(B(p,n)) = \left(2S_xQ_{-2}J(L_x+L_y)\right)f(x,y)\ at\ x=1$ gives

$$H_b(B(p,n)) = \frac{4(n-1)(n-2)}{3n-5} + \frac{4(n-1)}{3n-4} + \frac{4(n-1)}{3n-3} + \frac{4}{3n-2}$$

Theorem 2.3. If P_n is the pan graph, then

$$f(x,y) = (n-2)x^2y^2 + 2x^2y^3 + xy^3$$

Proof: By definition of pan graph and by computation, we find that P_n has $(n+1)$ vertices and $(n+1)$ edges. Based on the degrees of end vertices, the edge set of P_n can be tabulated in Table 4.

Table 4: Details of the degrees of vertices and number of edges in P_n

(d_u, d_v)	(2,2)	(2,3)	(3,1)
Total number of edges	$(n-2)$	2	1

The M-polynomial of P_n is given as

$$M(P_n: x,y) = f(x,y) = \sum_{\delta \le i \le j \le \Delta} m_{ij}(G)\, x^i y^j$$

$$f(x,y) = (n-2)x^2y^2 + 2x^2y^3 + xy^3$$

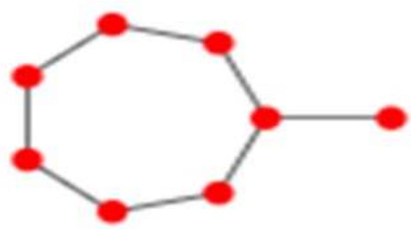

Figure 2: The pan graph P_n for n = 7

Theorem 2.4. Topological indices of pan graph P_n are given by the following.

1. $M_1(P_n) = 4n + 6$
2. $M_2(P_n) = 4n + 7$

3. $m_{M_2}(P_n) = \frac{3n+2}{12}$
4. $R_\alpha(P_n) = 2^{2\alpha}(n-2) + 2^{\alpha+1}3^\alpha + 3^\alpha$
5. $RR_\alpha(P_n) = \frac{n-2}{2^{2\alpha}} + \frac{2}{3^\alpha 2^\alpha} + \frac{1}{3^\alpha}$
6. $H(P_n) = \frac{5n+3}{10}$
7. $SSD(P_n) = \frac{6n+11}{3}$
8. $A(P_n) = \frac{1728n+729}{216}$
9. $I(P_n) = \frac{20n+23}{20}$
10. $ABC(P_n) = \frac{\sqrt{3}(n-2)+\sqrt{6}+2}{\sqrt{6}}$
11. $GA(P_n) = \frac{10n-20+8\sqrt{6}+5\sqrt{3}}{10}$
12. $B_1(P_n) = 8n + 14$
13. $B_2(P_n) = 8n + 22$
14. $HB_1(P_n) = 32n + 92$
15. $HB_2(P_n) = 32n + 210$
16. $mB_1(P_n) = \frac{15n+8}{30}$
17. $mB_2(P_n) = \frac{18n+8}{36}$
18. $H_b(P_n) = \frac{15n+8}{15}$

Proof:

1. First Zagreb Index
   $D_x f(x,y) = 2(n-2)x^2y^2 + 4x^2y^3 + xy^3$
   $D_y f(x,y) = 2(n-2)x^2y^2 + 6x^2y^3 + 3xy^3$
   Thus $M_1(P_n) = (D_x + D_y)f(x,y)\ at\ x = 1\ and\ y = 1$ gives
   $M_1(P_n) = 4n + 6$
2. Second Zagreb Index
   $D_y f(x,y) = 2(n-2)x^2y^2 + 6x^2y^3 + 3xy^3$
   $(D_x . D_y)f(x,y) = 4(n-2)x^2y^2 + 12x^2y^3 + 3xy^3$
   Thus $M_2(P_n) = (D_x . D_y)f(x,y)\ at\ x = 1\ and\ y = 1$ gives
   $M_2(P_n) = 4n + 7$
3. Modified Second Zagreb Index
   $S_y(f(x,y)) = \frac{(n-2)}{2}x^2y^2 + \frac{2x^2y^3}{3} + \frac{xy^3}{3}$
   $(S_x . S_y)f(x,y) = \frac{(n-2)}{4}x^2y^2 + \frac{x^2y^3}{3} + \frac{xy^3}{3}$
   Thus $m_{M_2}(P_n) = (S_x . S_y)f(x,y)\ at\ x = 1\ and\ y = 1$ gives
   $$m_{M_2}(P_n) = \frac{3n+2}{12}$$
4. General Randic Index
   $D_y^\alpha(f(x,y)) = 2^\alpha(n-2)x^2y^2 + 2.3^\alpha x^2y^3 + 3^\alpha xy^3$
   $(D_x^\alpha . D_y^\alpha)f(x,y) = 2^{2\alpha}(n-2)x^2y^2 + 2^{\alpha+1}.3^\alpha x^2y^3 + 3^\alpha xy^3$
   Thus $R_\alpha(P_n) = (D_x^\alpha . D_y^\alpha)f(x,y)\ at\ x = 1\ and\ y = 1$ gives

$R_\alpha(P_n) = 2^{2\alpha}(n-2) + 2^{\alpha+1}.3^\alpha + 3^\alpha$

5. Inverse Randic Index

$S_y^\alpha(f(x,y)) = \frac{(n-2)}{2^\alpha}x^2y^2 + \frac{2x^2y^3}{3^\alpha} + \frac{xy^3}{3^\alpha}$

$(S_x^\alpha \,.\, S_y^\alpha)f(x,y) = \frac{(n-2)}{2^{2\alpha}}x^2y^2 + \frac{2x^2y^3}{2^\alpha 3^\alpha} + \frac{xy^3}{3^\alpha}$

Thus $RR_\alpha(P_n) = (S_x^\alpha \,.\, S_y^\alpha)f(x,y)\ at\ x = 1\ and\ y = 1$ gives

$RR_\alpha(P_n) = \frac{(n-2)}{2^{2\alpha}} + \frac{2}{2^\alpha 3^\alpha} + \frac{1}{3^\alpha}$

6. Harmonic Index

$Jf(x,y) = (n-2)x^4 + 2x^5 + x^5$

$2S_x\,Jf(x,y) = \left[\frac{(n-2)}{2} + \frac{1}{2}\right]x^4 + \frac{4}{5}x^5$

Thus $H(P_n) = 2S_x\,Jf(x,y)\ at\ x = 1$ gives

$H(P_n) = \frac{5n+3}{10}$

7. Symmetric Division Index

$(D_xS_y)f(x,y) = (n-2)x^2y^2 + \frac{4x^2y^3}{3} + \frac{xy^3}{3}$

$(S_xD_y)f(x,y) = (n-2)x^2y^2 + 3x^2y^3 + 3xy^3$

$(D_xS_y + S_xD_y)f(x,y) = 2(n-2)x^2y^2 + \frac{10}{3}xy^3 + \left[\frac{4}{3} + 3\right]x^2y^3$

Thus $SSD(P_n) = (D_xS_y + S_xD_y)f(x,y)\ at\ x = y = 1$ gives

$SSD(P_n) = \frac{6n+11}{3}$

8. Augmented Zagreb Index

$D_y^3(f(x,y)) = 8(n-2)x^2y^2 + 54x^2y^3 + 27xy^3$

$D_x^3D_y^3(f(x,y)) = 64(n-2)x^2y^2 + 432x^2y^3 + 27xy^3$

$JD_x^3D_y^3(f(x,y)) = 64(n-2)x^4 + 432x^5 + 27x^4$

$Q_{-2}JD_x^3D_y^3(f(x,y)) = 64(n-2)x^2 + 432x^3 + 27x^2$

$(S_x^3Q_{-2}JD_x^3D_y^3)f(x,y) = 8(n-2)x^2 + \frac{432}{27}x^3 + \frac{27}{8}x^2$

Thus $A(P_n) = (S_x^3Q_{-2}JD_x^3D_y^3)f(x,y)\ at\ x = 1$ gives

$A(P_n) = \frac{1728n + 729}{216}$

9. Inverse Sum Index

$(D_xD_y)f(x,y) = 4(n-2)x^2y^2 + 12x^2y^3 + 3xy^3$

$(JD_xD_y)f(x,y) = 4(n-2)x^4 + 12x^5 + 3x^4$

$(S_xJD_xD_y)f(x,y) = (n-2)x^4 + \frac{12}{5}x^5 + \frac{3}{4}x^4$

Thus $I(P_n) = (S_xJD_xD_y)f(x,y)\ at\ x = 1$ gives

$$I(P_n) = \frac{20n + 23}{20}$$

10. Atom-bond Connectivity Index

$\left(S_y^{1/2}\right) f(x,y) = \frac{(n-2)x^2y^2}{\sqrt{2}} + \frac{2x^2y^3}{\sqrt{3}} + \frac{xy^3}{\sqrt{3}}$

$\left(S_x^{1/2} S_y^{1/2}\right) f(x,y) = \frac{(n-2)x^2y^2}{2} + \frac{2x^2y^3}{\sqrt{6}} + \frac{xy^3}{\sqrt{3}}$

$\left(J S_x^{1/2} S_y^{1/2}\right) f(x,y) = \frac{(n-2)x^4}{2} + \frac{2x^5}{\sqrt{6}} + \frac{x^4}{\sqrt{3}}$

$\left(Q_{-2} J S_x^{1/2} S_y^{1/2}\right) f(x,y) = \frac{(n-2)x^2}{2} + \frac{2x^3}{\sqrt{6}} + \frac{x^2}{\sqrt{3}}$

$\left(D_x^{1/2} Q_{-2} J S_x^{1/2} S_y^{1/2}\right) f(x,y) = \frac{(n-2)x^2}{\sqrt{2}} + \sqrt{2}x^3 + \frac{\sqrt{2}}{\sqrt{3}}x^2$

Thus $ABC(P_n) = \left(D_x^{1/2} Q_{-2} J S_x^{1/2} S_y^{1/2}\right) f(x,y)\ at\ x = 1$ gives

$$ABC(P_n) = \frac{\sqrt{3}(n-2) + \sqrt{6} + 2}{\sqrt{6}}$$

11. Geometric Arithmetic Index

$\left(D_y^{1/2}\right) f(x,y) = \sqrt{2}(n-2)x^2y^2 + 2\sqrt{3}x^2y^3 + \sqrt{3}xy^3$

$\left(D_x^{1/2} D_y^{1/2}\right) f(x,y) = 2(n-2)x^2y^2 + 2^{3/2}\sqrt{3}x^2y^3 + \sqrt{3}xy^3$

$\left(J D_x^{1/2} D_y^{1/2}\right) f(x,y) = 2(n-2)x^4 + 2^{3/2}\sqrt{3}x^5 + \sqrt{3}x^4$

$\left(2 S_x J D_x^{1/2} D_y^{1/2}\right) f(x,y) = (n-2)x^4 + \frac{2^{5/2}\sqrt{3}x^5}{5} + \frac{\sqrt{3}}{2}x^4$

Thus $GA(P_n) = \left(2 S_x J D_x^{1/2} D_y^{1/2}\right) f(x,y)\ at\ x = 1$ gives

$$GA(P_n) = \frac{10n - 20 + 8\sqrt{6} + 5\sqrt{3}}{10}$$

12. First K-Banahatti Index

$(2D_x Q_{-2} J) f(x,y) = 4(n-2)x^2 + 12x^3 + 4x^2$

$(D_x + D_y) f(x,y) = 4(n-2)x^2y^2 + 10x^2y^3 + 4xy^3$

Thus $B_1(P_n) = (D_x + D_y + 2D_x Q_{-2} J) f(x,y)\ at\ x = y = 1$ gives

$B_1(P_n) = 8n + 14$

13. Second K-Banahatti Index

$(Q_{-2} J (D_x + D_y)) f(x,y) = 4(n-2)x^2 + 10x^3 + 4x^2$

$\left(D_x Q_{-2} J (D_x + D_y)\right) f(x,y) = 8(n-2)x^2 + 30x^3 + 8x^2$

Thus $B_2(P_n) = (D_x Q_{-2} J (D_x + D_y)) f(x,y)\ at\ x = 1$ gives

$B_2(P_n) = 8n + 22$

14. First K-hyper Banahatti Index
$(D_x^2 + D_y^2)f(x,y) = 8(n-2)x^2y^2 + 26x^2y^3 + 10xy^3$
$(2D_x{}^2Q_{-2}J)f(x,y) = 8(n-2)x^2 + 36x^3 + 8x^2$
$\left(2D_xQ_{-2}J(D_x + D_y)\right)f(x,y) = 16(n-2)x^2 + 60x^3 + 16x^2$
Thus $HB_1(P_n) = \left(D_x^2 + D_y^2 + 2D_x{}^2Q_{-2}J + 2D_xQ_{-2}J(D_x + D_y)\right)f(x,y)\ at\ x = 1$ gives
$HB_1(P_n) = 32n + 92$

15. Second K-hyper Banahatti Index
$(D_x^2 + D_y^2)f(x,y) = 8(n-2)x^2y^2 + 26x^2y^3 + 10xy^3$
$\left(J(D_x^2 + D_y^2)\right)f(x,y) = 8(n-2)x^4 + 26x^5 + 10x^4$
$\left(Q_{-2}J(D_x^2 + D_y^2)\right)f(x,y) = 8(n-2)x^2 + 26x^3 + 10x^2$
$\left(D_x^2Q_{-2}J(D_x^2 + D_y^2)\right)f(x,y) = 32(n-2)x^2 + 234x^3 + 40x^2$
Thus $HB_2(P_n) = \left(D_x^2Q_{-2}J(D_x^2 + D_y^2)\right)f(x,y)\ at\ x = 1$ gives
$HB_2(P_n) = 32n + 210$

16. Modified First K-hyper Banahatti Index
$\left(Q_{-2}J(L_x + L_y)\right)f(x,y) = 2(n-2)x^4 + 3x^5 + 2x^6 + x^3$
$\left(S_xQ_{-2}J(L_x + L_y)\right)f(x,y) = \frac{(n-2)x^4}{2} + \frac{3x^5}{5} + \frac{x^6}{3} + \frac{x^3}{3}$
Thus $mB_1(P_n) = \left(S_xQ_{-2}J(L_x + L_y)\right)f(x,y)\ at\ x = 1$ gives
$mB_1(P_n) = \frac{15n + 8}{30}$

17. Modified Second K-hyper Banahatti Index
$\left(Q_{-2}J(S_x + S_y)\right)f(x,y) = (n-2)x^2 + \frac{5x^3}{3} + \frac{4x^2}{3}$
$\left(S_xQ_{-2}J(S_x + S_y)\right)f(x,y) = \frac{(n-2)x^2}{2} + \frac{5x^3}{9} + \frac{x^2}{2} + \frac{x^2}{6}$
Thus $mB_2(P_n) = \left(S_xQ_{-2}J(S_x + S_y)\right)f(x,y)\ at\ x = 1$ gives
$mB_2(P_n) = \frac{18n + 8}{36}$

18. Harmonic K-Banahatti Index
$\left(Q_{-2}J(L_x + L_y)\right)f(x,y) = \frac{(n-2)x^4}{2} + \frac{3x^5}{5} + \frac{x^6}{3} + \frac{x^3}{3}$
$\left(2S_xQ_{-2}J(L_x + L_y)\right)f(x,y) = (n-2)x^4 + \frac{6x^5}{5} + \frac{2x^6}{3} + \frac{2x^3}{3}$
Thus$H_b(P_n) = \left(2S_xQ_{-2}J(L_x + L_y)\right)f(x,y)\ at\ x = 1$ gives
$H_b(P_n) = \frac{15n + 8}{15}$

Theorem 2.5. If $S(n)$ is the sun graph, then

$$f(x,y) = 2n\, x^2y^{n+1} + \left(\frac{n^2-n}{2}\right)x^{n+1}y^{n+1}$$

Proof: By definition of sun graph and by computation, we find that $S(n)$ has $2n$ vertices and $\frac{n^2+3n}{2}$ edges. Based on the degrees of end vertices, the edge set of $S(n)$ can be tabulated in Table 5.

Table 5: Details of the degrees of vertices and number of edges in S(n)

(d_u, d_v)	$(2, n+1)$	$(n+1, n+1)$
Total number of edges	$2n$	$\frac{n^2-n}{2}$

The M-polynomial of $S(n)$ is given as

$$M(S(n){:}\,x,y) = f(x,y) = \sum_{\delta\le i\le j\le\Delta} m_{ij}(G)\, x^i y^j$$

$$f(x,y) = 2n\, x^2y^{n+1} + \left(\frac{n^2-n}{2}\right)x^{n+1}y^{n+1}$$

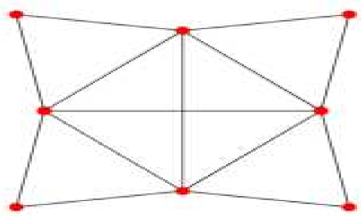

Figure 3: The n-sun graph $S(n)$

Theorem 2.6. Topological indices of sun graph $S(n)$ are given by the following.

1. $M_1(S(n)) = n^3 + 2n^2 + 5n$
2. $M_2(S(n)) = \frac{n^4+n^3+7n^2+7n}{2}$
3. $m_{M_2}(S(n)) = \frac{3n^2+n}{2(n+1)^2}$
4. $R_\alpha(S(n)) = 2^{\alpha+1}\, n(n+1)^\alpha + \left(\frac{n^2-n}{2}\right)(n+1)^{2\alpha}$
5. $RR_\alpha(S(n)) = \frac{2^{1-\alpha}\, n}{(n+1)^\alpha} + \left(\frac{n^2-n}{2}\right)\frac{1}{(n+1)^{2\alpha}}$
6. $H(S(n)) = \frac{4n}{n+3} + \left(\frac{n^2-n}{2}\right)\frac{1}{n+1}$
7. $SSD(S(n)) = \frac{2n(n^2+n+2)}{n+1}$
8. $A(S(n)) = 16n + \frac{(n^2-n)(n+1)^6}{16n^3}$

9. $I(S(n)) = \frac{n^4+3n^3+15n^2+13n}{4(n+3)}$
10. $ABC(S(n)) = \frac{2n}{\sqrt{2}} + (2n)^{1/2}\left(\frac{n^2-n}{2}\right)\frac{1}{n+1}$
11. $GA(S(n)) = \frac{4n\sqrt{2}\sqrt{n+1}}{n+3} + \left(\frac{n^2-n}{2}\right)$
12. $B_1(S(n)) = 3n^3 + 4n^2 + 9n$
13. $B_2(S(n)) = 2n^4 + 2n^3 + 6n^2 + 6n$
14. $HB_1(S(n)) = 9n^4 + 7n^3 + 23n^2 + 25n$
15. $HB_2(S(n)) = (n+1)^2(4n^4 - 2n^3 + 4n^2 + 10n)$
16. $mB_1(S(n)) = \frac{2n}{2n+2} + \frac{n^2-n}{3n+1} + \frac{2n}{n+3}$
17. $mB_2(S(n)) = \frac{n(n+3)}{(n+1)^2} + \frac{n^2-n}{2n(n+1)}$
18. $H_b(S(n)) = \frac{4n}{2n+2} + \frac{2(n^2-n)}{3n+1} + \frac{4n}{n+3}$

Proof:

1. First Zagreb Index

$$D_x f(x,y) = 4nx^2y^{n+1} + (n+1)\left(\frac{n^2-n}{2}\right)x^{n+1}y^{n+1}$$

$$D_y f(x,y) = 2n(n+1)x^2y^{n+1} + (n+1)\left(\frac{n^2-n}{2}\right)x^{n+1}y^{n+1}$$

Thus $M_1(S(n)) = (D_x + D_y)f(x,y)\ at\ x = 1\ and\ y = 1$ gives

$$M_1(S(n)) = n^3 + 2n^2 + 5n$$

2. Second Zagreb Index

$$D_y f(x,y) = 2n(n+1)x^2y^{n+1} + (n+1)\left(\frac{n^2-n}{2}\right)x^{n+1}y^{n+1}$$

$$(D_x . D_y)f(x,y) = 4n(n+1)x^2y^{n+1} + (n+1)^2\left(\frac{n^2-n}{2}\right)x^{n+1}y^{n+1}$$

Thus $M_2(S(n)) = (D_x . D_y)f(x,y)\ at\ x = 1\ and\ y = 1$ gives

$$M_2(S(n)) = \frac{n^4 + n^3 + 7n^2 + 7n}{2}$$

3. Modified Second Zagreb Index

$$S_y(f(x,y)) = \frac{2nx^2y^{n+1}}{(n+1)} + \left(\frac{n^2-n}{2(n+1)}\right)x^{n+1}y^{n+1}$$

$$(S_x . S_y)f(x,y) = \frac{nx^2y^{n+1}}{(n+1)} + \left(\frac{n^2-n}{2(n+1)^2}\right)x^{n+1}y^{n+1}$$

Thus $m_{M_2}(S(n)) = (S_x . S_y)f(x,y)\ at\ x = 1\ and\ y = 1$ gives

$$m_{M_2}(S(n)) = \frac{3n^2 + n}{2(n+1)^2}$$

4. General Randic Index

$$D_y^{\alpha}\big(f(x,y)\big) = 2n(n+1)^{\alpha}x^2y^{n+1} + (n+1)^{\alpha}\left(\frac{n^2-n}{2}\right)x^{n+1}y^{n+1}$$

$$\big(D_x^{\alpha}\,.\,D_y^{\alpha}\big)f(x,y) = 2n2^{\alpha}(n+1)^{\alpha}x^2y^{n+1} + (n+1)^{2\alpha}\left(\frac{n^2-n}{2}\right)x^{n+1}y^{n+1}$$

Thus $R_{\alpha}(S(n)) = \big(D_x^{\alpha}\,.\,D_y^{\alpha}\big)f(x,y)\ at\ x = 1\ and\ y = 1$ gives

$$R_{\alpha}(S(n)) = 2^{\alpha+1}\,n(n+1)^{\alpha} + (n+1)^{2\alpha}\left(\frac{n^2-n}{2}\right)$$

5. Inverse Randic Index

$$S_y^{\alpha}\big(f(x,y)\big) = \frac{2nx^2y^{n+1}}{(n+1)^{\alpha}} + \left(\frac{n^2-n}{2}\right)\frac{x^{n+1}y^{n+1}}{(n+1)^{\alpha}}$$

$$\big(S_x^{\alpha}\,.\,S_y^{\alpha}\big)f(x,y) = \frac{2nx^2y^{n+1}}{2^{\alpha}(n+1)^{\alpha}} + \left(\frac{n^2-n}{2}\right)\frac{x^{n+1}y^{n+1}}{(n+1)^{2\alpha}}$$

Thus $RR_{\alpha}(S(n)) = \big(S_x^{\alpha}\,.\,S_y^{\alpha}\big)f(x,y)\ at\ x = 1\ and\ y = 1$ gives

$$RR_{\alpha}(S(n)) = \frac{2^{1-\alpha}\,n}{(n+1)^{\alpha}} + \left(\frac{n^2-n}{2}\right)\frac{1}{(n+1)^{2\alpha}}$$

6. Harmonic Index

$$Jf(x,y) = 2nx^{n+3} + \left(\frac{n^2-n}{2}\right)x^{2n+2}$$

$$2S_x\,Jf(x,y) = \frac{4nx^{n+3}}{n+3} + \left(\frac{n^2-n}{2}\right)\frac{x^{2n+2}}{n+1}$$

Thus $H(S(n)) = 2S_x\,Jf(x,y)\ at\ x = 1$ gives

$$H(S(n)) = \frac{4n}{n+3} + \left(\frac{n^2-n}{2}\right)\frac{1}{n+1}$$

7. Symmetric Division Index

$$\big(D_xS_y\big)f(x,y) = \frac{4nx^2y^{n+1}}{n+1} + \left(\frac{n^2-n}{2}\right)x^{n+1}y^{n+1}$$

$$\big(S_xD_y\big)f(x,y) = n(n+1)x^2y^{n+1} + \left(\frac{n^2-n}{2}\right)x^{n+1}y^{n+1}$$

$$\big(D_xS_y + S_xD_y\big)f(x,y) = n\left((n+1) + \frac{4}{(n+1)}\right)x^2y^{n+1} + (n^2-n)x^{n+1}y^{n+1}$$

Thus $SSD(S(n)) = \big(D_xS_y + S_xD_y\big)f(x,y)\ at\ x = y = 1$ gives

$$SSD(S(n)) = \frac{2n(n^2+n+2)}{n+1}$$

8. Augmented Zagreb Index

$$D_y^3\big(f(x,y)\big) = 2n(n+1)^3x^2y^{n+1} + \left(\frac{n^2-n}{2}\right)(n+1)^3x^{n+1}y^{n+1}$$

$$D_x^3D_y^3\big(f(x,y)\big) = 16n(n+1)^3x^2y^{n+1} + \left(\frac{n^2-n}{2}\right)(n+1)^6x^{n+1}y^{n+1}$$

$$JD_x^3D_y^3\big(f(x,y)\big) = 16n(n+1)^3x^{n+3} + \left(\frac{n^2-n}{2}\right)(n+1)^6x^{2n+2}$$

$$Q_{-2}JD_x^3D_y^3\big(f(x,y)\big) = 16n(n+1)^3x^{n+1} + \left(\frac{n^2-n}{2}\right)(n+1)^6x^{2n}$$

$$\big(S_x^3Q_{-2}JD_x^3D_y^3\big)f(x,y) = 16nx^{n+1} + \left(\frac{n^2-n}{2}\right)\left(\frac{(n+1)^6}{(2n)^3}\right)x^{2n}$$

Thus $A(S(n)) = \big(S_x^3Q_{-2}JD_x^3D_y^3\big)f(x,y)\ at\ x=1$ gives

$$A(S(n)) = 16n + \frac{(n^2-n)(n+1)^6}{16n^3}$$

9. Inverse Sum Index

$$\big(D_xD_y\big)f(x,y) = 4n(n+1)x^2y^{n+1} + \left(\frac{n^2-n}{2}\right)(n+1)^2x^{n+1}y^{n+1}$$

$$\big(JD_xD_y\big)f(x,y) = 4n(n+1)x^{n+3} + \left(\frac{n^2-n}{2}\right)(n+1)^2x^{2n+2}$$

$$\big(S_xJD_xD_y\big)f(x,y) = \frac{4n(n+1)}{(n+3)}x^{n+3} + \left(\frac{n^2-n}{2}\right)\frac{(n+1)^2}{(2n+2)}x^{2n+2}$$

Thus $I(S(n)) = \big(S_xJD_xD_y\big)f(x,y)\ at\ x=1$ gives

$$I(S(n)) = \frac{n^4+3n^3+15n^2+13n}{4(n+3)}$$

10. Atom-bond Connectivity Index

$$\left(S_y^{1/2}\right)f(x,y) = \frac{2nx^2y^{n+1}}{\sqrt{n+1}} + \left(\frac{n^2-n}{2}\right)\frac{x^{n+1}y^{n+1}}{\sqrt{n+1}}$$

$$\left(S_x^{1/2}S_y^{1/2}\right)f(x,y) = \frac{2nx^2y^{n+1}}{\sqrt{2(n+1)}} + \left(\frac{n^2-n}{2}\right)\frac{x^{n+1}y^{n+1}}{(n+1)}$$

$$\left(JS_x^{1/2}S_y^{1/2}\right)f(x,y) = \frac{2nx^{n+3}}{\sqrt{2(n+1)}} + \left(\frac{n^2-n}{2}\right)\frac{x^{2n+2}}{(n+1)}$$

$$\left(Q_{-2}JS_x^{1/2}S_y^{1/2}\right)f(x,y) = \frac{2nx^{n+1}}{\sqrt{2(n+1)}} + \left(\frac{n^2-n}{2}\right)\frac{x^{2n}}{(n+1)}$$

$$\left(D_x^{1/2}Q_{-2}JS_x^{1/2}S_y^{1/2}\right)f(x,y) = \frac{2nx^{n+1}}{\sqrt{2}} + \sqrt{2n}\left(\frac{n^2-n}{2}\right)\frac{x^{2n}}{(n+1)}$$

Thus $ABC(S(n)) = \left(D_x^{1/2}Q_{-2}JS_x^{1/2}S_y^{1/2}\right)f(x,y)\ at\ x=1$ gives

$$ABC(S(n)) = \frac{2n}{\sqrt{2}} + (2n)^{1/2}\left(\frac{n^2-n}{2}\right)\frac{1}{n+1}$$

11. Geometric Arithmetic Index

$$\left(D_y^{1/2}\right)f(x,y) = 2n\sqrt{(n+1)}x^2y^{n+1} + \left(\frac{n^2-n}{2}\right)\sqrt{(n+1)}\,x^{n+1}y^{n+1}$$

$$\left(D_x^{1/2}D_y^{1/2}\right)f(x,y) = 2n\sqrt{2(n+1)}x^2y^{n+1} + \left(\frac{n^2-n}{2}\right)(n+1)\,x^{n+1}y^{n+1}$$

$$\left(JD_x^{1/2}D_y^{1/2}\right)f(x,y) = 2n\sqrt{2(n+1)}x^{n+3} + \left(\frac{n^2-n}{2}\right)(n+1)\,x^{2n+2}$$

$$\left(2S_xJD_x^{1/2}D_y^{1/2}\right)f(x,y) = \frac{4n\sqrt{2(n+1)}x^{n+3}}{(n+3)} + \left(\frac{n^2-n}{2}\right)\,x^{2n+2}$$

Thus $GA(S(n)) = \left(2S_xJD_x^{1/2}D_y^{1/2}\right)f(x,y)\ at\ x=1$ gives

$$GA(S(n)) = \frac{4n\sqrt{2}\sqrt{n+1}}{n+3} + \left(\frac{n^2-n}{2}\right)$$

12. First K-Banahatti Index

$$(2D_xQ_{-2}J)f(x,y) = 4n(n+1)x^{n+1} + 2n(n^2-n)x^{2n}$$

$$\left(D_x+D_y\right)f(x,y) = 2n(n+3)x^2y^{n+1} + (n+1)(n^2-n)x^{n+1}y^{n+1}$$

Thus $B_1(S(n)) = \left(D_x+D_y+2D_xQ_{-2}J\right)f(x,y)\ at\ x=y=1$ gives

$$B_1(S(n)) = 3n^3+4n^2+9n$$

13. Second K-Banahatti Index

$$(Q_{-2}J(D_x+D_y))f(x,y) = 2n(n+3)x^{n+1} + (n+1)(n^2-n)x^{2n}$$

$$\left(D_xQ_{-2}J\left(D_x+D_y\right)\right)f(x,y) = 2n(n+1)(n+3)x^{n+1} + 2n(n+1)(n^2-n)x^{2n}$$

Thus $B_2(S(n)) = \left(D_xQ_{-2}J(D_x+D_y)\right)f(x,y)\ at\ x=1$ gives

$$B_2(S(n)) = 2n^4+2n^3+6n^2+6n$$

14. First K-hyper Banahatti Index

$$(D_x^2+D_y^2)f(x,y) = 2n((n+1)^2+4)x^2y^{n+1} + (n+1)^2(n^2-n)x^{n+1}y^{n+1}$$

$$\left(2D_x{}^2Q_{-2}J\right)f(x,y) = 4n(n+1)^2x^{n+1} + (2n)^2(n^2-n)x^{2n}$$

$$\left(2D_xQ_{-2}J\left(D_x+D_y\right)\right)f(x,y) = 4n(n+1)(n+3)x^{n+1} + 4n(n+1)(n^2-n)x^{2n}$$

Thus $HB_1(S(n)) = \left(D_x^2+D_y^2+2D_x{}^2Q_{-2}J + 2D_xQ_{-2}J\left(D_x+D_y\right)\right)f(x,y)\ at\ x=1$ gives

$$HB_1(S(n)) = 9n^4+7n^3+23n^2+25n$$

15. Second K-hyper Banahatti Index

$$(D_x^2+D_y^2)f(x,y) = 2n(n^2+2n+5)x^2y^{n+1} + (n+1)^2(n^2-n)x^{n+1}y^{n+1}$$

$$\left(J\left(D_x^2+D_y^2\right)\right)f(x,y) = 2n(n^2+2n+5)x^{n+3} + (n+1)^2(n^2-n)x^{2n+2}$$

$$\left(Q_{-2}J\left(D_x^2+D_y^2\right)\right)f(x,y) = 2n(n^2+2n+5)x^{n+1} + (n+1)^2(n^2-n)x^{2n}$$

$$\left(D_x^2Q_{-2}J\left(D_x^2+D_y^2\right)\right)f(x,y) = 2n(n+1)^2(n^2+2n+5)x^{n+1} + (n+1)^2(2n)^2(n^2-n)x^{2n}$$

Thus $HB_2(S(n)) = \left(D_x^2 Q_{-2} J\left(D_x^2 + D_y^2\right)\right) f(x,y)\ at\ x = 1$ gives

$$HB_2(S(n)) = (n+1)^2(4n^4 - 2n^3 + 4n^2 + 10n)$$

16. Modified First K-hyper Banahatti Index

$$\left(Q_{-2}J\left(L_x + L_y\right)\right) f(x,y) = 2nx^{2n+2} + (n^2-n)x^{3n+1} + 2nx^{n+3}$$

$$\left(S_x Q_{-2}J\left(L_x + L_y\right)\right) f(x,y) = \frac{2nx^{2n+2}}{2n+2} + \frac{(n^2-n)x^{3n+1}}{3n+1} + \frac{2nx^{n+3}}{n+3}$$

Thus $mB_1(S(n)) = (S_x Q_{-2}J(L_x + L_y)) f(x,y)\ at\ x = 1$ gives

$$mB_1(S(n)) = \frac{2n}{2n+2} + \frac{n^2-n}{3n+1} + \frac{2n}{n+3}$$

17. Modified Second K-hyper Banahatti Index

$$\left(Q_{-2}J\left(S_x + S_y\right)\right) f(x,y) = \frac{n(n+3)x^{n+1}}{n+1} + \frac{(n^2-n)x^{2n}}{n+1}$$

$$\left(S_x Q_{-2}J\left(S_x + S_y\right)\right) f(x,y) = \frac{n(n+3)x^{n+1}}{(n+1)^2} + \frac{(n^2-n)x^{2n}}{(n+1)(2n)}$$

Thus $mB_2(S(n)) = (S_x Q_{-2}J(S_x + S_y)) f(x,y)\ at\ x = 1$ gives

$$mB_2(S(n)) = \frac{n(n+3)}{(n+1)^2} + \frac{n^2-n}{2n(n+1)}$$

18. Harmonic K-Banahatti Index

$$\left(Q_{-2}J\left(L_x + L_y\right)\right) f(x,y) = 2nx^{2n+2} + (n^2-n)x^{3n+1} + 2nx^{n+3}$$

$$\left(2S_x Q_{-2}J\left(L_x + L_y\right)\right) f(x,y) = \frac{4nx^{2n+2}}{2n+2} + \frac{2(n^2-n)x^{3n+1}}{3n+1} + \frac{4nx^{n+3}}{n+3}$$

Thus$H_b(S(n)) = (2S_x Q_{-2}J(L_x + L_y)) f(x,y)\ at\ x = 1$ gives

$$H_b(S(n)) = \frac{4n}{2n+2} + \frac{2(n^2-n)}{3n+1} + \frac{4n}{n+3}$$

3. Conclusion

In this paper, we have obtained M-polynomial of 18 different topological indices of Barbell graph, Pan Graph and Sun graph. We can find M-polynomial on different graphs like honey comb graph and its derived graphs etc.

Acknowledgement. The authors are thankful to the reviewer for their comments and suggestions.

Author Contribution: All the authors contributed equally to writing of this paper.

Conflict of Interest: The authors declare that they have no conflict of interest.

REFERENCES

1. A. Graovac, T. Pisanski On the Wiener index of a graph, *J. Math. Chem.*, 8 (1991) 53-62.

2. A. Bollobas and P. Erdos. Graphs of extremal weights, *Ars Combinatoria*, 50 (1998) 225–233.
3. A. Jalil M. Khalaf, S. Hussain, F. Afzal, A. Maqbool, D. Afzal. M-Polynomial and Topological Indices of Book Graph, *Journal of Discrete Mathematical Sciences and Cryptography*. 23(6) (2020) 1217-1237.
4. B. Furtula, A. Graovac, and D. VukičeviĆ. Augmented Zagreb index, *Journal of Mathematical Chemistry*, 48(2) (2010) 370–380.
5. B. Basavanagoud and P. Jakkannavar, Kulli-Basava indices of graphs, *Int. J. Appl. Eng. Res.*, 14(1) (2019) 325−342.
6. B. Basavanagoud and P. Jakkannavar, M-polynomial and degree-based topological indices of graphs, *Electronic J. Math. Anal. Appl.*, 8 (1) (2020) 75−99.
7. B. Liu, I. Gutman, On general Randi´c indices, *MATCH Commun. Math. Com-put. Chem.*, 58 (2007) 147–154.
8. E. Deutsch and S. Klavžar, M-Polynomial and degree-based topological indices, *Iran. J. Math. Chem.*, 6 (2) (2015) 93−102.
9. E. Estrada, L. Torres, L. Rodríguez and I. Gutman. An atom-bond connectivity index: Modelling the enthalpy of formation of alkanes, *Indian Journal of Chemistry*, 37(10) (1998) 849-855.
10. F. Harary, *Graph Theory*, Addison-Wesley, Reading- Mass (1969).
11. F. Afzal, S. Hussain, D. Afzal, S. Razaq. Some new degree based topological indices via m-Polynomial, *Journal of Information and Optimization Sciences*, 41(4) (2020) 1061-1076.
12. H. Zhang and F. Zhang, The Clar covering polynomial of hexagonal systems I, *Discrete Appl. Math.*, 69 (1996) 147−167.
13. I. Gutman and N. Trinajstic, *Chemical Physics Letters*, 17 (1972) 535-538.
14. I. Gutman, K.C. Das, The first Zagreb index 30 years after, *MATCH Commun. Math. Comput. Chem.*, 50 (2004) 83-92.
15. K. C. Das, Atom–bond connectivity index of graphs, *Discr. Appl. Math.*, 158 (2010) 1181–1188.
16. M. S. Anjum and M. U. Safdar, K Banhatti and K hyper-Banhatti indices of nanotubes, *Eng. Appl. Sci. Lett.*, 2 (1) (2019) 19-37.
17. M. Cancan, S. Ediz, H. Mutee-Ur-Rehman, and D. Afzal. M-polynomial and topological indices poly (Ethylene Amido Amine) dendrimers, *Journal of Information and Optimization Sciences*, 41(4) (2020) 1117-1131.
18. Mallikarjun Basanna Kattimani and Smita K. Phadnis, Maximal transitive domination number of a graph, *Journal of the Maharaja Sayajirao University of Baroda*, 54(2) (VI) (2020) 159-161.
19. Mallikarjun B Kattimani and Selastina Mary A, Transitive domination number of a graph, *Journal of the Maharaja Sayajirao University of Baroda*, 54(2) (VII) (2020) 41-43.
20. R. Khalid, N. Idrees, and M. J. Saif. Topological characterization of book graph and stacked bookgraph, *Computers, Materials & Amp; Continua*, 60(1) (2019) 41–54.
21. S.M. Kang, W. Nazeer, W. Gao, D. Afzal and S. N. Gillani, M-polynomials and topological indices of dominating David derived networks, *Open Chem.*, 16 (2018) 201−213.

22. S. Ragavi, The First and Second Zagreb indices of some special types of graphs, *Indian Journal of Applied Research*, 9(4) (2019).
23. S. Ragavi, The First and Second Zagreb Indices of Degree Splitting of Graphs, *IJMTT*, 65(2) (2019) 31-36.
24. S. Fajtlowicz. On conjectures of graffiti II., *Congr*, 60 (1987) 189–197.
25. T. Balaban. Highly discriminating distance based numerical descriptor, *Chemical Physics Letters*, 89(5) (1982) 399-404.
26. V.R. Kulli, The (a,b)-status index of graphs, *Annals of Pure and Applied Mathematics*, 21(2) (2020) 113-118.
27. V.R Kulli, Connectivity neighbourhood Dakshyani indices POPAM dendrimers, *Annals of Pure and Applied Mathematics*, 2(1) (2019) 49-54.
28. V.R. Kulli, The Gourava indices and coindices of graphs, *Annals of Pure and Applied Mathematics,* 14(1) (2017) 33-38.
29. V.R. Kulli, New Kulli-Basava indices of graphs, *International Research Journal of Pure Algebra,* 9(7) (2019) 58-63.
30. V.R. Kulli, On the sum connectivity reverse index of oxide and honeycomb networks, *Journal of computer and Mathematical Sciences,* 8(9) (2017) 408- 413.
31. V.R. Kulli, Ravan indices and their polynomials of certain rhombus networks, *International Journal of Current Research in Life Sciences*, 7(5) (2018) 2110-2116.
32. V.R. Kulli, On K Banahatti indices of graphs. Journal of Computer and Mathematical Sciences, 7(4) (2016) 213–218.
33. V.R.Kulli, On KBanhatti indices and K hyper-Banhatti indices of V-Phenylenic nanotubes and nanotorus, *Journal of Computer and Mathematical Sciences,* 7(6) (2016) 302-307.
34. V.R.Kulli, On KBanhatti indices and K hyper-Banhatti indices of V-Phenylenic nanotubes and nanotorus, *Journal of Computer and Mathematical Sciences,* 7(6) (2016) 302-307.
35. V.R.Kulli, On KBanhatti indices and K hyper-Banhatti indices of V-Phenylenic nanotubes and nanotorus, *Journal of Computer and Mathematical Sciences,* 7(6) (2016) 302-307.
36. V.R.Kulli, On K hyper-Banhatti indices and coindices of graphs, *International Research Journal of Pure Algebra*, 6(5) (2016) 300-304.
37. V.R. Kulli, On K hyper-Banahatti indices and coindices of graphs, *International Research Journal of Pure Algebra*, 6(5) (2016) 300-304.
38. V.R. Kulli, B. Cheluvaraju and H.S. Baregowda, Some bounds of sum connectivity Banahatti index of graphs, *Palestine Journal of Mathematics*, 8(2) (2019) 355-364.
39. V.R. Kulli ,On hyper Gourava indices and coindices, *International Journal of Mathematical Archive,* 8(12) (2017).116-120.
40. X. Li and I. Gutman, *Mathematical Aspects of Randi´c-Type Molecular Structure Descriptors,* Mathematical Chemistry Monographs No.1, Kragujevac, 2006.
41. Y. C. Kwun, M. Munir, W. Nazeer, R. Rafique and S. M. Kang, M-polynomials and topological indices of V-phenylenic nanotubes and nanotori, *Sci. Reports*, 7 (2017) Art. 8756.