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Equitable Symmetric *n*-Sigraphs

K. M. Manjula[1], C. N. Harshavardhana[2*] and R. Kemparaju[3]

[1]Department of Mathematics, Government First Grade College for Women
Hassan-573 202, India. E-mail: manjula.km.gowda@gmail.com
[2]Department of Mathematics, Government First Grade College for Women
Holenarasipura-573 211, India. E-mail: cnhmaths@gmail.com
[3]Department of Mathematics, Government First Grade College
T.Narasipura-571 124, India. E-mail: kemps007@gmail.com

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Abstract. An *n-tuple* $(a_1, a_2, ..., a_n)$ is *symmetric*, if $a_k = a_{n-k+1}$, $1 \leq k \leq n$. Let $H_n = \{(a_1, a_2, ..., a_n) : a_k \in \{+, -\}, a_k = a_{n-k+1}, 1 \leq k \leq n\}$ be the set of all symmetric n-tuples. A *symmetric n-sigraph (symmetric n-marked graph)* is an ordered pair $S_n = (G, \sigma)$ $(S_n = (G, \mu))$, where $G = (V, E)$ is a graph called the *underlying graph* of S_n and

$$\sigma : E \rightarrow H_n \ (\mu : V \rightarrow H_n)$$

is a function. In this paper, we introduced a new notion equitable symmetric n-sigraph of a symmetric n-sigraph and its properties are obtained. Also, we obtained the structural characterization of equitable symmetric n-signed graphs.

Keywords: Symmetric n-sigraphs, Symmetric n-marked graphs, Balance, Switching, Equitable n-sigraphs, Complementation

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1. Introduction

Unless mentioned or defined otherwise, for all terminology and notion in graph theory the reader is refer to [3]. We consider only finite, simple graphs free from self-loops.

Let $n \geq 1$ be an integer. An *n-tuple* $(a_1, a_2, ..., a_n)$ is *symmetric*, if $a_k = a_{n-k+1}$, $1 \leq k \leq n$. Let $H_n = \{(a_1, a_2, ..., a_n) : a_k \in \{+, -\}, a_k = a_{n-k+1}, 1 \leq k \leq n\}$ be the set of all symmetric n-tuples. Note that H_n is a group under coordinate wise multiplication, and the order of H_n is 2^m, where $m = \left\lceil \frac{n}{2} \right\rceil$.

A *symmetric n-sigraph (symmetric n-marked graph)* is an ordered pair $S_n = (G, \sigma)$ $(S_n = (G, \mu))$, where $G = (V, E)$ is a graph called the underlying graph of S_n and $\sigma : E \rightarrow H_n$ $(\mu : V \rightarrow H_n)$ is a function.

In this paper by an *n-tuple/n-sigraph/n-marked graph* we always mean a symmetric n-tuple/symmetric n-sigraph/symmetric n-marked graph.

An n-tuple $(a_1, a_2, ..., a_n)$ is the *identity n-tuple*, if $a_k = +$, for $1 \leq k \leq n$, otherwise it is a *non-identity n-tuple*. In an n-sigraph $S_n = (G, \sigma)$ an edge labelled with the identity n-tuple is called an *identity edge*, otherwise it is a *non-identity edge*.

Further, in an n-sigraph $S_n = (G, \sigma)$, for any $A \subseteq E(G)$ the *n-tuple* $\sigma(A)$ is the product of the n-tuples on the edges of A.

In [10], the authors defined two notions of balance in n-sigraph $S_n = (G, \sigma)$ as follows (See also R. Rangarajan and P.S.K.Reddy [6]):

Definition 1.1. Let $S_n = (G, \sigma)$ be an n-sigraph. Then,
(i) S_n is *identity balanced* (or *i-balanced*), if product of n-tuples on each cycle of S_n is the identity n-tuple and
(ii) S_n is *balanced*, if every cycle in S_n contains an even number of non-identity edges.

Note: An i-balanced n-sigraph need not be balanced and conversely.

The following characterization of i-balanced n-sigraphs is obtained in [10].

Theorem 1.1. (**E. Sampathkumar et al. [10]**) An n-sigraph $S_n=(G, \sigma)$ is i-balanced if, and only if, it is possible to assign n-tuples to its vertices such that the n-tuple of each edge uv is equal to the product of the n-tuples of u and v.

In [10], the authors also have defined switching and cycle isomorphism of an n-sigraph $S_n = (G, \sigma)$ as follows: (See also [1, 4, 5, 7–9, 12–23])

Let $S_n = (G, \sigma)$ and $S'_n = (G', \sigma')$ be two n-sigraphs. Then S_n and S'_n are said to be *isomorphic*, if there exists an isomorphism $\phi : G \to G'$ such that if uv is an edge in S_n with label $(a_1, a_2, ..., a_n)$ then $\phi(u)\phi(v)$ is an edge in S'_n with label $(a_1, a_2, ..., a_n)$.

Given an n-marking μ of an n-sigraph $S_n = (G, \sigma)$, *switching* S_n with respect to μ is the operation of changing the n-tuple of every edge uv of S_n by $\mu(u)\sigma(uv)\mu(v)$. The n-sigraph obtained in this way is denoted by $S_\mu(S_n)$ and is called the *μ-switched n-sigraph* or just *switched n-sigraph*.

Further, an n-sigraph S_n *switches* to n-sigraph S'_n (or that they are *switching equivalent* to each other), written as $S_n \sim S'_n$ whenever there exists an n-marking of S_n such that $S_\mu(S_n) \cong S'_n$.

Two n-sigraphs $S_n = (G, \sigma)$ and $S_n' = (G', \sigma')$ are said to be *cycle isomorphic*, if there exists an isomorphism $\phi : G \to G'$ such that the n-tuple $\sigma(C)$ of every cycle C in S_n equals to the n-tuple $\sigma(\phi(C))$ in S_n'.

We make use of the following known result (see [10]).

Theorem 1.2. (**E. Sampathkumar et al. [10]**) Given a graph G, *any two n-sigraphs* with G as underlying graph are *switching equivalent* if and only if, they are *cycle isomorphic*.

2. Equitable n-sigraph of an *n*-sigraph

A subset D of V is called an *equitable dominating set* if for every $v \in V - D$ there exists a vertex $u \in D$ such that $uv \in E(G)$ and $|deg(u) - deg(v)| \leq 1$. Further, a vertex $u \in V$ is said to be *degree equitable* with a vertex $v \in V$ if $|deg(u) - deg(v)| \leq 1$.

Let $u \in V(G)$. Then the number of vertices which are degree equitable with u, is called degree equitable number of u.

In [2], Dharmalingam introduced equitable graph of a graph as follows: Let $G = (V, E)$ be a graph. The equitable graph $E_t(G)$ of G is defined as the graph with vertex set as $V(G)$ and two vertices u and v are adjacent if and only if u and v are degree equitable.

Motivated by the existing definition of complement of an n-sigraph, we extend the notion of equitable graphs to n-sigraphs as follows: The *equitable n-sigraph* $E_t(S_n)$ of an n-sigraph $S_n=(G,\sigma)$ is an n-sigraph whose underlying graph is $E_t(G)$ and the n-tuple of any edge uv is $E_t(S_n)$ is $\mu(u)\mu(v)$, where μ is the canonical n-marking of S_n. Further, an n-sigraph $S_n=(G,\sigma)$ is called equitable n-sigraph, if $S_n \cong E_t(S_n')$ for some n-*sigraph* S_n'. The following result indicates the limitations of the notion $E_t(S_n)$ as introduced above, since the entire class of i-unbalanced n-sigraphs is forbidden to be equitable n-sigraphs.

Theorem 2.1. For any n-sigraph $S_n = (G, \sigma)$, its *equitable n-sigraph $E_t(S_n)$ is i-balanced.*

Proof: Since the n-tuple of any edge uv in $E_t(Sn)$ is $\mu(u)\mu(v)$, where μ is the canonical *n-marking* of S_n, by Theorem 1.1, *$E_t(S_n)$ is i-balanced.*

For any positive integer k, the k^{th} iterated equitable *n-sigraph* $E_t(S_n)$ of S_n is defined as follows:

$$(E_t)^0(S_n) = S_n,\ (E_t)^k(S_n) = E_t((E_t)^{k-1}(S_n)).$$

Corollary 2.2. For any *n-sigraph* $S_n = (G, \sigma)$ and any positive integer k, $(E_t)^k(S_n)$ is *i-alanced.*

Theorem 2.3. *An n-sigraph $S_n = (G, \sigma)$ is an equitable n-sigraph if and only if, S_n is i-balanced n-sigraph and its underlying graph G is an equitable graph.*

Proof: Suppose that S_n is i-balanced and G is a $E_t(G)$. Then there exists a graph H such that $E_t(H) \cong G$. Since S_n is i-balanced, by Theorem 1.1, there exists an n-marking μ of G such that each edge uv in S_n satisfies $\sigma(uv) = \mu(u)\mu(v)$. Now consider the n-sigraph $S_n' = (H, \sigma')$, where for any edge e in H, $\sigma'(e)$ is the n-marking of the corresponding vertex in G. Then clearly, $E_t(S_n') \cong S_n$. Hence S_n is an equitable n-sigraph.

Conversely, suppose that $S_n = (G, \sigma)$ is an equitable n-sigraph. Then there exists an n-sigraph $S_n' = (H, \sigma')$ such that $E_t(S_n') \cong S_n$. Hence G is the $E_t(G)$ of H and by Theorem 2.1, S_n is i-balanced.

In [2], Dharmalingam characterized graphs for which $E_t(G) \cong G$.

Theorem 2.4. (K. M. Dharmalingam [2])

For any graph $G = (V, E)$, $E_t(G) \cong G$ if and only if G is K_n.

We now characterize n-sigraphs which are switching equivalent to their equitable n-sigraphs.

Theorem 2.5. *For any n-sigraph* $S_n = (G, \sigma)$, $S_n \sim E_t(S_n)$ *if and only if G is* K_n *and* S_n *is i-balanced.*

Proof: Suppose $S_n \sim E_t(S_n)$. This implies, $G \cong E_t(G)$ and hence by Theorem 2.4, we see that G must be isomorphic to K_n. Now, if S_n is any n-sigraph with underlying graph as K_n, Theorem 2.1 implies that $E_t(S_n)$ is i-balanced and hence if S_n is i-unbalanced its $E_t(S_n)$ being i-balanced cannot be switching equivalent to S_n in accordance with Theorem 1.2. Therefore, S_n must be i-balanced.
Conversely, suppose that G is K_n and S_n is i-balanced. Since $E_t(S_n)$ is i- balanced as per Theorem 2.1, the result follows from Theorem 1.2 again.

Theorem 2.6. *For any two n-sigraphs* S_n *and* S_n' *with the same underlying graph their equitable n-sigraphs are switching equivalent.*

Proof: Suppose $S_n = (G, \sigma)$ and $S'_n = (G', \sigma')$ be two n-sigraphs with $G \cong G'$. By Theorem 2.1, $E_t(S_n)$ and $E_t(S_n')$ are i-balanced and hence, the result follows from Theorem 1.2.

3. Complementation

In this section, we investigate the notion of complementation of a graph whose edges have signs (a sigraph) in the more general context of graphs with multiple signs on their edges. We look at two kinds of complementation: complementing some or all of the signs, and reversing the order of the signs on each edge.

For any $m \in H_n$, the *m-complement* of $a = (a_1, a_2, ..., a_n)$ is: $a^m = am$. For any $M \subseteq H_n$, and $m \in H_n$, the *m-complement* of M is $M^m = \{a^m : a \in M\}$.

For any $m \in H_n$, the *m-complement* of an n-sigraph $S_n = (G, \sigma)$, written (S_n^m), is the same graph but with each edge label $a = (a_1, a_2, ..., a_n)$ replaced by a^m.

For an n-sigraph $S_n = (G, \sigma)$, the $E_t(S_n)$ is i-balanced. We now examine, the condition under which m-complement of $E_t(S_n)$ is i-balanced, where for any $m \in H_n$.

Theorem 3.1. Let $S_n=(G, \sigma)$ be an n-sigraph. Then, for any $m \in H_n$, if $E_t(G)$ is bipartite then $(E_t(S_n))^m$ is i-balanced.
Proof: Since, by Theorem 2.1, $E_t(S_n)$ is i-balanced, for each k, $1 \leq k \leq n$, the number of n-tuples on any cycle C in $E_t(S_n)$ whose k^{th} co-ordinate are $-$ is even. Also, since $E_t(G)$ is bipartite, all cycles have even length; thus, for each k, $1 \leq k \leq n$, the number of n-tuples on any cycle C in $E_t(S_n)$ whose k^{th} co-ordinate are $+$ is also even. This implies that the same thing is true in any m-complement, where for any $m, \in H_n$. Hence $(E_t(S_n))^t$ is i-balanced.

Theorem 2.5 and 2.6 provides easy solutions to other n-sigraph switching equivalence relations, which are given in the following results.

Corollary 3.2. *For an two n-sigraphs S_n and S_n' with the same underlying graph, $E_t(S_n)$ and $E_t((S_n')^m)$ are switching equivalent.*

Corollary 3.3. For any two *n-sigraphs* S_n and S_n' with the same underlying graph, graph, $E_t((S_n)^m)$ and $E_t(S_n')$ are *switching quivalent.*

Corollary 3.4. For any two *n-sigraphs* S_n and S_n' with the same underlying graph, graph, $E_t((S_n)^m)$ *and* $E_t((S_n')^m)$ are *switching equivalent.*

Corollary 3.5. For any two *n-sigraphs* $S_n = (G, \sigma)$ *and* $S_n' = (G', \sigma')$ with the $G \cong G'$ and G, G' are *bipartite,* $(E_t(S_n))^m$ and $E_t(S_n')$ are *switching equivalent.*

Corollary 3.6. For any two *n-sigraphs* $S_n = (G, \sigma)$ *and* $S_n' = (G', \sigma')$ with the $G \cong G'$ *and* G, G' are *bipartite,* $E_t(S_n)$ *and* $(E_t(S_n'))^m$ are *switching equivalent.*

Corollary 3.7. For any two *n-sigraphs* $S_n = (G, \sigma)$ *and* $S_n' = (G', \sigma')$ *with the* $G \cong G'$ *and* G, G' *are bipartite,* $(E_t(S_1))^m$ *and* $(E_t(S_2))^m$ *are switching equivalent.*

Corollary 3.8. For any *n-sigraph* $S_n = (G, \sigma)$, $S_n \sim \mathrm{E}_t((S_n)^m)$ *if and only if* G *is* K_n and S_n *is i-balanced.*

4. Conclusion

We have introduced a new notion for *n*-signed graphs called equitable *n*-sigraph of an *n*-signed graph. We have proved some results and presented the structural characterisation of the equitable *n*-signed graph. There is no structural characterization of the equitable graph, but we have obtained the structural characterisation of an equitable *n*-signed graph.

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