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Existence of Symmetric Positive Solutions for the Fourth-Order Boundary Value Problem

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Abstract. In this paper, we study the existence of positive solutions for a class of fourth-order two-point boundary value problems:

$$u^{(4)}(t) = f\big(u(t)\big), \quad t \in [0,1],$$
$$u(0) = u(1) = u'(0) = u'(1) = 0.$$

where $f: R \to [0,\infty)$ is continuous. When the nonlinear f satisfies appropriate growth conditions, the problem is transformed into the existence of fixed points of a fully continuous operator on a special cone by using the properties of Green's function. By using the generalized Leggett-Williams fixed point theorem, we obtain that there are at least three symmetric solutions to the problem.

Keywords: Boundary value problem, greens function, multiple solution, fixed point theorem.

AMS Mathematics Subject Classification (2010): 30E25

1. Introduction

In the past 20 years, there has been attention focused on the existence of positive solutions to boundary value problems for ordinary differential equations, see [1-8]. In 2012, Sun and Zhao [9] proved the existence of three positive solutions for a third-order three-point BVP with sign-changing Green's function by apply the Leggett-Williams fixed point theorem

$$u'''(t) = f(t, u(t)), t \in [0,1],\ u'(0) = u''(\eta) = u(1) = 0,$$

where $f \in C([0,1]\times[0,+\infty)), \eta \in [2-\sqrt{2},1)$.

In 2015, Zhou and Zhang [10] by using Leggett-Williams fixed point theorem and Holder inequality, the existence of three positive solutions for the fourth-order impulsive differential equations with integral boundary conditions

$$u^{(4)}(t) = w(t) f(t, x(t)), 0 < t < 1, t \neq t_k,$$

$$\Delta x\big|_{t=t_k} = I_k(t_k, x(t_k)), \Delta x'\big|_{t=t_k} = 0, k = 1,2,\cdots,m, \quad x(0) = \int_0^1 g(s)x(s)ds, x'(1) = 0,$$

$$x''(0) = \int_0^1 h(s)x''(s)ds, x'''(1) = 0.$$

Here $w \in L^p[0,1]$ for some $1 \leq p \leq +\infty$, $t_k (k = 1,2,\cdots,m)$ (where m is fixed positive integer) are fixed points with $0 = t_0 < t_1 < t_2 < \cdots < t_k < t_{k+1} = 1$, $\Delta x\big|_{t=t_k}$ denotes the jump of $x(t)$ at $t = t_k$.

However, it is worth noticing there are few results about the generalization of the Leggett-Williams fixed point theorem, even higher-order problem. In 2015, Abdulkadir Dogan [11] using the generalization of the Leggett-Williams fixed point theorem studied the following boundary value problem:

$$u''(t) + f(t, u(t)) = 0, t \in [0,1],$$

$$u'(0) = 0, u(1) = 0,$$

where $f : R \to [0,\infty)$ is continuous. A solution $u \in C^2[0,1]$ is both nonnegative and concave on [0,1]. More relevant results, see [12-15].

So in this paper, we discuss the existence of at least three positive solutions to the following boundary value problem:

$$u^{(4)}(t) = f(u(t)), t \in [0,1], \tag{1.1}$$

$$u(0) = u(1) = u'(0) = u'(1) = 0, \tag{1.2}$$

where $f : R \to [0,\infty)$ is continuous. A solution u of (1.1)-(1.2) is both nonnegative and concave on [0,1]. We impose growth conditions on f which allows us to apply the generalization of the Leggett-Williams fixed point theorem in finding three symmetric positive solutions of (1.1)-(1.2).

2. Preliminaries

In this section, we give some background material concerning cone theory in a Banach space, and we give the generalization of the Leggett-Williams fixed-point theorem.

Definition 2.1. Let E be a real Banach space. A nonempty closed convex set P is called a cone of E if it satisfies the following conditions

(1) $x\in P, \lambda \geq 0$ imply $\lambda x \in P$;

(2) $x\in P, -x\in P$ imply $x=0$.

Every cone $P\subset E$ induces an ordering in E given by $x \leq y$ if and only if $y-x\in P$.

Definition 2.2. A map α is said to be a nonnegative continuous concave functional on a cone P in a real Banach space E if $\alpha : P \to [0,\infty)$ is continuous, and

$$\alpha(tx+(1-t)y) \geq t\alpha(x)+(1-t)\alpha(y),$$

for all $x, y\in P$ and $0\leq t\leq 1$. Similarly, we say the map β is a nonnegative continuous convex functional on a cone P in a real Banach space E if $\beta : P \to [0,\infty)$ is continuous, and

$$\beta(tx+(1-t)y) \leq t\beta(x)+(1-t)\beta(y),$$

for all $x, y\in P$ and $0\leq t\leq 1$.

Let γ, β, θ be nonnegative continuous convex functional on P, and α, ψ be nonnegative continuous concave functional on P. Then for nonnegative real numbers h, a, b, d and c, we define the following convex sets:

$$P(\gamma,c)=\{u\in P:\gamma(u)<c\},$$

$$P(\gamma,\alpha,a,c)=\{u\in P:a\le\alpha(u),\gamma(u)\le c\},$$

$$Q(\gamma,\beta,d,c)=\{u\in P:\beta(u)\le d,\gamma(u)\le c\},$$

$$P(\gamma,\theta,\alpha,a,b,c)=\{u\in P:a\le\alpha(u),\theta(u)\le b,\gamma(u)\le c\},$$

$$Q(\gamma,\beta,\psi,h,d,c)=\{u\in P:h\le\psi(u),\beta(u)\le d,\gamma(u)\le c\}.$$

We consider the boundary value problem

$$u^{(4)}(t)=h(t),t\in[0,1], \tag{2.1}$$

$$u(0)=u(1)=u'(0)=u'(1)=0, \tag{2.2}$$

Lemma 2.3. The boundary value problem (2.1)-(2.2) has a unique solution

$$u(t)=\int_0^1 G(t,s)h(s)ds$$

and let its Green's function $G(t,s)$ is

$$G(t,s)=\frac{1}{6}\begin{cases} t^2(1-s)^2[(s-t)+2(1-t)s],0\le t\le s\le 1,\\ s^2(1-t)^2[(t-s)+2(1-s)t],0\le s\le t\le 1.\end{cases}$$

The following is a generalization of the Leggett-Williams fixed-point theorem which will play an important role in the proof of our main results.

Theorem 2.4. ([12]) Let P be a cone in a real Banach space E. Suppose there exist positive numbers c and M, nonnegative continuous concave functional α and ψ on P, and nonnegative continuous convex functional γ,β and θ on P with $\alpha(u)\le\beta(u),\|u\|\le M\gamma(u)$, for all $u\in\overline{P(\gamma,c)}$. Suppose that $F:\overline{P(\gamma,c)}\to\overline{P(\gamma,c)}$ is a completely continuous operator and that there exist nonnegative numbers h,d,a,b with $0<d<a$, such that

(B1) $\{u\in P(\gamma,\theta,\alpha,a,b,c):\alpha(u)>a\}\ne\phi$ and $\alpha(Fu)>a$

for $u\in P(\gamma,\theta,\alpha,a,b,c)$;

(B2) $\{u\in Q(\gamma,\beta,\psi,h,d,c):\beta(u)<d\}\neq\phi$ and $\beta(Fu)<d$

for $u\in Q(\gamma,\beta,\psi,h,d,c)$;

(B3) $\alpha(Fu)>a$ for $u\in P(\gamma,\alpha,a,c)$ with $\theta(Fu)>b$;

(B4) $\beta(Fu)<d$ for $u\in Q(\gamma,\beta,d,c)$ with $\psi(Fu)<h$.

Then F has at least three fixed points $u_1,u_2,u_3\in\overline{P(\gamma,c)}$ such that $\beta(u_1)<d, a<\alpha(u_2)$ and $d<\beta(u_3)$, with $\alpha(u_3)<a.$

3. Main results

In this section, we give the growth conditions on f which allow us to apply the generalization of the Leggett-William fixed-point theorem in establishing the existence of at least three positive solutions of (1.1)-(1.2). We will make use of various properties of Green's function $G(t,s)$ which include

$$\int_0^1 G(t,s)ds=\frac{t^2(t-1)^2(1-6t^3)}{24},0\le t\le 1,$$

$$\int_0^{\frac{1}{r}} G(\frac{1}{2},s)ds=\int_{1-\frac{1}{r}}^1 G(\frac{1}{2},s)ds=\frac{r-1}{48r^4},r>2,$$

$$\int_{\frac{1}{r}}^{\frac{1}{2}} G(\frac{1}{2},s)ds=\int_{\frac{1}{2}}^{1-\frac{1}{r}} G(\frac{1}{2},s)ds=\frac{r^4-16r+16}{48\cdot 16r^4},r>2,$$

$$\int_{t_1}^{t_2} G(t_1,s)ds+\int_{1-t_2}^{1-t_1} G(t_1,s)ds=\frac{1}{6}t_1^{\,2}[\frac{1}{2}(t_2^{\,2}-t_1^{\,2})+t_2(t_2-t_1)+(t_1^{\,3}-t_2^{\,3})],0<t_1<t_2\le\frac{1}{2}.$$

$\min\limits_{r\in[0,1]}\dfrac{G(t_1,r)}{G(t_2,r)}=\dfrac{1}{8t(1-t)^2},\max\limits_{r\in[0,1]}\dfrac{G(\frac{1}{2},r)}{G(t,r)}=\dfrac{t_1^{\,3}}{t_2^{\,3}},0<t\le\dfrac{1}{2}$. Let $E=C[0,1]$ be endowed

with the maximum norm, $\|u\| = \max_{t\in[0,1]}|u(t)|$. Then for $0 < t_3 \le \frac{1}{2}$, we define the cone $P \subset E$ by

$$P = \left\{ \begin{array}{l} u \in E : u \text{ is concave, symmetric,} \\ \text{nonnegative, valued on } [0,1], \\ \min\limits_{\mathrm{t}\in[\mathrm{t}_3,1\text{-}\mathrm{t}_3],\mathrm{u(t)}\ge 2\mathrm{t}_3} \|u\| \end{array} \right\}.$$

We define the nonnegative, continuous concave functional α, ψ and nonnegative continuous convex functional β, θ, γ on the cone P by

$$\alpha(u) = \min_{t\in[t_1,t_2]\cup[1-t_2,1-t_1]} u(t) = u(t_1),$$

$$\beta(u) = \min_{t\in[\frac{1}{r},\frac{r-1}{r}]} u(t) = u(\frac{1}{2}),$$

$$\gamma(u) = \min_{t\in[0,t_3]\cup[1-t_3,1]} u(t) = u(t_3),$$

$$\theta(u) = \min_{t\in[t_1,t_2]\cup[1-t_2,1-t_1]} u(t) = u(t_2),$$

$$\psi(u) = \min_{t\in[\frac{1}{r},\frac{r-1}{r}]} u(t) = u(\frac{1}{r}),$$

where t_1, t_2 and r are nonnegative numbers such that

$$0 < t_1 \le t_2 \le \frac{1}{2} \text{ and } \frac{1}{r} \le t_2 .$$

We see that, for all $u \in P$,

$$\alpha(u) = u(t_1) \le u(\frac{1}{2}) = \beta(u), \tag{3.1}$$

$$\|u\| = u(\frac{1}{2}) \le \frac{1}{2t_3} u(t_3) = \frac{1}{2t_3} \gamma(u), \tag{3.2}$$

and also that $u \in P$ is d solution of (1.1)-(1.2) if and only if

$$u(t) = \int_0^1 G(t,s) f(u(s)) ds, t \in [0,1].$$

We now present our result of the paper:

Theorem 3.1. Assume that there exist nonnegative numbers a,b,c such that $0<a<b<\frac{ct_1^3}{t_2^3}$, and suppose that f satisfies the following growth conditions:

(C1) $f(w)<\frac{384r^4}{5r^4-24r^2+16}(a-\frac{(r-1)c}{t_3{}^2(t_3-1)^2(1-6t_3{}^3)})$, for $\frac{8a}{r^2(1-r^2)}\le w\le a$;

(C2) $f(w)\ge\frac{6b}{t_1{}^2[\frac{1}{2}(t_2{}^2-t_1{}^2)+t_2(t_2-t_1)+(t_1{}^3-t_2{}^3)}$, for $b\le w\le\frac{t_2{}^3b}{t_1{}^3}$;

(C3) $f(w)\le\frac{24c}{t^2(t-1)^2(1-6t^3)}$, for $0\le w\le\frac{c}{2t_3}$.

Then the boundary value problem (1.1)-(1.2) has three symmetric positive solutions u_1,u_2,u_3 satisfying

$$\max_{t\in[0,t_3]\cup[1-t_3,1]} u_i(t)\le c,\ \text{for}\ i=1,2,3,$$

$$\min_{t\in[t_1,t_2]\cup[1-t_2,1-t_1]} u_1(t)>b,\ \max_{t\in[\frac{1}{r},\frac{r-1}{r}]} u_2(t)<a,$$

$$\min_{t\in[t_1,t_2]\cup[1-t_2,1-t_1]} u_3(t)<b,\ \max_{t\in[\frac{1}{r},\frac{r-1}{r}]} u_3(t)>a.$$

Proof: Let us define the completely continuous operator F by

$$(Fu)(t)=\int_0^1 G(t,s)f(u(s))ds.$$

We will seek fixed points of F in the cone. We note that, if $u\in P$, then from properties of $G(t,s)$, $Fu(t)\ge 0$ and $Fu(t)=Fu(t-1),0\le t\le\frac{1}{2}$, and

$(Fu)''(t)\le 0,0\le t\le 1, Fu(t_3)\ge 2t_3Fu(\frac{1}{2})$.

This implies that $Fu\in P$, and so $F:P\to P$. Now, for all $u\in P$, from (5), we get

$\alpha(u) \le \beta(u)$ and from (6), $\|u\| \le \frac{1}{2t_3}\gamma(u).$

If $u \in \overline{P(\gamma,c)},$ then $\|u\| \le \frac{1}{2t_3}\gamma(u) \le \frac{c}{2t_3}$ and from (C3) we get,

$$\gamma(Fu) = \max_{t\in[0,t_3]\cup[1-t_3,1]} \int_0^1 G(t,s) f(u(s))ds = \int_0^1 G(t_3,s) f(u(s))ds$$

$$\le \frac{24c}{t^2(t-1)^2(1-6t^3)} \int_0^1 G(t_3,s)ds = c.$$

Thus, $F: \overline{P(\gamma,c)} \to \overline{P(\gamma,c)}.$ It is immediate that

$$\left\{u \in P(\gamma,\theta,\alpha,b,\frac{bt_2^{\ 3}}{t_1^{\ 3}},c) : \alpha(u) > b\right\} \ne \phi \text{ and}$$

$$\left\{u \in Q(\gamma,\beta,\psi,\frac{8a}{r^2(1-r)^2},a,c) : \beta(u) < a\right\} \ne \phi.$$

We will show the remaining conditions of Theorem 2.4.:

(1) If $u \in Q(r,\beta,a,c)$ with $\psi(Fu) < \frac{8a}{r^2(1-r)^2}$, then $\beta(Fu) < a.$

$$\beta(Fu) = \max_{t\in[\frac{1}{r},\frac{r-1}{r}]} \int_0^1 G(t,s) f(u(s))ds$$

$$= \int_0^1 G(\frac{1}{2},s) f(u(s))ds$$

$$= \int_0^1 \frac{G(\frac{1}{2},s)}{G(\frac{1}{r},s)} G(\frac{1}{r},s) f(u(s))ds$$

$$\le \frac{1}{8r(1-r)^2} \int_0^1 G(\frac{1}{r},s) f(u(s))ds$$

$$\le \frac{1}{8r(1-r)^2} \psi(Fu) < a.$$

(2) If $u \in Q(r,\beta,\psi,\frac{8a}{r^2(1-r)^2},a,c)$, then $\beta(Fu) < a$.

$$\beta(Fu) = \max_{t\in[\frac{1}{r},\frac{r-1}{r}]} \int_0^1 G(t,s) f(u(s))ds$$

$$= \int_0^1 G(\frac{1}{2},s) f(u(s))ds$$

$$= 2\int_0^{\frac{1}{r}} G(\frac{1}{2},s) f(u(s))ds + 2\int_{\frac{1}{r}}^{\frac{1}{2}} G(\frac{1}{2},s) f(u(s))ds$$

$$< \frac{(r-1)c}{t_3^{\,2}(t_3-1)^2(1-6t_3^{\,3})}$$

$$+\frac{r^4-16r+16}{384r^4}\cdot\frac{384r^4}{5r^4-24r^2+16}\cdot(a-\frac{(r-1)c}{t_3^{\,2}(t_3-1)^2(1-6t_3^{\,3})}) = a.$$

(3) If $u \in Q(r,\beta,\psi,\frac{8a}{r^2(1-r)^2},a,c)$ with $\theta(Fu) > \frac{t_2^{\,3}b}{t_1^{\,3}}$, then $\alpha(Fu) > b$.

$$\alpha(Fu) = \max_{t\in[t_1,t_2]\cup[1-t_2,1-t_1]} \int_0^1 G(t,s) f(u(s))ds$$

$$= \int_0^1 G(t_1,s) f(u(s))ds$$

$$= \int_0^1 \frac{G(t_1,s)}{G(t_2,s)} G(t_2,s) f(u(s))ds$$

$$\geq \frac{t_1^{\,3}}{t_2^{\,3}} \int_0^1 G(t_2,s)ds = \theta(Fu) > b.$$

(4) If $u \in Q(r,\theta,\alpha,b,\frac{t_2^{\,3}b}{t_1^{\,3}})$, then $\alpha(Fu) > b$.

$$\alpha(Fu) = \max_{t\in[t_1,t_2]\cup[1-t_2,1-t_1]} \int_0^1 G(t,s) f(u(s))ds$$

$$= \int_0^1 G(t_1, s) f(u(s)) ds$$
$$> \int_{t_1}^{t_2} G(t_1, s) f(u(s)) ds + \int_{1-t_2}^{1-t_1} G(t_1, s) f(u(s)) ds$$
$$\geq \frac{6b}{t_1^{\,2}[\frac{1}{2}(t_2^{\,2} - t_1^{\,2}) + t_2(t_2 - t_1) + (t_1^{\,3} - t_2^{\,3})]}$$
$$\cdot \left[\int_{t_1}^{t_2} G(t_1, s) ds + \int_{1-t_2}^{1-t_1} G(t_1, s) ds \right] = b.$$

Since all the conditions of the generalized Leggett-Williams fixed point theorem are satisfied, (1.1)-(1.2) has three positive solutions $u_1, u_2, u_3 \in \overline{P(\gamma, c)}$, such that $\beta(u_1) < d$, $\alpha < \alpha(u_2)$ and $d < \beta(u_3)$, with $\alpha(u_3) < a$.

4. Concluding remarks

In this paper, we have chosen to perform the analysis when f is autonomous. However, if $f = f(t, y)$ and in addition, for each fixed y, $f(t, y)$ is symmetric about $t = 1/2$, then an analogous theorem would be valid with respect to the same cone P.

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