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Dual Mixed Gaussian Quadrature Based Adaptive Scheme for Analytic Functions

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Abstract. An efficient adaptive scheme based on a dual mixed quadrature rule of precision eleven for approximate evaluation of line integral of analytic functions has been constructed. At first, the precision of Gauss-Legendre four point transformed rule is enhanced by using Richardson extrapolation. A suitable convex combination of the resulting rule $RGL_4(f)$ and the Gauss-Legendre five point rule further enhances the precision producing a new mixed quadrature rule $SM_{GLRGL}(f)$. This mixed rule is termed as dual mixed Gaussian quadrature rule as it acquires a very high precision eleven using Gaussian quadrature rule in two steps. An adaptive quadrature scheme is designed .Some test integrals having analytic function integrands have been evaluated using the dual mixed rule and its constituent rules in non- adaptive mode. The same set of test integrals have been evaluated using those rules as base rules in the adaptive scheme. The dual mixed rule based adaptive scheme is found to be most effective.

Keywords: Gauss-Legendre 5-point rule, Mixed quadrature rule, Richardson extrapolation, $SM_{GLRGL}(f)$.

AMS Mathematics Subject Classification (2010): 60D30, 60D32

1. Introduction

There are several rules for the approximate evaluation of real definite integral

$$I(f) = \int_a^b f(x)dx \text{ and } \int_{-1}^{1} f(z)dz \tag{1.1}$$

However there are only few quadrature rules for evaluating an integral of type

$$I(f) = \int_L f(z)dz \tag{1.2}$$

where L is a directed line segment from the point $(z_0 - h)$ to $(z_0 + h)$ in the domain of f. Using the transformation $z = z_0 + ht, t \in [-1, 1]$ (due to [3]), we transformed the integral (1.2) to the form

$$h\int_{-1}^{1} f(z_0+ht)dt \tag{1.3}$$

and made the approximation of the integral by applying standard quadrature rule meant for approximate evaluation of real definite integral (1.1). The rules so formed are termed as *transformed rules* for numerical integration of (1.2).

The integral (1.1) have been successfully approximated by several Authors [1, 2, 10] by applying the mixed quadrature rule in the Real and complex planes. These rules are limited to precision upto nine. In literature, precision of a quadrature rule has been enhanced through Richardson extrapolation and Kronrod extension [2, 5, 8]. These methods of precision enhancement are very much cumbersome and each having single base rule. But the enhancement of precision by mixed quadrature approach is very much simple with the aid of two rules and easy to handle.

In this paper, keeping in view the improvement of precision method proposed by earlier authors, a mixed quadrature rule of precision eleven has been designed by the linear combination of following two rules.

I. The rule $RGL_4(f)$ obtained by Richardson extrapolation of Gauss-Legendre 4-point rule.

II. Gauss-Legendre 5-point rule.

Gauss-Legendre rule

The (n+1) point Gauss-Legendre rule [6,7,9,11] is given by

$$\int_{-1}^{1} f(z)dz = \sum_{k=0}^{n} \omega_k f(z_k) \tag{1.3}$$

where $\omega_k{}'s$ are $(n+1)$ weights and $z_k{}'s$ are $(n+1)$nodes. The $(2n+2)$ unknowns can be obtained by assuming the rule is exact for all polynomials of degree $(2n+1)$.

For $n=3$ the Gauss-Legendre 4-point rule is

$$\int_{-1}^{1} f(z)dz = \sum_{k=0}^{3} \omega_k f(z_k) \tag{1.4}$$

Assuming the rule (1.4) is exact for all polynomial of degree-7. Taking $f(z) = 1, z, z^2 \dots z^7$, we get 8-equations. On solving these 8-equtions, we can obtain all eight unknowns $\omega_k' s \; and \; z_k{}'s$. Using the values of unknowns in (2.2), we get the Gauss-Legendre 4-point transformed rule as

$$GL_4(f) = \frac{h}{36}\left[\left(18+\sqrt{30}\right)\{f(z_0-\alpha h)+f(z_0+\alpha h)\} + (18-\sqrt{30})\{f(z_0-\beta h) + f(z_0+\beta h)\}\right] \tag{1.5}$$

where $\alpha = \sqrt{\frac{3-2\sqrt{\frac{6}{5}}}{7}}$, $\beta = \sqrt{\frac{3+2\sqrt{\frac{6}{5}}}{7}}$ and f is infinitely differentiable in its domain.

Similarly For $n=4$ the Gauss-Legendre 5-point rule is

$$\int_{-1}^{1} f(z)dz = \sum_{k=0}^{4} \omega_k f(z_k) \tag{1.6}$$

Assuming the rule (1.6) is exact for all polynomial of degree-9. Taking $f(z) = 1, z, z^2 \dots z^9$, we get 10-equations. On solving these 10-equtions, we can obtain all ten unknowns $\omega_k' s \; and \; z_k' s$. Using the values of unknowns in (1.6), we get the Gauss-Legendre 5-point transformed rule as

$$I(f) = \int_{-1}^{1} f(z)dz \cong GL_5(f) = \frac{h}{900}\left[(322 + 13\sqrt{70})\{f(z_0 - \mu h) + f(z_0 + \mu h)\} + (322 - 13\sqrt{70})\{f(z_0 - \gamma h) + f(z_0 + \gamma h)\} + 512 f(z_0)\right] \quad (1.7)$$

where $\mu = \sqrt{\frac{5-2\sqrt{\frac{10}{7}}}{9}}$ and $\gamma = \sqrt{\frac{5+2\sqrt{\frac{10}{7}}}{9}}$

2. Richardson extrapolation of Gauss-Legendre 4-point transformed rule

Consider the Gauss-Legendre 4-point rule (1.5)

$$GL_4(f) = \frac{h}{36}\left[(18 + \sqrt{30})\{f(z_0 - \alpha h) + f(z_0 + \alpha h)\} + (18 - \sqrt{30})\{f(z_0 - \beta h) + f(z_0 + \beta h)\}\right]$$

We can write $(18 + \sqrt{30}) = \frac{1}{2}(21 + 35\beta^2)$ and $(18 - \sqrt{30}) = \frac{1}{2}(21 + 35\alpha^2)$.

Applying Taylor's theorem, we get

$$GL_4(f) = 2h\left[f(z_0) + \frac{h^2}{3!} f^{ii}(z_0) + \frac{h^4}{5!} f^{iv}(z_0) + \frac{h^6}{7!} f^{vi}(z_0) + \frac{3\times 301}{7^3\times 5^2}\frac{h^8}{8!} f^{viii}(z_0) + \frac{3\times 1561}{7^4\times 5^2}\frac{h^{10}}{10!} f^{x}(z_0) + \frac{9\times 13503}{7^5\times 5^3}\frac{h^{12}}{12!} f^{xii}(z_0) + \cdots\right] \quad (2.1)$$

The exact value of the integral due to Taylor

$$I(f) = 2h\left[f(z_0) + \frac{h^2}{3!} f^{ii}(z_0) + \frac{h^4}{5!} f^{iv}(z_0) + \frac{h^6}{7!} f^{vi}(z_0) + \frac{h^8}{9!} f^{viii}(z_0) + \frac{h^{10}}{11!} f^{x}(z_0) + \frac{h^{12}}{13!} f^{xii}(z_0) + \cdots\right] \quad (2.2)$$

Error Bounds for $GL_4(f)$

Let us denote the truncation error due to the rule $GL_4(f)$ by $EGL_4(f)$

We have $I(f) = GL_4(f) + EGL_4(f)$

$$\Rightarrow EGL_4(f) = I(f) - GL_4(f) \quad (2.3)$$

Using (2.1) and (2.2) on (2.3) we obtain $EGL_4(f) = \frac{128}{7^2\times 5^2}\frac{h^9}{9!} f^{viii}(z_0) + \frac{128\times 19}{7^3\times 5^2}\frac{h^{11}}{11!} f^{x}(z_0) + \frac{128\times 1163}{7^4\times 5^3}\frac{h^{13}}{13!} f^{xii}(z_0) + \cdots$ (2.4)

The expression (2.4) is known as Error bounds or the truncation error of the rule $GL_4(f)$, from the error term we concluded that the degree of precision of the Gauss-Legendre 4-point rule is 7.

Richardson extrapolation of $GL_4(f)$

Changing the number of sub-division from n to $n/2$ simultaneously the step length h becomes $2h$.

Denoting the corresponding formula by $GL_{4\frac{h}{2}}(f)$, we have

$$GL_{4\frac{h}{2}}(f) = \frac{2h}{36}\left[\left(18+\sqrt{30}\right)\{f(z_0-2\alpha h)+f(z_0+2\alpha h)\}+\left(18-\sqrt{30}\right)\{f(z_0-2\beta h)+f(z_0+2\beta h)\}\right] \tag{2.5}$$

Using the values of α, β and applying Taylors theorem, we get

$$GL_{4\frac{h}{2}}(f) = 4h\left[f(z_0)+\frac{2^2h^2}{3!}f^{ii}(z_0)+\frac{2^4h^4}{5!}f^{iv}(z_0)+\frac{2^6h^6}{7!}f^{vi}(z_0)+\frac{3\times 301}{7^3\times 5^2}\frac{2^8h^8}{8!}f^{viii}(z_0)+\frac{3\times 1561}{7^4\times 5^2}\frac{2^{10}h^{10}}{10!}f^{x}(z_0)+\frac{9\times 13503}{7^5\times 5^3}\frac{2^{12}h^{12}}{12!}f^{xii}(z_0)+\cdots\right] \tag{2.6}$$

For this case the exact value of the integral, due to the Taylors theorem becomes

$$I_{\frac{h}{2}}(f) = 4h\left[f(z_0)+\frac{2^2h^2}{3!}f^{ii}(z_0)+\frac{2^4h^4}{5!}f^{iv}(z_0)+\frac{2^6h^6}{7!}f^{vi}(z_0)+\frac{2^8h^8}{9!}f^{viii}(z_0)+\frac{2^{10}h^{10}}{11!}f^{x}(z_0)+\frac{2^{12}h^{12}}{13!}f^{xii}(z_0)+\cdots\right] \tag{2.7}$$

Denoting the truncation error of the rule $GL_{4\frac{h}{2}}(f)$ by $EGL_{4\frac{h}{2}}(f)$,

We have $I_{\frac{h}{2}}(f) = GL_{4\frac{h}{2}}(f) + EGL_{4\frac{h}{2}}(f) \Rightarrow EGL_{4\frac{h}{2}}(f) = I_{\frac{h}{2}}(f) - GL_{4\frac{h}{2}}(f)$ (2.8)

Using (2.6) and (2.7) on (2.8), we get

$$EGL_{4\frac{h}{2}}(f) = \frac{128\times 2^9}{7^2\times 5^2}\frac{h^9}{9!}f^{viii}(z_0)+\frac{1216\times 2^{12}}{7^3\times 5^2}\frac{h^{11}}{11!}f^{x}(z_0)+\frac{74432\times 2^{14}}{7^4\times 5^3}\frac{h^{13}}{13!}f^{xii}(z_0)+\cdots \tag{2.9}$$

Resuming the original integral, we have

$$I(f) = GL_4(f) + EGL_4(f) \tag{2.10}$$

$$I(f) = GL_{4\frac{h}{2}}(f) + EGL_{4\frac{h}{2}}(f) \tag{2.11}$$

Subtracting (2.11) from the 2^9 times of (2.10), we get

$$(2^9-1)I(f) = \left[2^9GL_4(f) - GL_{4\frac{h}{2}}(f)\right] + \left[2^9EGL_4(f) - EGL_{4\frac{h}{2}}(f)\right]$$

$$\Rightarrow I(f) = \frac{1}{511}\left[2^9GL_4(f) - GL_{4\frac{h}{2}}(f)\right] + \frac{1}{511}\left[2^9EGL_4(f) - EGL_{4\frac{h}{2}}(f)\right]$$

$$\Rightarrow I(f) = RGL_4(f) + ERGL_4(f)$$

where $RGL_4(f) = \frac{1}{511}\left[2^9GL_4(f) - GL_{4\frac{h}{2}}(f)\right]$ (2.13)

and $ERGL_4(f) = \left[2^9EGL_4(f) - EGL_{4\frac{h}{2}}(f)\right]$ (2.14)

Using (1.5) and (2.5) in (2.14), we get

$RGL_4(f) = \frac{2h}{18396}\left[2^9\left(18+\sqrt{30}\right)\{f(z_0-\alpha h)+f(z_0+\alpha h)\}+2^9\left(18-\sqrt{30}\right)\{f(z_0-\beta h)+f(z_0+\beta h)\}-\left(18+\sqrt{30}\right)\{f(z_0-2\alpha h)+f(z_0+2\alpha h)\}-(18-\sqrt{30})\{f(z_0-2\beta h)+f(z_0+2\beta h)\}\right]$ (2.15)

Here (2.13) and (2.14) are called Modified Gauss-Legendre 4-pont rule due to Richardson Extrapolation and truncation error in the Modified Gauss-Legendre 4-point rule respectively.

Error bounds for $RGL_4(f)$

Using (2.4) and (2.9) in (2.14), we get

$$ERGL_4(f) = -\frac{2^{16}\times 57}{7^3\times 5^2}\frac{h^{11}}{11!}f^{x}(z_0) - \frac{2^{16}\times 3489}{7^4\times 5^3}\frac{h^{13}}{13!}f^{xii}(z_0) + \cdots \quad (2.16)$$

The expression (2.16) is known as Error bounds or the truncation error of the rule $RGL_4(f)$, from the error term we concluded that the degree of precision of the Modified Gauss-Legendre 4-pont rule due to Richardson Extrapolation is 9.

3. Gauss-Legendre 5-point transformed rule

Recalling the Gauss-Legendre 5-point transformed rule (1.7), using the values of $\mu\ and\ \gamma$ We can write $\left(322+13\sqrt{70}\right) = (189+819\gamma^2)$ and $\left(322-13\sqrt{70}\right) = (189+819\mu^2)$, using these values and applying Taylors theorem on (1.7) we get

$$GL_5(f) = 2h\left[f(z_0)+\frac{h^2}{3!}f^{ii}(z_0)+\frac{h^4}{5!}f^{iv}(z_0)+\frac{h^6}{7!}f^{vi}(z_0)+\frac{h^8}{9!}f^{viii}(z_0)+\frac{355}{3969}\frac{h^{10}}{10!}f^{x}(z_0)+\frac{1897125}{9^6\times 7^2}\frac{h^{12}}{12!}f^{xii}(z_0)+\cdots\right] \quad (3.1)$$

Error bounds of $GL_5(f)$

Let us denote the truncation error of the rule $GL_5(f)$ by $EGL_5(f)$, we have

$$I(f) = GL_5(f) + EGL_5(f)$$

$$\Rightarrow EGL_5(f) = I(f) - GL_5(f) \quad (3.2)$$

Using (2.2) and (3.5) on (3.6) we obtain

$$EGL_5(f) = \frac{128}{3969}\frac{h^{11}}{11!}f^{x}(z_0) + \frac{2755968}{9^6\times 7^2}\frac{h^{13}}{13!}f^{xii}(z_0) + \cdots \quad (3.3)$$

The error term established that the degree of precision of $GL_5(f)$ is 9.

4. Formulation of the Dual Mixed quadrature rule of precision eleven

The proposed mixed quadrature rule can be constructed by using Gauss-Legendre 5-point rule and Richardson Extrapolation of Gauss-Legendre 4-point rule [1], [2],[5] as follows

$$I(f) = GL_5(f) + EGL_5(f) \quad (4.1)$$

$$I(f) = RGL_4(f) + ERGL_4(f) \quad (4.2)$$

Adding 175 times of (4.2) with $3^4\times 2^9\times 57$ times of (4.1), we get

$2364079\, I(f) = [2363904\, GL_5(f) + 175\, RGL_4(f)] + [2363904\, EGL_5(f) + 175\, ERGL_4(f)]$

$\Rightarrow I(f) = \frac{1}{2364079}[2363904\, GL_5(f) + 175\, RGL_4(f)] + \frac{1}{2364079}[2363904\, EGL_5(f) + 175\, ERGL_4(f)]$

$\Rightarrow I(f) = SM_{GLRGL}(f) + ESM_{GLRGL}(f)$

where $SM_{GLRGL}(f) = \frac{1}{2364079}[2363904 GL_5(f) + 175\, RGL_4(f)]$ (4.3)

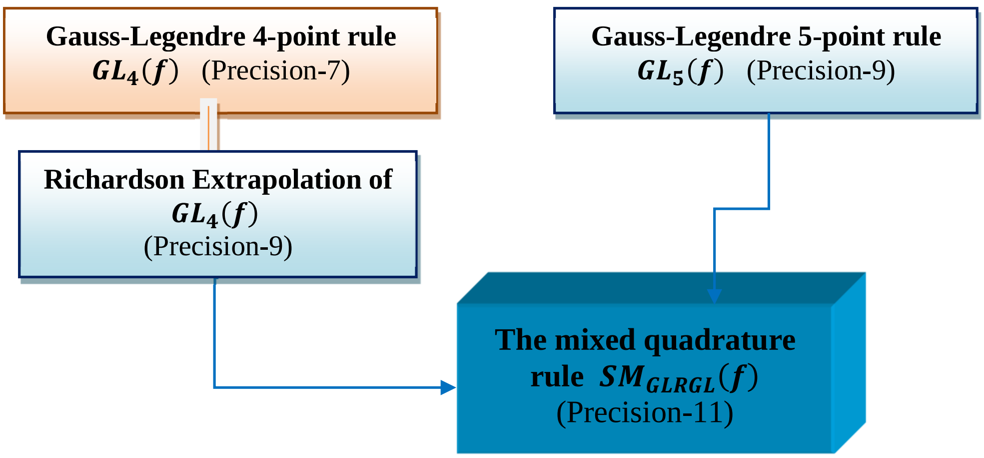
and $ESM_{GLRGL}(f) = \frac{1}{2364079}[2363904 EGL_5(f) + 175\, ERGL_4(f)]$ (4.4)

The expression (4.3) is the desired mixed quadrature rule and (4.4) is the truncation error associated due to the rule.

Using (1.7) and (2.15) on (4.4), we get

$SM_{GLRGL}(f) = \frac{65664}{2364079}\frac{h}{25}[(322 + 13\sqrt{70})\{f(z_0 - \mu h) + f(z_0 + \mu h)\} + (322 - 13\sqrt{70})\{f(z_0 - \gamma h) + f(z_0 + \gamma h)\} + 512 f(z_0)] + \frac{25}{2364079}\frac{h}{1314}[2^9(18 + \sqrt{30})\{f(z_0 - \alpha h) + f(z_0 + \alpha h)\} + 2^9(18 - \sqrt{30})\{f(z_0 - \beta h) + f(z_0 + \beta h)\} - (18 + \sqrt{30})\{f(z_0 - 2\alpha h) + f(z_0 + 2\alpha h)\} - (18 - \sqrt{30})\{f(z_0 - 2\beta h) + f(z_0 + 2\beta h)\}]$ (4.5)

In the constructed mixed rule $SM_{GLRGL}(f)$, the number of function evaluations is thirteen.



Dual mixed

Figure 1: Diagrammatic Representation of construction of the rule

5. Error analysis

An error analysis of the constructed rule has been obtained by the following Theorems.

Theorem 1. If $f(z)$ is analytic in the given domain $\Omega \supset [z_0 - h, z_0 + h]$, then the truncation error due to the rule $SM_{GLRGL}(f)$ is denoted by $ESM_{GLRGL}(f)$ and

$$|ESM_{GLRGL}(f)| \cong \frac{4942879268}{10^{11}}\frac{h^{13}}{13!} f^{xii}(z_0)$$

Proof: Using (2.16) and (3.3) on (4.4), we get

$$ESM_{GLRGL}(f) = \frac{4942879268}{10^{11}}\frac{h^{13}}{13!}f^{xii}(z_0)+...$$

$$\Rightarrow ESM_{GLRGL}(f) \cong \frac{4942879268}{10^{11}}\frac{h^{13}}{13!}f^{xii}(z_0) \text{ [Since truncation error= o}(h^{13})]$$

$$\Rightarrow |ESM_{GLRGL}(f)| \cong \frac{4942879268}{10^{11}}\frac{h^{13}}{13!}f^{xii}(z_0) \qquad \square$$

Theorem 2. The Error bound of the constructed quadrature rule is

$|ESM_{GLRGL}(f)| \leq \frac{3735552}{115839871}\frac{h^{11}}{11!}|\xi_2 - \xi_1|, \quad \xi_1, \xi_2 \epsilon[-1,1]$, where $M = \max_{-1\leq z\leq 1} |f^{xi}(z)|$

Proof: From (3.3), we get $EGL_5(f) \cong \frac{128}{3969}\frac{h^{11}}{11!}f^{x}(\xi_1),\ \xi_1\epsilon[-1,1]$

and from (2.16), we get $ERGL_4(f) \cong -\frac{2^{16}\times 57}{7^3\times 5^2}\frac{h^{11}}{11!}f^{x}(z_0)f^{x}(\xi_2),\ \xi_2\epsilon[-1,1]$

Using above two values in (4.4), we can write

$$ESM_{GLRGL}(f) \cong \frac{1}{2364079}\left[\left\{\frac{29184\times 128}{49}\frac{h^{11}}{11!}f^{x}(\xi_1)\right\} - \left\{\frac{2^{16}\times 57}{49}\frac{h^{11}}{11!}f^{x}(\xi_2)\right\}\right]$$

$$= \frac{3735552}{115839871}\frac{h^{11}}{11!}\{f^{x}(\xi_1) - f^{x}(\xi_2)\} = \frac{-3735552}{115839871}\frac{h^{11}}{11!}\{f^{x}(\xi_2) - f^{x}(\xi_1)\}$$

$$= \frac{-3735552}{115839871}\frac{h^{11}}{11!}\int_{\xi_1}^{\xi_2} f^{xi}(z)dz$$

$$\Rightarrow |ESM_{GLRGL}(f)| \cong \frac{3735552}{115839871}\frac{h^{11}}{11!}\left|\int_{\xi_1}^{\xi_2} f^{xi}(z)dz\right| \leq \frac{3735552}{115839871}\frac{h^{11}}{11!}\int_{\xi_1}^{\xi_2}\left|f^{xi}(z)\right|dz$$

$$\leq \frac{3735552}{115839871}\frac{h^{11}}{11!}\int_{\xi_1}^{\xi_2} Mdz, \quad \text{where } M = \max_{-1\leq z\leq 1} f(z)$$

$$\Rightarrow |ESM_{GLRGL}(f)| \leq \frac{3735552}{115839871}\frac{h^{11}}{11!}|\xi_2 - \xi_1| \qquad (5.1)$$

Since ξ_1 and ξ_2 are arbitrarily chosen points in the interval $[-1,1]$, (5.1) shows that the absolute value of the truncation error will be less if the points ξ_1 and ξ_2 are closure to each other. $\square$

Corollary 1. The error bound for the truncation error is

$|ESM_{GLRGL}(f)| \leq \frac{7471104M}{115839871}\frac{h^{11}}{11!}, \quad M = \max_{-1\leq z\leq 1} |f^{xi}(z)|$

Proof: From the theorem-2

$|ESM_{GLRGL}(f)| \leq \frac{3735552M}{115839871}\frac{h^{11}}{11!}|\xi_2 - \xi_1|,\ \xi_1, \xi_2\epsilon[-1,1]$, where $M = \max_{-1\leq z\leq 1} |f^{xi}(z)|$

Again $|\xi_2 - \xi_1| \leq 2$, using on the above inequation, we get

$|ESM_{GLRGL}(f)| \leq \frac{7471104M}{115839871}\frac{h^{11}}{11!}$ $\square$

Theorem 3. The error committed due to the mixed quadrature rule $SM_{GLRGL}(f)$ is less than its constituent rules.

Proof: From (3.3) and Theorem-1 $|ESM_{GLRGL}(f)| \leq |EGL_5(f)|$

From (2.4) and Theorem-1 $|ESM_{GLKEL}(f)| \leq |\mathrm{G}L_4(f)|$

From (2.16) and Theorem-1 $|ESM_{GLKEL}(f)| \leq |ERGL_4(f)|$ $\square$

6. Numerical verification

The effectiveness of the rule is verified by applying it in different integrals given in the table.

Table 1:

Integrals	Exact value	Values obtained by quadrature rules		
		$GL_4(f)$	$GL_5(f)$	$SM_{GLRGL}(f)$
$I_1=\int_{-i}^{i}\cos z\,dz$	2.350402387287602913i	2.350402092156377i	2.3504023864628259i	2.35040238644061372i
$I_2=\int_{-i}^{i}e^z dz$	1.6829419696157930133i	1.6829419686347704 i	1.68294196961648186 i	1.68294196961964 i
$I_3=\int_{-\pi i}^{\pi i}\cos z\,dz$	23.097478714515496i	23.0865572669713985 i	23.0971877270045254 i	23.097478714684501 i
$I_4=\int_{0}^{2i}\sinh z\,dz$	-1.41614683654714238	-1.416146660016126516	-1.41614683721308171	-1.4161468371954084
$I_5=\int_{1-\frac{i}{4}}^{1+\frac{i}{4}}\ln z\,dz$	0.00511348170783701898765i	0.005113486673587732i	0.0051134816470075i	0.00511348164737754i
$I_6=\int_{-\frac{i}{3}}^{\frac{i}{3}}\cosh z\,dz$	0.65438939359230448i	0.6543893935777153i	0.654389393592309i	0.654389393592222089i

7. Application of the quadrature rule in adaptive quadrature routines

A simple adaptive strategy given in following Algorithm [4, 10].

Algorithm

The input to this scheme is $a, b, \in, n, f$. The output is $P \cong \int_a^b f(x)dx$ with error less than $\in$, n is the number of interval initially chosen. The adaptive strategy is outlined in the following four steps.

Step-1: An approximation I_1 to $I = \int_a^b f(x)dx$ is computed.

Step-2: The interval is divided into pieces, [a, c] and [c, b] where $c = \frac{a+b}{2}$, and then $I_2 \approx \int_a^c f(x)dx$ and $I_3 \approx \int_c^b f(x)dx$ are computed.
Step-3: $I_2 + I_3$ is compared with I_1, to estimate error in $I_2 + I_3$.
Step-4: If |estimated error| $\leq \frac{\epsilon}{2}$ (termination criterion), then $I_2 + I_3$ is accepted as an approximation to $\int_a^b f(x)dx$. Otherwise the same procedure is applied to [a, c] and [c, b], allowing each piece to a tolerance of $\frac{\epsilon}{2}$.
Applying quadrature routines to the proposed quadrature rule to each of the sub intervals covering [a, b] until the termination criterion is satisfied. If the termination criterion is not satisfied in one or more of the sub intervals, then those sub intervals must be further subdivided and entire process repeated.

Table 2: Approximation of the integrals given in the Table-1 by the constructed rule $SM_{RGLKEL}(f)$ and the constituent rule $GL_4(f)$ in the adaptive quadrature routines.
Let us consider the prescribed tolerance $\epsilon = 1.0 \times 10^{-8}$.

Integrals	For the Mixed rule $SM_{GLRGL}(f)$			For the constituent rule $GL_4(f)$		
	Approximate value (P)	No of steps required	|Error|= |P-I|	Approximate value (P)	No of steps required	|Error|= |P-I|
$I_1 = \int_{-i}^{i} \cos z\, dz$	2.35040238728626283 i	01	2.34×10^{-12}	2.350402387282485i	03	5.118×10^{-12}
$I_2 = \int_{-i}^{i} e^z dz$	1.68294196961556576 i	01	2.272×10^{-13}	1.682941969612063i	01	3.73×10^{-12}
$I_3 = \int_{-\pi i}^{\pi i} \cos z\ dz$	23.0974787145112096 i	07	4.286×10^{-12}	23.0974787145081877i	15	7.308×10^{-12}
$I_4 = \int_{0}^{2i} \sinh z\, dz$	-1.41614683654748182	01	3.394×10^{-13}	-1.416146836544004	03	3.139×10^{-12}
$I_5 = \int_{1-\frac{i}{4}}^{1+\frac{i}{4}} \ln z\, dz$	0.00511348170781666199 i	01	2.035×10^{-14}	0.005113481718729i	01	1.0892×10^{-11}
$I_6 = \int_{-\frac{i}{3}}^{\frac{i}{3}} \cosh z\, dz$	0.654389393592214885 i	01	8.959×10^{-14}	0.654389393592248	01	5.6×10^{-14}
$I_7 = \int_{-\sqrt{3}i}^{\sqrt{3}i} z^{10} dz$	-76.5251538616650488 i	15	1.443×10^{-11}	-76.5251538616128784 i	31	6.6609×10^{-11}

8. Conclusions

From the tables it is evident that the mixed quadrature rule when applied each of the test integrals gives better result than that of constituent rules (Gauss-Legendre 4- point and Gauss-Legendre 5-point transformed rules) in non-adaptive mode. This mixed quadrature rule $SM_{GLRGL}(f)$ also gives better results than its constituent's when used as base rules in adaptive scheme. In some cases also the number of steps required to achieve the desired accuracy is reduced.

It is important to note that the results obtained in the table-1 are much better than the results of the same set of test integrals obtained in previous study in the papers [1,2, 5].

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