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On Star-critical ($K_{1,n}$, $K_{1,m} + e$) Ramsey Numbers

C.J.Jayawardene[1*]***, J.N.Senadheera***[2]***, K.A.S.N.Fernando***[2] **and** ***W.C.W.Navaratna***[2]

[1]Department of Mathematics, University of Colombo Sri Lanka
email: c_jayawardene@maths.cmb.ac.lk
[2]Department of Mathematics, The Open University of Sri Lanka Sri Lanka,
E-mail: jnsen@ou.ac.lk
[*]Corresponding author

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Abstract. We say that $K_n \rightarrow (G,H)$, if for every red/blue colouring of edges of the complete graph K_n, there exists a red copy of G, or a blue copy of H in the colouring of K_n. The Ramsey number $r(G,H)$ is the smallest positive integer n such that $K_n \rightarrow (G,H)$. Let $r(n,m)=r(K_n, K_m)$. A closely related concept of Ramsey numbers is the Star-critical Ramsey number $r^*(G, H)$ defined as the largest value of k such that $K_{r(G,H)-1} \vee K_{1,k} \rightarrow (G,H)$. Literature on survey papers in this area reveals many unsolved problems related to these numbers. One of these problems is the calculation of Ramsey numbers for certain classes of graphs. The primary objective of this paper is to calculate the Star critical Ramsey numbers for the case of Stars versus $K_{1,m}+e$. The methodology that we follow in solving this problem is to first find a closed form for the Ramsey number $r^*(K_{1,n}, K_{1,m}+e)$ for all $n, m \geq 3$. Based on the values of $r^*(K_{1,n}, K_{1,m}+e)$ for different n, m we arrive at a general formula for $r^*(K_{1,n}, K_{1,m}+e)$. Henceforth, we show that $r^*(K_{1,n}, K_{1,m}+e) = n+m-1$ is defined by a piecewise function related to the three disjoint cases of n, m both even and $n \leq m - 2$, n or m is odd and $n \leq m-2$ and $n > m-2$.

Keywords: Graph theory, Ramsey theory, Ramsey critical graphs

AMS Mathematics Subject Classification (2010): 05C55, 05C38, 05D10

1. Introduction

Given two graphs G and H, we say that $K_n \rightarrow (G, H)$, if any red and blue two colouring of K_n contains a copy of G in red or a copy of H in bue. Studies on Ramsey numbers/Multipartite Ramsey numbers related to different classes have been studied extensively in the past few decades (see [4-7,9]). Studies on Star- critical Ramsey numbers related to different classes of graphs are trees vs complete graphs [3], paths vs. paths [2], stars vs. stripes [1] and complete graphs vs stripes are some such examples (also see [8,10]). In this paper, we extend this list by calculating Star-critical Ramsey numbers related to stars versus $K_{1,m} + e$.

2. Notation

Consider a simple graph G and let $v \in V(G)$. We denote the neighborhood of v by $\Gamma(v)$ which represents the set of vertices adjacent to v. The degree of v which is equal to $|\Gamma(v)|$

is denoted by $d(v)$. Consider a red/blue colouring of the complete graph K_n given by $K_n = H_R \oplus H_B$ where H_R and H_B denote the red and blue graphs with vertex set $V(G)$. Likewise, the degree of vertex v in H_R and H_B are denoted by $d_R(v)$ and $d_B(v)$ respectively. Then clearly, we get $n - 1 = d_R(v) + d_B(v)$.

3. The exact values of $r_*(K_{1,n}, K_{1,m} + e)$ for $n, m \geq 3$

In order to find lower bounds for Star critical Ramsey numbers, we deal with constructions of graphs generated by regular K_n convex n-gons drawn in an Euclidean plane. Label the vertices of K_n by $v_0, v_1, v_2, \ldots, v_{n-1}$ in the anti-clockwise order. Given any $0 \leq i, k \leq n - 1$, $v_{i+k \pmod n}$ and $v_{i-k \pmod n}$ are represented by the two vertices separated from v_i by a path of length k along the outer cycle of the n-gon, in the anti-clockwise direction and the clockwise direction respectively. The red/blue colorings of K_n in such a scenario are called *standard regular colorings of K_n*. The following lemma plays an crucial role in finding $r_*(K_{1,n}, K_{1,m} + e)$ for $n, m \geq 3$.

Lemma 2.1. *Given $n, m \geq 3$*

$$r(K_{1,n}, K_{1,m} + e) = \begin{cases} n + m - 1 & \text{if } n \text{ and } m \text{ are are both even and } n \leq m - 2 \\ n + m & \text{if } n \text{ or } m \text{ is odd and } n \leq m - 2 \\ 2n + 1 & \text{if } n > m - 2 \end{cases}$$

Proof. We break up the proof in to 4 parts correspondingly.

Case 1. *If n and m are both even and $n \leq m - 2$*

Consider a standard coloring on K_{n+m-2} such that each $v_i \in V(H_R)$ $(0 \leq i \leq n + m - 3)$ is adjacent in red to all vertices of $\{ v_{(i \pm k) \bmod (n+m-2)} \mid 0 < k \leq (n-2)/2 \}$ and adjacent in blue to all the other vertices of $V(K_{n+m-2}) \setminus \{ v_i \}$ except for the $(m-2)/2$ diagonal red edges joining v_i to the diametrically opposite vertex $v_{(i + (n+m-2)/2)} \bmod (n+m-2)$ when $i = 0, 1, \ldots, ((m-2)/2)-1$ (see Figure 1). We note that there are many alternative colorings with different number of red diagonals. However, this particular coloring was selected as the same coloring can be used to find Star-critical Ramsey numbers. Such a coloring is well defined, since by definition, (v_i, v_j) is a red edge iff (v_j, v_i) is a red edge. In such a construction, any vertex of K_{n+m-2} will be adjacent in red to $(n-2)/2$ vertices immediately left of it, $(n-2)/2$ vertices immediately right of it and at most one vertex opposite it. Therefore, the red degree of any vertex adjacent in red to its opposite vertex is equal to $2 \times (n-2)/2 + 1 = n - 1$. Similarly, the red degree of any vertex not adjacent in red to its opposite vertex is equal $2 \times (n-2)/2 = n - 2$. Accordingly, the blue degree will be $(n + m - 3) - (n - 1) = m - 2$ or else $(n + m - 3) - (n - 2) = m - 1$, respectively. In this coloring, H_R has no red $K_{1,n}$. Also, H_B has no blue $K_{1,m} + e$. That is, $K_{n+m-2} \nrightarrow (K_{1,n}, K_{1,m} + e)$. Hence, $r(K_{1,n}, K_{1,m} + e) \geq n + m - 1$.

Next, we need to show that, $r(K_{1,n}, K_{1,m} + e) \leq n + m - 1$. Suppose there exists a red/blue coloring of K_{n+m-1} such that H_R contains no $K_{1,n}$ and H_B contains no $K_{1,m} + e$. In order to avoid a red $K_{1,n}$, every vertex $v \in V(K_{n+m-1})$ must satisfy $d_R(v) \leq n - 1$. However, by Handshaking lemma, all vertices of $V(K_{n+m-1})$ cannot have $d_R(v) = n-1$ since otherwise it will force H_R to have an odd number of odd degree vertices. Therefore, there exists a vertex $v_0 \in V(K_{n+m-1})$ such that $d_R(v_0) \leq n - 2$. Hence $d_B(v_0) \geq m$. In order to avoid a blue $K_{1,m} + e$, all vertices of $\Gamma_B(v_0)$ must be adjacent to each other in red. That is, the vertices of $\Gamma_B(v_0)$ induce a red complete graph of order at least m.

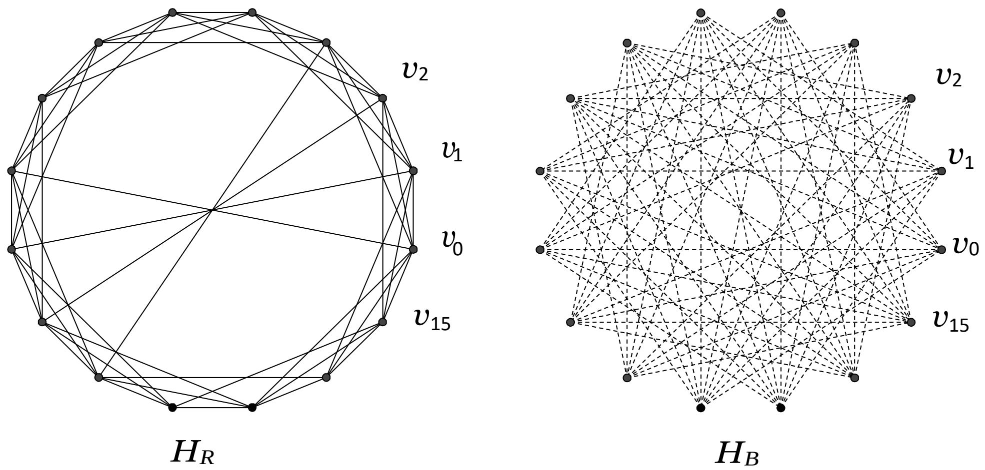


Figure 1: A Ramsey critical ($K_{1,8}$, $K_{1,10} + e$) coloring of $K_{16} = H_R \oplus H_B$

Let $w \in \Gamma_B(v_0)$. Then, $d_R(w) \geq m - 1 \geq n$. That is, H_R contains a red $K_{1,n}$, a contradiction. Therefore, $K_{n+m-1} \longrightarrow (K_{1,n}, K_{1,m} + e)$. Hence, $r(K_{1,n}, K_{1,m} + e) \leq n + m - 1$. Combining with the earlier result, we find $r(K_{1,n}, K_{1,m} + e) = n + m - 1$, as required.

Case 2. *If n is odd and $n \leq m - 2$*

As before, consider a standard coloring on K_{n+m-1} such that each $v_i \in V(H_R)$ ($0 \leq i \leq n + m - 2$) is adjacent to $\{v_{(i \pm k) \bmod (n+m-1)} \mid 0 < k \leq (n\text{-}1)/2\}$ in red and adjacent to all the other vertices of $V(K_{n+m-1}) \setminus \{v_i\}$ in blue. This coloring is also well defined. In such a construction, any vertex of K_{n+m-1} will be adjacent in red to (n-1)/2 vertices immediately left of it, (n-1)/2 vertices immediately right of it. The red degree of any vertex is equal to $2 \times (n\text{-}1)/2 = n - 1$ and the blue degree of any vertex is $(n + m - 2) - n - 1 = m - 1$. Therefore, H_R has no red $K_{1,n}$. Also, H_B has no blue $K_{1,m} + e$. That is, $K_{n+m-1} \nrightarrow (K_{1,n}, K_{1,m} + e)$. Hence, $r(K_{1,n}, K_{1,m} + e) \geq n + m$.

Next we need to show that, $r(K_{1,n}, K_{1,m} + e) \leq n+m$. Suppose there exists a red/blue coloring of K_{n+m} such that H_R contains no $K_{1,n}$ and H_B contains no $K_{1,m} + e$.

In order to avoid a red $K_{1,n}$, every vertex $v \in V(K_{n+m})$ must satisfy $d_R(v) \leq n-1$. That is, for any vertex $v \in V(K_{n+m})$, $d_B(v) \geq m$. Let $v_0 \in V(K_{n+m})$. In order to avoid a blue $K_{1,m} + e$, all vertices of $\Gamma_B(v_0)$ must be adjacent to each other in red. However, as $n + 1 \leq m$, we argue that $\Gamma_B(v_0)$ contains a red $K_{1,n}$, a contradiction. Hence, $r(K_{1,n}, K_{1,m}+e) \leq n+m$. Combining with the earlier result, $r(K_{1,n}, K_{1,m}+e) = n+m$, as required.

Case 3. *If n is even, m is odd and $n \leq m - 2$*

Now consider a standard coloring on K_{n+m-1} such that each $v_i \in V(H_B)$ ($0 \leq i \leq n + m - 2$) is adjacent to $\{v_{(i \pm k) \bmod (n+m-1)} \mid 0 < k \leq (m\text{-}1)/2\}$ in blue and adjacent to all the other vertices of $V(K_{n+m-1}) \setminus \{v_i\}$ in red. This coloring is also well defined. In such a construction, any vertex of K_{n+m-1} will be adjacent in blue to (m-1)/2 vertices immediately left of it, (m-1)/2 vertices immediately right of it. Therefore, the blue degree of any vertex is equal to $2 \times (m-1)/2 = m - 1$ and the red degree of any vertex is $(n + m - 2) - (m - 1) = n - 1$. Therefore, H_R has no red $K_{1,n}$. Also H_B has no $K_{1,m} + e$ since it has no blue $K_{1,m}$. That is, $K_{n+m-1} \nrightarrow (K_{1,n}, K_{1,m} + e)$. Hence, $r(K_{1,n}, K_{1,m} + e) \geq n + m$.

Next we need to show that, $r(K_{1,n}, K_{1,m} + e) \leq n + m$. Suppose there exists a red/blue coloring of K_{n+m} such that H_R contains no $K_{1,n}$ and H_B contains no $K_{1,m}+e$. In order to avoid a red $K_{1,n}$, every vertex $v \in V(K_{n+m})$ must satisfy $d_R(v) \leq n - 1$. Hence, for any vertex $v \in V(K_{n+m})$, $d_B(v) \geq m$. Let $v_0 \in V(K_{n+m})$. In order to avoid a blue $K_{1,m} + e$, all vertices of $\Gamma_B(v_0)$ must be adjacent to each other in red. As $n + 1 \leq m$, $\Gamma_B(v_0)$ contains a red $K_{1,n}$, a contradiction. Hence, $r(K_{1,n}, K_{1,m} + e) = n + m$.

Case 4. $n > m - 2$

Consider a standard regular coloring of $K_{2n} = H_R \oplus H_B$ such that each $v_i \in V(K_{2n})$ $(0 \leq i \leq n)$ forms a red clique of size n and each $v_i \in V(K_{2n})$ $(n + 1 \leq i \leq 2n)$ also forms an independent red clique of size n. That is, $H_R = 2K_n$ and $H_B = K_{n,n}$ (see Figure 2).

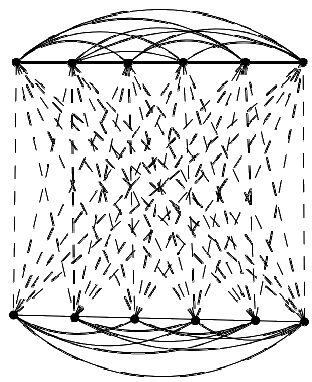
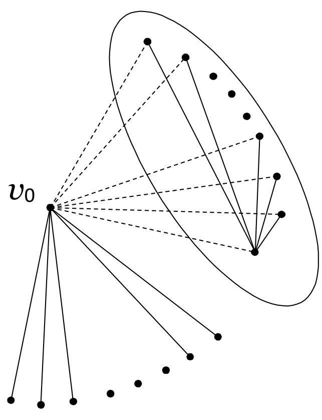


Figure 2: A Red/blue graph coloring of K_{12} with no red $K_{1,6}$ and no blue $K_{1,7} + e$



Blue neighborhood will be forced to induce a red K_{n+1}

Figure 3: Neighborhood of a vertex of K_{2n+1} used in the argument containing no red $K_{1,n}$

Clearly, H_R has no $K_{1,n}$. Furthermore, H_B has no $K_{1,m} + e$, since it has no blue C_3. That is, $K_{2n} \nrightarrow (K_{1,n}, K_{1,m} + e)$. Hence, $r(K_{1,n}, K_{1,m} + e) \geq 2n + 1$.

Next we need to show that, $r(K_{1,n}, K_{1,m}+e) \leq 2n+1$. Suppose there exists a red/blue coloring of K_{2n+1} such that H_R contains no $K_{1,n}$ and H_B contains no $K_{1,m} + e$.

Let $v_0 \in V(K_{2n+1})$. In order to avoid a red $K_{1,n}$, v_0 must satisfy $d_B(v_0) \geq 2n - (n-1) = n + 1 \geq m$. To avoid a blue $K_{1,m} + e$, all vertices of $\Gamma_B(v_0)$ must be adjacent to each other in red. That is, the vertices of $\Gamma_B(v_0)$ induces a red complete graph of order at least $n + 1$ (see Figure 3). Hence, $V(K_{2n+1})$ will contain a vertex of red degree n, a contradiction.

Lemma 2.1. *Given n, m* ≥ 3

$$r^*(K_{1,n}, K_{1,m} + e) = \begin{cases} n+m-2 & \text{if } n \text{ and } m \text{ are are both even and } n \leq m-2 \\ 1 & \text{if } n \text{ or } m \text{ is odd and } n \leq m-2 \\ n+1 & \text{if } n > m-2 \end{cases}$$

Proof. We break up the proof in to 3 cases.

Case 1. *n and m are even and* $n \leq m - 2$

To show that, $r_*(K_{1,n}, K_{1,m} + e) \geq n + m - 2$, consider the coloring of $K_{n+m-2} \vee K_{1,n+m-3}$ introduced in Case 1 of Lemma 1.

Add a vertex (say x) and connect it in blue to all the vertices v_i and the diametrically opposite vertices $v_{j \bmod (n+m-2)}$ for $i = 0, 1, ..., (m\text{-}2)/2 - 1$ where $j = i + (n+m\text{-}2)/2$. Connect all the other vertices excluding the vertex $v_{(n+m-4)/2}$ to x in red (see Figure 4).

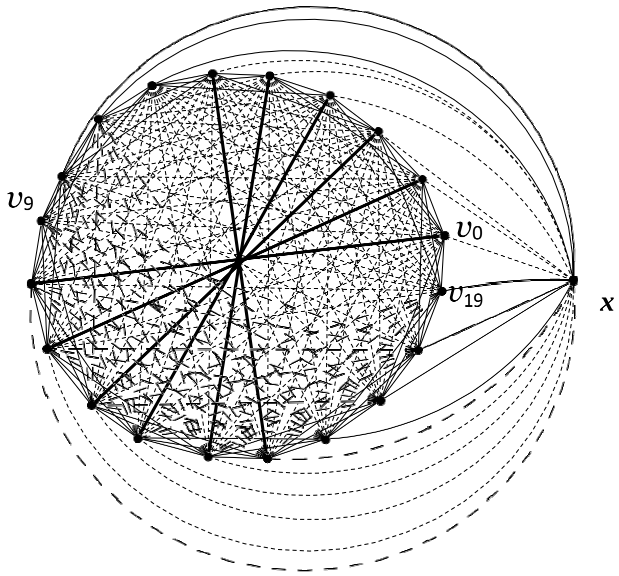


Figure 4: A red/blue coloring of $K_{n+m-2} \vee K_{1,n+m-3}$ when $n = 8$ and $m = 14$

This coloring of $K_{n+m-2} \vee K_{1,n+m-3}$ contains neither red $K_{1,n}$ nor blue $K_{1,m} +e$. Thus, $K_{n+m-2} \vee K_{1,n+m-3} \nrightarrow (K_{1,n}, K_{1,m} + e)$. Therefore, $r_*(K_{1,n}, K_{1,m} + e) \geq n + m - 2$. Finally, using $r_*(K_{1,n}, K_{1,m} + e) \leq r(K_{1,n}, K_{1,m} + e) - 1 = n + m - 2$, we conclude that $r_*(K_{1,n}, K_{1,m} + e) = n + m - 2$.

Case 2. *n or m is odd and* $n \leq m - 2$
We first show that, $r_*(K_{1,n}, K_{1,m} + e) \leq 1$. Suppose there exists a red/blue coloring of $K_{n+m-1} \vee K_{1,1}$ such that H_R contains no $K_{1,n}$ and H_B contains no $K_{1,m} + e$. First let us restrict our attention to the red/blue coloring of K_{n+m-1}. In order to avoid a red $K_{1,n}$, any vertex $v \in K_{n+m-1}$ must satisfy $d_R(v) \leq n - 1$ and hence $d_B(v) \geq m - 1$. Suppose that there exists a vertex $v_0 \in K_{n+m-1}$ such that $d_R(v_0) \leq n - 2$. That is, $d_B(v_0) \geq m$. In order to avoid a blue $K_{1,m} + e$, $\Gamma_B(v_0)$ must induce a red complete graph. Since $n < m - 1$, the vertices of $\Gamma_B(v_0)$ will contain a red complete graph of order at least $n + 1$. Hence, H_R contains a red $K_{1,n}$, a contradiction. Thus, we can assume that, any vertex $v \in K_{n+m-1}$ must satisfy $d_R(v) = n - 1$ and $d_B(v) = m - 1$. Choose the point outside of K_{n+m-1}. In order to avoid a red $K_{1,n}$, this vertex cannot be adjacent in red to any vertex of K_{n+m-1}. Furthermore, if this vertex v_0 is adjacent to some vertex in blue, then since $n \leq m-2$, $\Gamma_B(v_0)$ will contain a red complete graph of order at least $n + 1$, a contradiction. Therefore, if the vertex outside of K_{n+m-1} is adjacent in any colour to a vertex of K_{n+m-1}, we will get a red $K_{1,n}$ or a blue $K_{1,m} + e$.

Hence, $r_*(K_{1,n}, K_{1,m} + e) \leq 1$. Since by definition, $r_*(K_{1,n}, K_{1,m} + e) \geq 1$, we conclude that $r_*(K_{1,n}, K_{1,m} + e) = 1$.

Case 3. $\boldsymbol{n > m - 2}$
Consider the regular standard coloring of $K_{2n} = H_R \oplus H_B$ given in Case 4 of Lemma 1. Extend this coloring to a coloring of $K_{2n} \vee K_{1,n}$ such that the new vertex (say x) of degree n is adjacent in blue to all vertices of one partite set of $H_B = K_{n,n}$ (see Figure 4). Observe that, H_R has no $K_{1,n}$. Furthermore, H_B has no $K_{1,m} + e$ since it has no blue C_3. That is, $K_{2n} \vee K_{1,n} \nrightarrow (K_{1,n}, K_{1,m} + e)$. Hence, $r_*(K_{1,n}, K_{1,m} + e) \geq n + 1$.

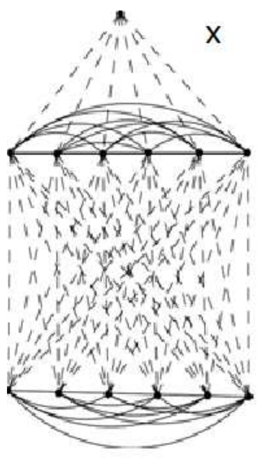


Figure 5: The blue graph of $K_{2n} \vee K_{1,n}$ considered in proving $r_*(K_{1,n}, K_{1,m} + e) \geq n + 1$ when $n = 6$ and $m \leq 7$

Next we show that, $r_*(K_{1,n}, K_{1,m} + e) \leq n + 1$. Suppose there exists a red/blue coloring of $K_{2n} \vee K_{1,n+1}$ such that H_R contains no $K_{1,n}$ and H_B contains no $K_{1,m} + e$. Let us first restrict our attention to a red/blue coloring of K_{2n}. In order to avoid a red $K_{1,n}$, any vertex $v \in K_{2n}$ must satisfy $d_R(v) \leq n - 1$ and hence $d_B(v) \geq n$. Next, suppose that there exists a vertex $v_0 \in K_{2n}$ such that $d_R(v_0) \leq n - 2$. Then, $d_B(v_0) \geq n + 1 \geq m$. In order to avoid a blue $K_{1,m} + e$, all vertices of $\Gamma_B(v_0)$ must be adjacent to each other in red. Thus, the vertices of $\Gamma_B(v_0)$ will contain a red complete graph of order at least n+1. Hence, H_R contains a red $K_{1,n}$, a contradiction. Therefore, we can assume that, any vertex $v \in K_{2n}$ must satisfy $d_R(v) = n - 1$ and $d_B(v) = n$.

Let the vertex outside of K_{2n} in $K_{2n} \vee K_{1,n+1}$ be denoted by x. In order to avoid a red $K_{1,n}$, x cannot be adjacent in red to any vertex of K_{2n}. If the vertex x is adjacent to n + 1 vertices of K_{2n} in blue, then since $n + 1$ m, $\Gamma_B(x)$ will contain a red complete graph of order at least $n + 1$, a contradiction. Hence, x cannot be adjacent to $n + 1$ vertices of K_{2n} in any color. Therefore, $r_*(K_{1,n}, K_{1,m} + e) \leq n+1$. Since by definition, $r_*(K_{1,n}, K_{1,m} + e) \geq n + 1$, we can conclude that $r_*(K_{1,n}, K_{1,m} + e) = n + 1$.

4. Results and discussion

In this paper, we proved that the Ramsey number $r(K_{1,n}, K_{1,m} + e)$ is $2n + 1$ for $n > m - 2$. When $n > m - 2$, $r(K_{1,n}, K_{1,m} + e)$ is $n + m + 1$ or $n + m$ depending on whether n and m are both even or at least one of them is odd, respectively. Furthermore, we showed that the Star critical Ramsey number $r_*(K_{1,n}, K_{1,m} + e)$ is $n + 1$ for $n > m$ 2. When $n < m - 2$, $r_*(K_{1,n}, K_{1,m} + e)$ is $n + m$- 2 or 1 depending on whether n and m are both even or at least one of them is odd, respectively. This result is consistent with the known result that, Star critical Ramsey number $r_*(G, H)$ for any two simple graphs G and H, satisfies $1 \leq r_*(G, H) \leq r(G, H)$-1. These findings are in agreement with the known result that, Star-critical Ramsey number $r_*(G, H)$ for any two simple graphs G and H, satisfies $1 \leq r_*(G, H) \leq r(G, H) - 1$.

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