Annals of Pure and Applied Mathematics
Vol. 22, No. 2, 2020, 71-73
ISSN: 2279-087X (P), 2279-0888(online)
Published on 4 October 2020
www.researchmathsci.org
DOI: http://dx.doi.org/10.22457/apam.v22n2a01701

Annals of Pure and Applied Mathematics

Brief Note

Verification of a Conjecture Proposed by N. Burshtein on a Particular Diophantine Equation

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Received 3 September 2020; accepted 2 October 2020

Abstract. In [1] among other equations, the author considered the equation $p^x + (p + 1)^y + (p + 2)^z = M^2$ when $p = 4N + 3$ is prime, $x = 1, y = z = 2$ and M is a positive integer. For all values $0 \le N \le 50$, he established that the equation has exactly one solution when $N = 2$, namely when $p = 11$. In [1 – Conjecture 1] he stated that the equation has no solutions for all values $N > 50$. In this note we verify that Conjecture 1 is indeed true for all values $N > 50$.

Keywords: Diophantine equations

AMS Mathematics Subject Classification (2010): 11D61

1. Introduction

The field of Diophantine equations is ancient, vast, and no general method exists to decide whether a given Diophantine equation has any solutions, or how many solutions.

The famous general equation

$$p^x + q^y = z^2$$

has many forms. The literature contains a very large number of articles on non-linear such individual equations involving particular primes and powers of all kinds.

In [1], we extended the above equation, and considered equations of the form $p^x + (p + 1)^y + (p + 2)^z = M^2$ for all primes $p \ge 2$ and integers x, y, z satisfying $1 \le x, y, z \le 2$. The value M is a positive integer. All the possibilities for infinitely many solutions, no solution cases and unique solutions have been determined, except for the equation $p + (p + 1)^2 + (p + 2)^2 = M^2$ when p is of the form $4N + 3$. In this case, it was established that $p = 11$ is the only solution when $3 \le p \le 199$. We have conjectured [1 – Conjecture 1] that for all primes $p > 199$, the equation has no solutions. In this note, we provide a formal proof as to the validity of our conjecture in [1] implying now that the solution with $p = 11$ is unique.

2. All the solutions of $p + (p + 1)^2 + (p + 2)^2 = M^2$ when $p = 4N + 3$

In the following theorem we will show that the equation has a unique solution.

Theorem 2.1. Suppose that $p = 4N + 3$ $(N \geq 0)$ is prime. Then the equation $p + (p + 1)^2 + (p + 2)^2 = M^2$ has a unique solution when $p = 11$ $(N = 2)$.

Proof: The left side of the equation yields

$$p + (p + 1)^2 + (p + 2)^2 = 2p^2 + 7p + 5 = (p + 1)(2p + 5) = (p + 1)\big(2(p + 1) + 3\big). \qquad (1)$$

If $(p + 1)\big(2(p + 1) + 3\big) = M^2$ has a solution for some value p, then the two factors $(p + 1)$, $\big(2(p + 1) + 3\big)$ in (1) must satisfy simultaneously the two conditions in each of the following cases, namely:

(a) $p + 1 = A^2$, $2(p + 1) + 3 = B^2$.
(b) $p + 1 \neq A^2$, $2(p + 1) + 3 \neq B^2$.

Suppose **(a)**: $p + 1= A^2$, $2(p + 1) + 3 = B^2$.
The equality $p + 1= A^2$ implies that $p = A^2 - 1 = A^2 - 1^2 = (A - 1)(A + 1)$. When $A = 2$, then $p = 3$. But $2 (3 + 1) + 3 = 11 \neq B^2$. Thus $A \neq 2$. For all values $A > 2$, the prime $p = (A - 1)(A + 1)$ is a product of two distinct factors which is impossible. The two conditions in **(a)** are not satisfied simultaneously.

Hence case **(a)** does not exist.

Suppose **(b)**: $p + 1 \neq A^2$, $2(p + 1) + 3 \neq B^2$.
We have two cases, namely $\gcd\big(p + 1, 2(p + 1) + 3\big) = 1$, $\gcd\big(p + 1, 2(p + 1) + 3\big) = 3$.

If $\gcd\big(p + 1, 2(p + 1) + 3\big) = 1$, and $(p + 1)\big(2(p + 1) + 3\big) = M^2$, it then follows that $p + 1 = A^2$ and $2(p + 1) + 3 = B^2$ must exist simultaneously. But this contradicts our supposition, and hence $\gcd\big(p + 1, 2(p + 1) + 3\big) \neq 1$.

If $\gcd\big(p + 1, 2(p + 1) + 3\big) = 3$, denote $p + 1 = 3K$, and $2(p + 1) + 3 = 2\cdot 3K + 3 = 3(2K + 1)$ where $\gcd(K, 2K + 1) = 1$. If $(p + 1)\big(2(p + 1) + 3\big) = (3K)\cdot 3(2K + 1) = 3^2\cdot K(2K + 1) = M^2$, it now follows that the two conditions $K = H^2$ and $2K + 1 = 2H^2 + 1 = L^2$ exist simultaneously. In order to achieve the smallest possible difference $L^2 - 2H^2 = 1$, set H as the largest possible value $H = L - 1$. We then obtain

$$L^2 - 2H^2 = L^2 - 2(L - 1)^2 = -L^2 + 4L - 2 = L(4 - L) - 2. \qquad (2)$$

Since for all values $L \geq 4$, it follows from (2) that $L(4 - L) - 2 < 0$, therefore L may assume only the two values $L = 2, 3$. When $L = 2$, then in (2) $L^2 - 2H^2 = 2 > 1$. Thus $L \neq 2$. When $L = 3$, then $L^2 - 2H^2 = L(4 - L) - 2 = 1$, and hence $H = 2$. This in turn implies that $K = H^2 = 4$, $p + 1 = 3K = 12$ and $p = 11$ for which $M = 18$. When $p = 11$, it follows that the two conditions in which $p + 1 = 12 \neq A^2$, and $2(p + 1) + 3 = 27 \neq B^2$ are indeed satisfied simultaneously.

The equation $p + (p + 1)^2 + (p + 2)^2 = M^2$ has a unique solution in which $p = 11$ and $M = 18$.

This concludes the proof of Theorem 2.1. □

Final remark. In [1] we have shown that when $3 \leq p \leq 199$**,** the equation $p + (p + 1)^2 + (p + 2)^2 = M^2$ has exactly one solution with $p = 11$. Theorem 2.1 establishes that

Conjecture 1 in [1] which stated that for all $p > 199$ the equation has no solutions is indeed true now, and the solution with $p = 11$ is therefore unique.

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