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Fibonacci Cordial Labeling of Some Special Families of Graphs

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Abstract. An injective function f from vertex set $V(G)$ of a graph G to the set $\{F_0, F_1, F_2, \cdots, F_n\}$, where F_i is the i^{th} Fibonacci number $(i = 0,1,\cdots,n)$, is said to be Fibonacci cordial labeling if the induced function f^* from the edge set $E(G)$ the set $\{0,1\}$ defined by $f^*(uv) = (f(u) + f(v))(mod\ 2)$ satisfies the condition $|e_f(0) - e_f(1)| \leq 1$, where $e_f(0)$ is the number of edges with label 0 and $e_f(1)$ is the number of edges with label 1. A graph that admits Fibonacci cordial labeling is called Fibonacci cordial graph. In this paper we discuss Fibonacci cordial labeling of the families of complete graphs K_n, paths P_n, cycles C_n, and corona product of C_m and K_p for $p = 1,2$, and 3.

Keywords: Fibonacci Cordial, Corona

AMS Mathematics Subject Classification (2010): 05C78

1. Introduction

Assume G to be a simple connected graph with n vertices.

Definition 1.1. *A function $f: V(G) \to \{0,1\}$ is said to be Cordial Labeling if the induced function $f^*: E(G) \to \{0,1\}$ defined by*

$$f^*(uv) = |f(u) - f(v)|$$

satisfies the conditions $|v_f(0) - v_f(1)| \leq 1$, as well as $|e_f(0) - e_f(1)| \leq 1$, where

$v_f(0) :=$ number of vertices with label 0,
$v_f(1) :=$ number of vertices with label 1,
$e_f(0) :=$ number of edges with label 0,
$e_f(1) :=$ number of edges with label 1.

Fibonacci Cordial labeling is an extension of Cordial labeling, where we label the vertices with Fibonacci numbers instead of 0 and 1.

Definition 1.2. *The sequence F_n of Fibonacci numbers is defined by the recurrence relation:*

$$F_n = F_{n-1} + F_{n-2}; F_0 = 0, F_1 = F_2 = 1,$$

Definition 1.3. *An injective function* $f: V(G) \to \{F_0, F_1, \cdots, F_n\}$ *is said to be Fibonacci cordial labeling if the induced function* $f^*: E(G) \to \{0,1\}$ *defined by*

$$f^*(uv) = (f(u) + f(v))(\text{mod } 2)$$

satisfies the condition $|e_f(0) - e_f(1)| \leq 1$.

For other definitions and graph theoretic terminologies we follow [1]. Variations of graph labelings using Fibonacci numbers were investigated by various researchers over the last decade [3, 6, 7]. Reader may refer to [2] for an extensive and upadted survey. This particular labeling was first introduced by Rokad and Ghodasara [3] in 2010. Later in 2017, it was explored for more families of graphs by Rokad [2]. We have considered the families of complete graphs K_n, paths P_n, cycles C_n, and corona product of K_n and C_m.

2. Fibonacci cordial labeling for complete graphs

In this section we will provide a complete list of complete graphs those are Fibonacci cordial. We begin with the observation that placing of the labeling (Fibonacci numbers) are immaterial for K_n. As every Fibonacci number F_k is even if $3|k$, we will consider three different cases, $n = 3m, 3m+1$, and $3m+2$ for all $m \in \mathbb{Z}^+$.

Lemma 2.1. *The only Fibonacci Cordial complete graphs of the form* K_{3m} *are* K_3, K_6, K_9, *and* K_{18}.

Proof: First note that the vertex labeling can be chosen from $F_0, F_1, \cdots, F_{3m}$, out of which $2m$ labels are odd and remaining $m+1$ are even. Since we only need $3m$ many labeling, we drop either an odd or an even Fibonacci number from the list.

First we assume that we use all of the Fibonacci numbers, except an even one. As there are $2m$ many odd and m many even vertex labels, $e_f(1) = 2m^2$, and $e_f(0) = \binom{2m}{2} + \binom{m}{2}$. Hence in order to Fibonacci cordial we must have

$$\left|2m^2 - \binom{2m}{2} - \binom{m}{2}\right| \leq 1$$

It simplifies to $|m^2 - 3m| \leq 2$, which is only possible for $m = 1,2,3$.

Now if we consider all Fibonacci numbers, except an odd one, similar argument leads us to the conclusion that the graph will be Fibonacci cordial for $m = 1,6$.

Similar argument can be applied to the cases $n = 3m+1$, and $3m+2$, and we get next two lemmas.

Lemma 2.2. *The only Fibonacci Cordial complete graphs of the form* K_{3m+1} *are* K_4, K_7, *and* K_{22}.

Lemma 2.3. *The only Fibonacci Cordial complete graphs of the form* K_{3m+2} *are* K_2 *and* K_{11}.

The next theorem follows immediately from the previous lemmas.

Theorem 2.4. *The complete list of Fibonacci cordial complete graphs are* K_1, $K_2, K_3, K_4, K_6, K_7, K_9, K_{11}, K_{18}$ *and* K_{22}.

3. Fibonacci cordial labeling for cycles and path graphs

In this section we will provide a list cyclic graphs C_n, as well as introduce an iterative method that will produce Fibonacci cordial labeling for C_{n+1} from C_n. As a consequence we will inherit the Fibonacci cordial labeling for paths P_n. Let us denote $\tilde{e}_f = |e_f(0) - e_f(1)|$ and $\tilde{e}_g = |e_g(0) - e_g(1)|$. We begin this section with this following result.

Theorem 3.1. *For any two Fibonacci labeling (injective) $f: V(C_n) \rightarrow \{F_0, F_1, \cdots, F_n\}$ and $g: V(C_n) \rightarrow \{F_0, F_1, \cdots, F_n\}$, $\tilde{e}_f - \tilde{e}_g \equiv 0 \pmod 4$.*

Proof: Without loss of generality, f and g are same Fibonacci labeling except at $v_0 \in V(C_n)$. That is $f(v_0) = g(v_0)$. Let us consider $e_f(0) = m$, and hence $e_f(1) = n - m$. Also consider v_L and v_R are two adjacent vertices of v_0.

Now $f^*(v_L v_0) \equiv 1 (\mathrm{m}od \quad 2)$ and $f^*(v_0 v_R) \equiv 0 (\mathrm{m}od \quad 2)$, then it is clear that

$$\tilde{e}_f = \tilde{e}_g$$

Otherwise without loss of generality, we may assume that

$$f^*(v_L v_0) = f^*(v_0 v_R) \equiv 0 (\mathrm{m}od \ 2)$$

In this case, clearly $e_g(0)$ will be either m or $m - 2$. Respectively $e_g(1)$ will be either $n - m$, or $n - m + 2$. Thus

$$\tilde{e}_g - \tilde{e}_f = |e_g(0) - e_g(1)| - |e_f(0) - e_f(1)| \equiv 0 (\mathrm{m}od \quad 4)$$

Corollary 3.2. *For any injective function $f: V(G) \rightarrow \{F_0, F_1, \cdots, F_{2m}\}$ on cyclic graph C_{2m}, if $|e_f(0) - e_f(1)| = 2 \pmod 4$, then C_{2m} can not be converted FC.*

Lemma 3.3. *If the cyclic graph C_n is Fibonacci Cordial with $e_f(0) = e_f(1)$, then C_{n+1} will aslo be Fibonacci Cordial with $e_f(0) = e_f(1) + 1$.*

Proof: First note that n must be even, and hence $e_f(0) = e_f(1) = n/2$. From C_n we choose any edge $uv \in E(G)$ such that $f^*(uv) = 1$. Now we delete that edge and add another vertex w, and then connect $\{u, w\}$ and $\{v, w\}$. Note that as a consequence $e_f(0) = n/2 + 1$, and $e_f(1) = n/2$ which shows that C_{n+1} is also Fibonacci Cordial with $e_f(0) = e_f(1) + 1$. We also conclude that this is the only case (that is $e_f(0) = e_f(1) - 1$ is not possible), using Theorem 3.1.

Lemma 3.4. *If the cyclic graph C_n is Fibonacci Cordial with $e_f(1) = e_f(0) + 1$, then C_{n+1} will aslo be Fibonacci Cordial with $e_f(0) = e_f(1)$.*

We omit the proof of Theorem 3.4, as it will be very similar to the previous one.

Lemma 3.5. *If the cyclic graph C_n is Fibonacci Cordial with $e_f(0) = e_f(1) + 1$, then C_{n+1} will be Fibonacci Cordial.*

Theorem 3.6. *For m being an positive integer,*

1. C_{4m+k} is Fibonacci cordial, for all $k \in \{0,1,3\}$
2. C_{4m+2} is not Fibonacci cordial

Proof: We start with the following function that provides a Fibonacci cordial labeling for C_{4m}.

$$f(i)=\begin{cases}F_i, & \text{if} i\in\{12k,12k+1,\cdots,12k+5,12k+10,12k+11\}\\ F_{12k+7}, & \text{if} i=12k+6\\ F_{12k+6}, & \text{if} i=12k+7\\ F_{12k+9}, & \text{if} i=12k+8\\ F_{12k+8}, & \text{if} i=12k+9\end{cases}$$

Now note that for C_{4m}, $e_f(0)=e_f(1)$. Thus by applying Lemma 3.3, we know immediately that C_{4m+1} is Fibonacci cordial with $e_f(0)=e_f(1)+1$, and therefore (by Lamma 3.5 and Theorem 3.1) C_{4m+2} is not Fibonacci cordial. Now it only remains to show that C_{4m+3} is Fibonacci cordial.

We consider the Fibonacci cordial graph C_{4m+1}. Now there are two possibilities that we need to consider, that is both of F_{4m+2} and F_{4m+3} are odd, or one of them is even. For the former case, first we make an observation that there are edges $uv\in E(G)$, where both $f(u)$ and $f(v)$ are even. Otherwise (if an even vertex label has odd labels on it both sides) $e_f(1)=2m\geq 2(4m+2)/3$, which is impossible. Now in C_{4m+1} by replacing the edge uv by a path $uxyv$ we produce the Fibonacci cordial labeled graph C_{4m+3} with $e_f(0)=2m+1$ and $e_f(1)=2m+2$.

For the latter case, choose any edge $uv\in E(G)$, such that both of $f(u)$ and $f(v)$ are odd. Now again by replacing the edge uv by a path $uxyv$ in the same fashion, which also produce the Fibonacci cordial labeled graph C_{4m+3} with $e_f(0)=2m+1$ and $e_f(1)=2m+2$.

Theorem 3.7. *All Paths are Fibonacci cordial.*

Proof: For $n\in\{4m,4m+1,4m+3\}$, we know that C_n is Fibonacci cordial. Thus by deleting the appropriate edge we can get the Fibonacci cordial labeling of the path P_n. Otherwise if C_n is not Fibonacci cordial (that is $n=4m+2$) then first achieve the labeling with $|e_f(0)-e_f(1)|=2$. Without loss of generality let us assume $e_f(0)=e_f(1)+2$. Now by delete one of the edges with even labels we obtain the Fibonacci cordial path graph P_{4m+2}, for m being an integer.

4. Fibonacci cordial labeling for corona graphs

The *corona*$G_1\odot G_2$ of two graphs is the graph obtained by taking one copy of G_1, and p_1 copies of G_2 (where $|V(G_1)|=p_1$), and then joining the i^{th} vertex of G_1 by an edge to every vertex in the i^{th} copy of G_2.

The number of vertices of $C_n\odot K_p$ is $n(p+1)$, and number of edges are $n(p(p+1)/2+1)$. In the graph $G=C_n\odot K_p$, let $V=\{v_i^t|i\in\mathbb{N}\}$, where $t=0$ denote the vertices of C_n, and $t\in\mathbb{Z}_p^+$ denote the pendant vertices.

Theorem 4.1. $C_n\odot K_1$ *is Fibonacci cordial for all* $n\in\mathbb{N}$.

Proof: First we show that $C_3\odot K_1$ is Fibonacci cordial. Now we show that by induction that if $C_m\odot K_1$ is Fibonacci Cordial, then $C_m\odot K_1$ will be Fibonacci cordial. Let $f:V(C_m\odot K_1)\to\{F_0,F_1,\cdots,F_{2n}\}$ assign a Fibonacci cordial labeling to the graph $C_m\odot K_1$. Let F_k is the unused label among $\{F_0,F_1,\cdots,F_{2n}\}$. Now we will consider two cases based on the set of labels $\{F_k,F_{2n+1},F_{2n+2}\}$, all are odd, or at least one is even.

Case 1. First we consider the case at least one of the labels $\{F_k, F_{2n+1}, F_{2n+2}\}$ is even. We choose any edge $\{v_j^0 - v_{j+1}^0\} \in E(G)$. Without loss of generality let us assume that both $f(v_j^0)$ and $f(v_{j+1}^0)$ are odd. Now replace that edge by the path $\{v_j^0 - u - v_{j+1}^0\}$, and add another w as pendant vertex of u. Now label u by an odd and w by an even Fibonacci Number from the list of $\{F_k, F_{2n+1}, F_{2n+2}\}$. Hence the resultant corona product $C_{n+1} \odot K_1$ is also Fibonacci cordial with $e_f(0) = e_f(1) = n + 1$.

Case 2. Next we will consider the case, where all of $\{F_k, F_{2n+1}, F_{2n+2}\}$ are odd. First note that there is there is at least one pendant edge $\{v_k^0 - v_k^1\} \in E(G)$, such that both have even labels or $f(v_k^0)$ is odd and $f(v_k^1)$ is even.

Subcase (a).

If there is a pendant edge $\{v_k^0 - v_k^1\}$ such that $f(v_k^0)$ and $f(v_k^1)$ both are even. Let us choose an edge $v_j^0 - v_{j+1}^0 \in E(G)$. Without loss of generality let us assume that both $f(v_j^0)$ and $f(v_{j+1}^0)$ are even. Now replace that edge by the path $\{v_j^0 - u - v_{j+1}^0\}$, and add another w as pendant vertex of u. Now we relabel in the following manner,$f(u) = F_k, f(v) = F_{2n+1}, f(v_{k,1}) = F_{2n+2}$.

Subcase (b). Now we will consider the subcase that $f(v_k^0)$ is odd and $f(v_k^1)$ is even. Note that at least one edge $\{v_j^0 - v_{j+1}^0\} \in E(G)$, such that $f^*(v_j^0 v_{j+1}^0) = 1$, because otherwise $f(v_j^0)$ are either all even or all odd, and both are impossible. We replace that edge by the path $\{v_j^0 - u - v_{j+1}^0\}$, and add another w as pendant vertex of u. Now we relabel in the following manner,$f(u) = F_k, f(w) = F_{2n+1}, f(v_{k,1}) = F_{2n+2}$.

Hence in both (sub)cases we produce the Fibonacci cordial labeling of the graph $C_{n+1} \odot K_1$.

Now we iteratively produce a Fibonacci cordial labeling for the $C_{n+1} \odot K_2$ from the Fibonacci cordial labeling of $C_n \odot K_2$. First we show that $C_3 \odot K_2$ is Fibonacci cordial.

Theorem 4.2. *$C_n \odot K_2$ is Fibonacci cordial.*

Proof: We prove this by induction. We have already observed that $C_3 \odot K_2$ is Fibonacci cordial. Now we assume that $C_n \odot K_2$ is Fibonacci cordial, and we show that $C_{n+1} \odot K_2$ is also Fibonacci cordial. Note that in the list of the Fibonacci numbers $\{F_{3n+1}, F_{3n+2}, F_{3n+3}\}$, there are two are odd and one is even. Now we start by choosing an edge $(v_j^0 v_{j+1}^0)$, where $f^*(v_j^0 v_{j+1}^0) = 1$. Now we replace that edge by a path $\{v_j^0 - u - v_{j+1}^0\}$ and add a triangle graph with u be its one vertex. Let x and y be the other two vertices of that triangle graph. now by labeling u by the even number and x, y by the odd numbers from the Fibonacci set $\{F_{3n+1}, F_{3n+2}, F_{3n+3}\}$, we produce a Fibonacci labeling of the $C_{n+1} \odot K_2$.

Theorem 4.3. *$C_n \odot K_3$ is Fibonacci cordial.*

Proof: Following the pattern, we need to show that $C_{n+1} \odot K_3$ has a Fibonacci cordial labeling whenever $C_n \odot K_3$ is Fibonacci cordial. Considering that in $C_n \odot K_3$, we have used all Fibonacci number except F_k , note that in $C_{n+1} \odot K_3$ we have

$\{F_k, F_{4n+1}, F_{4n+2}, F_{4n+3}, F_{4n+4}\}$. Now in this list there are three possibilities in terms of the numbers of odd and even, viz $\{(4,1),(3,2),(2,3)\}$. Without loss of generality we consider the first case, and second case follows similarly. We start by choosing an edge $(v_j^0 v_{j+1}^0)$, where $f^*(v_j^0 v_{j+1}^0) = 1$. Now replace that edge by the path $\{v_j^0 - u - v_{j+1}^0\}$, and add a Complete graph K_4 with u be its one vertex. Let x, y and z be the other three vertices of that K_4 graph. Finally we label u by the even number and x, y, z by the odd numbers from the Fibonacci set $\{F_k, F_{4n+1}, F_{4n+2}, F_{4n+3}, F_{4n+4}\}$, we produce a Fibonacci labeling of the $C_{n+1} \odot K_2$.

If there are three even numbers in the set $\{F_k, F_{4n+1}, F_{4n+2}, F_{4n+3}, F_{4n+4}\}$, then we label $u, x,$ and y by the even numbers and z by an odd number.

5. Conclusion

In this paper we have provided the complete list of complete graphs that are Fibonacci cordial and prove that the remaining are not. We have also investigated the Fibonacci Cordial-ness of paths and cycles. At the end, we formally proved that the corona product of the cycles and complete graphs, given by $C_n \odot K_p$, is Fibonacci graceful for $p = 1,2,$ and 3. In future we would like to extend our work to investigate whether $C_n \odot K_p$ for $p \geq 4$. As well as, we wish to verify the the Fibonacci Cordial labeling for some other known graphs.

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