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Stabilizer Graph and Orbit Graph of Finite Groups

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Abstract. The interplay between groups and graphs have been the most famous and productive area of algebraic graph theory. In this paper, we define new graphs namely stabilizer graph, orbit graph, non-stabilizer graph and non-orbit graph for each permutation group G. Our aim is to translate properties of graphs into algebraic properties and then using the results and methods of algebra to declare theorem about graphs. We investigate the interconnection between the group theoretic properties of G and the graph theoretic properties of the above said graphs. Some of the graphs are illustrated with example. Finally we obtain some important results including theorem and discuss the relationship between group isomorphism and graph isomorphism.

1. Introduction

Graph theory is one of the foremost probing field in mathematics principally because of its applications in various fields which include biochemistry, electrical engineering, computer science and operations research. These both twigs of mathematics are playing a prominent role in modern mathematics.

Groups are the focal mathematical equipment for studying symmetries of an object and symmetries are usually associated to graph automorphisms. Several structures in abstract algebra are special cases of groups. In group theory we study and examine different groups and their structures while in graph theory we concentrate on the graphs that indicates the structure of materials and objects. The forceful combinatorial methods found in graph theory have also been utilized to demonstrate significant and well-known results in a variety of areas in mathematics comprising group theory.

All finite groups can be delineated as the automorphism group of a connected graph [1]. Relating a graph to a group provides a method of visualizing a group and connects two important branches of mathematics. It gives a review of cyclic groups, dihedral groups, direct products, generators and relations [2]. In the past few decades the researchers have concentrated on the modified form of groups and graphs by inter relating their properties. The literature of algebraic graph theory itself has grown

enormously since 1974 [3]. The study of the algebraic structures utilizing the properties of graphs has become a rousing research topic in the last twenty years, foremost to several interesting results and questions.

The concept of zero-divisor graph was first introduced by Beck in [4] while discussing the coloring of a commutative ring. In his work all elements of the ring were considered as vertices of the graph. Anderson and Naseer used this same concept in [5] and Mahadevi and Babujee also done their work on special structures of zero-divisor graphs [6]. Muneshwar and Bondar gave an introduction and overview of some nice ideas from group theory by using graph theory [7]. Chalapathi and Kiran Kumar explained the concept of order divisor graph of finite groups [8]. Cameron and Ghosh done their work on power graph of a finite group [9]. Omer, Sarmin and Erfanian introduced orbit graph for some finite solvable groups [10]. Shigehalli and Uppin discussed some properties of glue graph [11]. Varkey and Rajan studied on the spectrum and energy of concatenated singular graphs [12]. Revankar et al. [13] explained on eccentricity sum eigenvalue and eccentricity sum energy of a graph. In this paper, we construct a stabilizer graph and orbit graph for finite groups.

In section 2, we discuss stabilizer graph and non-stabilizer graph. Orbit graph and non-orbit graph are discussed in section 3. In each section related theorems and results are obtained. Also, some of graphs are illustrated with suitable examples. Section 4 concludes the paper.

2. Stabilizer graph of finite group
A. Stabilizer graph of G

Definition 2.1. Let G be a group of permutations of a set A. For each x in A, let $Stab_G(x) = \{ \phi \in G \mid \phi(x) = x \}$. We call $Stab_G(x)$ the stabilizer of x in G and is denoted by $S_G(x)$. (See [14]).

Definition 2.2. Let A be any finite set and G be a group of permutations. We define a stabilizer graph of G, denoted by $\Gamma(S_G(A))$ is a simple graph (V, E) with vertex set V = { $S_G(x) \mid \forall x \in A$} and edge set E = {$(S_G(x), S_G(y)) \mid S_G(x) = S_G(y) \; \forall$ x, y$\in$ A}.

Example 2.1. Let A = U(7) = {1, 2, 3, 4, 5, 6}. Let us calculate the left rectangular representation $\overline{U(7)}$ for U(7) = {1, 2, 3, 4, 5, 6}. Writing the permutations of U(7) in array form, we have (T_x is just a multiplication by x),

$$T_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix}, \qquad T_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 6 & 1 & 3 & 5 \end{pmatrix},$$

$$T_3 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 6 & 2 & 5 & 1 & 4 \end{pmatrix}, \qquad T_4 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 1 & 5 & 2 & 6 & 3 \end{pmatrix},$$

$$T_5 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 3 & 1 & 6 & 4 & 2 \end{pmatrix}, \qquad T_6 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 5 & 4 & 3 & 2 & 1 \end{pmatrix}.$$

Let G = $\overline{U(7)}$ = {(1), (1 2 4)(3 6 5), (1 3 2 6 4 5), (1 4 2)(3 5 6), (1 5 4 6 2 3),(1 6)(2 5)(3 4)}. Then, we have,

$Stab_G(1) = Stab_G(2) = Stab_G(3) = Stab_G(4) = Stab_G(5) = Stab_G(6) = \{(1)\}$.Stabilizer graph of U(7) in $\overline{U(7)}$ is shown in Figure 1.
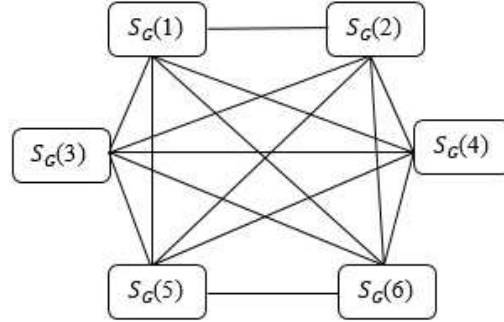
Figure 1: $\Gamma(S_{\overline{U(7)}}U(7))$

Definition 2.3. $\Gamma(S_G(A))$ is complete if and only if $S_G(x) = S_G(y)$ for all distinct x, y ∈ A.

Result 2.1. Stabilizer graph of the group $\overline{U(n)}$ is always complete. Because the stabilizer point of U(n) in $\overline{U(n)}$ is trivially equal to (1).

Result 2.2. If G is a symmetric group S_n where n ≥ 3, then the stabilizer graph is always null graph.
Proof: Let A be any finite set and G be a group of permutations of a set A. For each x in A, let $Stab_G(x) = \{\phi \in G \mid \phi(x) = x\}$. If G is a symmetric group S_n where $n \geq 3$, we get different stabilizer of each x in G. i.e. $S_G(x) \neq S_G(y) \; \forall \, x, y \in A$. Then there does not exist an edge between every pair of vertices, $S_G(x) \; \forall \, x \in A$. Therefore, the stabilizer graph $\Gamma(S_G(A))$ is a null graph.

Result 2.3. For abelian groups, the stabilizer graph is obviously complete.
Proof: Let G be an abelian group. i.e., S_n , $n \leq 2$ and $\overline{U(n)}$. For each x in A, we have $Stab_G(x) = \{(1)\}$ so that $S_G(x) = S_G(y) \; \forall \, x, y \in A$. Then there exist an edge between every pair of vertices, $S_G(x) \; \forall \, x \in A$. Therefore, the stabilizer graph $\Gamma(S_G(A))$ is complete.

Result 2.4. If G is the set of all even permutations of a symmetric group, then the stabilizer graph is complete for S_n where n ≤ 3 and null graph for S_n where n ≥ 4.
Proof: Let G be the set of all even permutations of a symmetric group S_n.
Case(i): If G is the set of all even permutations of a symmetric group S_n where $n \leq 3$, then for each x in A, we have $Stab_G(x) = \{(1)\}$ so that $S_G(x) = S_G(y) \; \forall \, x, y \in A$. Then there exist an edge between every pair of vertices, $S_G(x) \; \forall \, x \in A$. Therefore, the stabilizer graph $\Gamma(S_G(A))$ is complete.
Case(ii): If G is the set of all even permutations of a symmetric group S_n where $n \geq 4$, then for each x in A, $S_G(x) \neq S_G(y)$ for all $x, y \in A$. Then there does not exist an edge between every pair of vertices, $S_G(x)$ for all $x \in A$. Therefore, the stabilizer graph $\Gamma(S_G(A))$ is a null graph.

Result 2.5. If G is the set of all odd permutations of a symmetric group, then the stabilizer graph is null graph for S_n where n ≥ 3 and empty graph for S_2.

Proof: Let G be the set of all odd permutations of a symmetric group S_n.

Case (i): If G is the set of all odd permutations of a symmetric group S_n where $n \geq 3$, then for each x in A, $S_G(x) \neq S_G(y) \ \forall x, y \in A$. Then there does not exist an edge between every pair of vertices, $S_G(x) \forall \ x \in A$. Therefore, the stabilizer graph $\Gamma(S_G(A))$ is a null graph.

Case (ii): If G is the set of all odd permutations of a symmetric group S_n where $n = 2$, then for each x in A, stabilizer point does not exist. So there is no any vertex. Thus we get an empty graph.

Result 2.6. If G is the set of all even permutations of a group $\overline{U(n)}$, then the stabilizer graph is complete.

Proof: Let G be a group of even permutations of the group $\overline{U(n)}$. Then, by Result 2.1, for each x in A, $Stab_G(x) = \{(1)\}$ so that $S_G(x) = S_G(y) \ \forall \ x, y \in A$. Then there exist an edge between every pair of vertices, $S_G(x) \ \forall \ x \in A$. Therefore, the stabilizer graph $\Gamma(S_G(A))$ is complete.

Result 2.7. If G is the set of all odd permutations of a group $\overline{U(n)}$, then the stabilizer graph is empty graph.

Example 2.2. Let A = U(5) = {1, 2, 3, 4, 5}. If we take G as an odd permutation then G = $\overline{U(5)}$ = {(1 2 4 3), (1 3 4 2)}.

Result 2.8. If we take G as a random permutations of a symmetric group, then the stabilizer graph maybe connected or disconnected.

Example 2.3. Let A = {1, 2, 3, 4, 5, 6, 7, 8} and G = {(1), (1 3 2) (4 6 5) (7 8), (1 3 2)(4 6 5), (1 2 3) (4 5 6), (1 2 3) (4 5 6) (7 8), (7 8)}. Then, we have,
$Stab_G(1) = Stab_G(2) = Stab_G(3) = Stab_G(4) = Stab_G(5) = Stab_G(6) = \{(1), (7 \ 8)\}$,
$Stab_G(7) = Stab_G(8) = \{(1), (1 3 2)(4 6 5), (1 2 3)(4 5 6)\}$.

Theorem 2.1. Let U(n_1) and U(n_2) be two finite groups. If U(n_1) $\cong$ U(n_2), then $\Gamma\left(S_{\overline{U(n_1)}}U(n_1)\right) \cong \Gamma\left(S_{\overline{U(n_2)}}U(n_2)\right)$.

Proof: Assume U(n_1) $\cong$ U(n_2). Then there exist an isomorphic map ϕ: U(n_1)→U(n_2). Since ϕ is a group isomorphism that preserves stabilizer point of every element of a set, ϕ maps stabilizer point of U(n_1) to a stabilizer point of U(n_2). Let x_i and y_i be the stabilizer points of U(n_1) and U(n_2) respectively, where i = 1, 2, 3…n. Then we see that $\phi(x_i) = y_i$ for all i. Also, if x_i is adjacent to x_j, then $\phi(x_i)$ is adjacent to $\phi(x_j)$, for all $i \neq$ j. Thus ϕ is a graph isomorphism. Hence $\Gamma\left(S_{\overline{U(n_1)}}U(n_1)\right) \cong \Gamma\left(S_{\overline{U(n_2)}}U(n_2)\right)$ □

Example 2.4. Let U(8) = {1, 3, 5, 7} and G = $\overline{U(8)}$ = {(1), (1 3)(5 7), (1 5)(3 7), (1 7)(3 5)}. Then, we have $Stab_G(1) = Stab_G(3) = Stab_G(5) = Stab_G(7) = \{(1)\}$. Similarly, let U(12) = {1, 5, 7, 11} and G = $\overline{U(12)}$ = {(1), (1 5)(7 11), (1 7)(5 11), (1 11)(5 7)}. Then, we have,
$Stab_G(1) = Stab_G(5) = Stab_G(7) = Stab_G(11) = \{(1)\}$. Stabilizer graph of U(8) in $\overline{U(8)}$ and Stabilizer graph of U(12) in $\overline{U(12)}$ is shown in Figure 2.
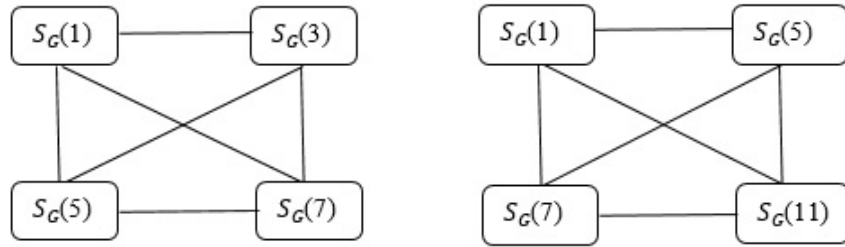
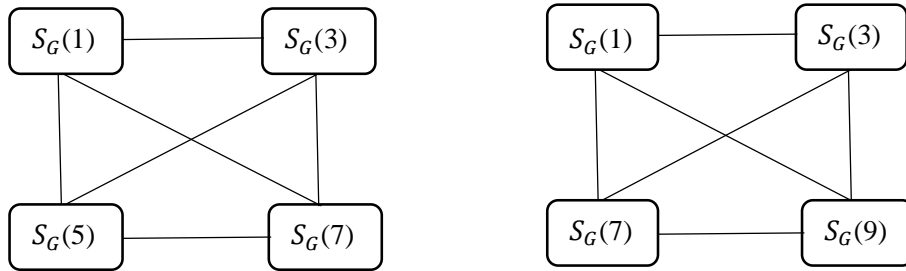**Figure 2:** Graph isomorphism between $\Gamma(S_{\overline{U(8)}}(U(8)))$ and $\Gamma(S_{\overline{U(12)}}(U(12)))$.

Corollary 2.1. The converse of the above theorem is not true.
Proof: Consider the finite groups U(8) and U(10). Let U(8) = {1, 3, 5, 7} and G = $\overline{U(8)}$ = {(1), (1 3)(5 7), (1 5)(3 7), (1 7)(3 5)} and Let U(10) = {1, 3, 7, 9} and G = $\overline{U(10)}$ = {(1), (1 3 9 7), (1 7 9 3), (1 9)(3 7)}.

Then stabilizer point of U(8) in $\overline{U(8)}$ and stabilizer point of U(10) in $\overline{U(10)}$ are $Stab_G(1) = Stab_G(3) = Stab_G(5) = Stab_G(7) = \{(1)\}$ and $Stab_G(1) = Stab_G(3) = Stab_G(7) = Stab_G(9) = \{(1)\}$ respectively.

Here $\Gamma(S_{\overline{U(8)}}(U(8)))$ and $\Gamma(S_{\overline{U(10)}}(U(10)))$ are isomorphic, but U(8) and U(10) are not isomorphic. Stabilizer graph of U(8) in $\overline{U(8)}$ and Stabilizer graph of U(10) in $\overline{U(10)}$ is shown in Figure 3.

Figure 3: Graph isomorphism between $\Gamma(S_{\overline{U(8)}}(U(8)))$ and $\Gamma(S_{\overline{U(10)}}(U(10)))$.

B. Non-stabilizer graph of G
Definition 2.4. Let A be any finite set and G be a group of permutations. We define a non-stabilizer graph of G, denoted by $\Gamma_N(S_G(A))$ is a simple graph (V, E) with vertex set V = $\{S_G(x) \mid \forall x \in A\}$ and edge set E = $\{((S_G(x), S_G(y)) \mid S_G(x) \neq S_G(y) \forall x, y \in A\}$.
Note: Null graph and disconnected graph can be converted into a connected graph by using the non-stabilizer graph definition.

Result 2.9. If G is a symmetric group S_n where n ≥ 3, then the non-stabilizer graph is always complete.

Proof: Let G be a symmetric group S_n where $n \geq 3$. Then, for each x in A, $S_G(x) \neq S_G(y) \ \forall \ x, y \in A$. By the definition of non-stabilizer graph, there exist an edge between every pair of vertices,$S_G(x) \ \forall \ x \in A$. Therefore, the non-stabilizer graph $\Gamma_N(S_G(A))$ is complete.

Result 2.10. Non-stabilizer graph need not be complete.

Note: Disconnected graph becomes connected when using non stabilizer graph definition but this need not be complete.

4. Orbit graph of finite group
A. Orbit graph of G
Definition 3.1. Let G be a group of permutations of a set A. For each x in A, let$Orb_G(x)$ = { ϕ(x) | $\phi \in$ G}. Then the set $Orb_G(x)$ is a subset of A called the orbit of x under G and is denoted by $O_G(x)$. We use $| Orb_G(x) |$ to denote the number of elements in $Orb_G(x)$. (See [14]).

Definition 3.2. Let A be any finite set and G be a group of permutations. We define a orbit graph of G, denoted by $\Gamma(O_G(A))$ is a simple graph (V, E) with vertex set V = { $O_G(x) | \forall x \in$ A} and edge set E = {$(O_G(x), O_G(y)) | O_G(x) = O_G(y) \forall$ x, y∈ A}.

Example 3.1. Let A = {1, 2, 3} and G =S_3= {(1), (1 2 3), (1 3 2), (2 3), (1 3), (1 2)}. Then,$Orb_G(1) = Orb_G(2) = Orb_G(3)$ = {1, 2, 3}. The Orbit graph of A under S_3 is shown in Figure 4.
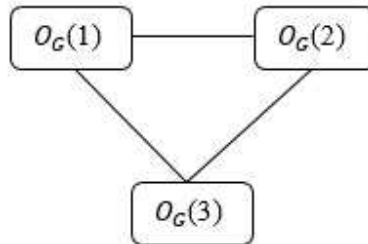
Figure 4: $\Gamma(O_{S_3}(A))$

Definition 3.3. $\Gamma(O_G(A))$ is complete if and only if O_G(x) = O_G(y) for all distinct x, y∈ A.

Result 3.1. If G is a symmetric group, then the orbit graph is always complete.
Proof: Let A be any finite set and G be a group of permutations of a set A. For each x in A, let $Orb_G(x)$ = {$\phi(x) | \phi \in G$}. If G is a symmetric group, then for each x in A we get $O_G(x)$ = {1, 2, 3, … ,n} so that $O_G(x) = O_G(y) \forall x, y \in A$. Then there exist an edge between every pair of vertices,$O_G(x) \forall x \in A$. Therefore, the Orbit graph $\Gamma(O_G(A))$ is complete.

Result 3.2. Orbit graph of the group $\overline{U(n)}$ is always complete.

Proof: Let G be the group $\overline{U(n)}$. Then for each x in A, we get $O_G(x) = \{1, 2, 3, \dots ,n\}$ so that $O_G(x) = O_G(y) \, \forall \, x, y \in A$. Then there exist an edge between every pair of vertices, $O_G(x) \, \forall \, x \in A$. Therefore, the Orbit graph $\Gamma(O_G(A))$ is complete.

Result 3.3. If G is the set of all even permutations of a symmetric group, then the orbit graph is always complete, except S_2.

Proof: Let G be the set of all even permutations of a symmetric group S_n.

Case (i): If G is the set of all even permutations of a symmetric group S_n where $n \neq 2$, then for each x in A, we get $O_G(x) = \{1, 2, 3, \dots ,n\}$ so that $O_G(x) = O_G(y)$, $\forall \, x, y \in A$. Then there exist an edge between every pair of vertices, $O_G(x) \, \forall \, x \in A$. Therefore, the Orbit graph $\Gamma(O_G(A))$ is complete.

Case (ii): If G is the set of all even permutations of a symmetric group S_n where $n = 2$, then for each x in A, we get different orbit points so that $O_G(x) \neq O_G(y) \, \forall \, x, y \in A$. Then there does not exist an edge between the pair of vertices, $O_G(x)$ and $O_G(y)$. Therefore, the Orbit graph $\Gamma(O_G(A))$ is a null graph.

Result 3.4. If G is the set of all odd permutations of a symmetric group, then the orbit graph is always complete, except S_2.

Proof: Similar to the proof of Result 3.3.

Theorem 3.1. Let $U(n_1)$ and $U(n_2)$ be two finite groups. If $U(n_1) \cong U(n_2)$, then $\Gamma\left(O_{\overline{U(n_1)}}U(n_1)\right) \cong \Gamma\left(O_{\overline{U(n_2)}}U(n_2)\right)$.

Proof: Similar to the proof of Theorem 2.1.

Corollary 3.1. The converse of the above theorem is not true.

Proof: Similar to the proof of Corollary 2.1.

B. Non-orbit graph of G

Definition 3.4. Let A be any finite set and G be a group of permutations. We define a non-orbit graph of G, denoted by $\Gamma_N(O_G(A))$ is a simple graph (V, E) with vertex set V = $\{O_G(x) \mid \forall \, x \in A\}$ and edge set E = $\{(O_G(x), O_G(y)) \mid O_G(x) \neq O_G(y), \forall \, x, y \in A\}$.

Example 3.2. Let A = $\{1, 2\}$ and G = $\{(1 \ 2)\}$. Then $Orb_G(1) = \{2\}$, $Orb_G(2) = \{1\}$ and non-orbit graph of A under G is shown in Figure 5.
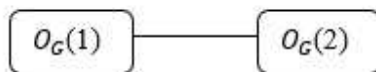
Figure 5: $\Gamma_N(O_G(A))$

4. Conclusion

In this paper, we established the stabilizer graph and orbit graph of finite groups and some of the related results were dealt with suitable examples. Next, non-stabilizer graph and non-orbit graph were found and discussed. We investigated the interconnection between the group theoretic properties of G and the graph theoretic properties of the

stabilizer graph, orbit graph, non-stabilizer graph and non-orbit graph. This proposed work can be extended to other graph theoretic properties with other finite groups.

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