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The Group Structure of Unit Γ-regular Ring Elements

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Abstract. In this paper we discuss a group structure of unit Γ-regular ring and unit inner inverse. Then we construct a unit Γ-regular ring and some of the theorem and lemma are discussed. We obtain some theorem about unit inner inverses, the concept of unit reflexive inverse is discussed. Also we declare some theorem and lemma about this concept.

Keywords: Group structure, unit elements, commutative ring, unit Γ-regular ring, unit inner inverse, unit reflexive inverse

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1. Introduction

An element a of a ring R is said to be regular if and only if there exists an element x of R such that α x α = α. The ring R is regular iff each element of R is regular. The idea of a regular ring introduced by von Neumann [12,14], but required as well regular ring contain a unit element [1]. If R is a ring with identity such that for every α x α = α and α^n x = x α^n, and if n is a unit of R, the every element of R is a sum of the bounded number of units [9, 10]. This concept is used in the theorem (3.9) and lemma (3.6). It is recognized that [6, 7] a ring R is strongly regular if and only if every α ∈ R is a group member. In this note we shall utilize the fundamental theorem for group members in a ring to exhibit locally that a ring element α ∈ R is unit regular precisely when there is a unit u ∈ R and a group G in R such that a ∈ uG. Thus unit regular rings are, so to speak locally a rotated version of strongly regular rings [5]. We remind that a ring is called regular if for every α ∈ R, α ∈ α R α and its unit regular if for every α ∈ R, there is a unit u ∈ R such that α u α = α [4]. A ring with unity is called finite if α β = 1 and β α = 1. Any solution α^- to α x α = α is called an inner or 1-inverse of [2], and any solution α^+ to α x α = α and x α x = x is called a reflexive or 1-2 inverse of α. If idempotent elements e and j in R, and e ~ j denotes the equivalence in [8] as contrasted with α = p α q, where p and q are invertible [5].

2. Preliminaries

Ring. A non-empty set R is said to be *ring*, together with two operation ⊕ and ∗ , which has the following properties:

(a) R is a commutative group under ⊕

(b) R is a associative under ∗

(c) Multiplicative identity: There is an element 1 such that r∗1=1∗r=r for all r ∈ R

(d) The operation ∗ distributes over ⊕: a ∗ (b ⊕ c) = (a ∗ b) ⊕ (a ∗ c).

Γ-Ring. Let R and Γ be two addition abelian group. If for all x, y, z ∈ R and for all α, β ∈ Γ the conditions:

1) x α y ∈ R
2) (x + y) α z = x α z + y α z and x (α +β) z = x α z + x β z
3) (x α y) β z = x α (y β z) are satisfied, then R is called a *Γ- ring*.

Regular. An element 'a' of a ring R is said to be *regular* and if there exists an element x of R such that a x a = a. The ring R is regular iff each element of R is regular.

Unit regular. Let R be a ring with identity. If a ∈ R, a is *unit regular* if there is a unit x ∈ R such that a x a = a.

Unit Γ-regular ring. A *Γ- ring(Γ, R) is a unit regular*, if for element a ∈ R , there is unit u ∈ R and a group G in Γ such that a ∈ uG.

Inverses. If A has a 1-inverse, A^- then it is not unique and that the most general 1-inverse is of the form A^- +H– A^- AHAA^- (or) A^- + (1– A^-) H+K(1–A A^-), where H and K is arbitrary [5].

1-inverse. a ∈ R is regular if there exist an element a^- such that a a^- a = a, the element a^- is called an *inside or 1-inverse* of a. Any solution a^- to a x a = a is called an inner or 1-inverse of a and any solution a^+ to a x a = a and x a x = x is called a reflexive or 1-2 inverse of [9].

Unit inner inverses. An element 'a' of a unit regular ring if a u a = a, with u is invertible, then $\mathcal{U}_a$ = u $\mathcal{U}_{a\,u}$ = $\mathcal{U}_{u\,a}$u. Indeed, if $w \in \mathcal{U}_{a\,u}$ then au$w$ a u = a u which implies that a u w a = a and hence u $w \in \mathcal{U}_a$ conversely, if a w a = a, w is a unit, then a u ($u^{-1}w$) a u = a u which implies that $u^{-1}w \in \mathcal{U}_{a\,u}$ and hence w ∈ u $\mathcal{U}_{a\,u}$.

Since u $\mathcal{U}_{a\,u}$ is independent of the choice of the unit inner inverse u of a, for any unit inner inverses u and v of a, such that $\mathcal{U}_a$ = u $\mathcal{U}_{a\,u}$ = v $\mathcal{U}_{av}$,in particular, u^{-1} v $\mathcal{U}_{a\,u}$ [5].

Unit inner inverse of idempotent element. The set of $\mathcal{U}_a$ is determined by the set of unit inner inverses $\mathcal{U}_e$. If $\mathcal{U}_e$ is the set of all unit of the form:

i. 1 + (1– e) x + y (1 − e) for some x , y;
ii. e+ (1 − e) v + s (1 − e) for some v , s;

iii. $1 + h - e h e$ for some h;

iv. $e + k - e k e$ for some k.

In general, the set $\mathscr{U}_a$ or even $\mathscr{U}_e$ will not be a union of semi groups [5].

Example 2.1. If e= $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in R_{2\times 2}$ where $R_{2\times 2}$ denotes the two by two matrix ring over a field, and if $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \in \mathscr{U}_e$, but $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^2 \notin \mathscr{U}_e$. It is only for idempotent elements possible to posses union of semi groups of unit inner inverses [5].

Regular Γ-ring. A Γ-ring (Γ,R) is regular if for each x ∈ R there exists $\delta \in \Gamma$ such that x δ x = x. we abbreviate this as R is regular.

Commutative Γ-regular ring. A Γ-regular ring (Γ, R) is said to be *commutative Γ-regular ring*, if α x = x α, α +x = x + α for α ∈ R and x ∈ Γ.

Zero element. A regular Γ-ring R is said to have a *zero element* if there exists an element 0 ∈ R such that 0+x = x+0 and 0 α x = x α 0 for all x ∈ R and α ∈ Γ. Also, a regular Γ-ring R is said to be commutative if x α y = y α x for all x, y ∈ R and α ∈ Γ.

Γ-Homomorphism. Let R and S be two Γ-regular rings. A mapping f of a Γ-regular ring R into a Γ-regular ring S is said to be a Γ-homomorphism of R into S if the following condition are satisfied:

1) (α + β)f =α f +β f
2) (α $\boldsymbol{\gamma}$ β)f =(α f)$\boldsymbol{\gamma}$(bf) ,

∀ α, β ∈ R and $\boldsymbol{\gamma}$ ∈ Γ. If f is one–one and onto then f is called a *Γ-homomorphism* from R into S.

Γ-regular ring homomorphism. A Γ-regular ring homomorphism is a mapping f of a Γ-regular ring R to Γ-regular ring R is said to be Γ-regular ring homomorphism and such that:

1) f(α + β) = f(α) + f(β)
2) f(α $\boldsymbol{\gamma}$ β) = f(α) $\boldsymbol{\gamma}$ f(β), for all α , β ∈ R and $\boldsymbol{\gamma}$ ∈ Γ.

Γ- regular endomorphism. Let R be a Γ-regular ring. A mapping f: R→R is called a *Γ-regular endomorphism* of R if x, y ∈ R, α ∈ Γ, then (x + y) f = x f +y f and (x α y) f = (x f) α (y f).

3. A note on the group structure of unit Γ-regular ring elements

Unit Γ-regular ring. A Γ- ring (Γ, R) is a unit regular, if for element a ∈ R , there is unit u ∈ R and a group G in Γ such that a ∈ uG.

Commutative unit Γ-regular ring. A Γ-regular ring (Γ,R) is said to be *unit commutative Γ-regular ring* and if there is a unit element α ∈ R and there exists an x ∈ Γ such that a x = x a.

Lemma 3.1. If (R, Γ) be a Γ-regular ring and x, a ∈ R. Then b = a – a x a has a 1-inverse y iff a has 1-inverse x + (1− x a) y (1−a x).

Proof: Let a ∈ R and there is a unit x ∈ Γ then a has 1-inversex + (1− x a) y (1−a x) is and it's enough to prove that b y b = b. Now we consider,

(a – a x a) y (a − a x a) = a − a y a − a x a + a y a = a– a x a = b. Hence b is 1-inverse y.

If b = a – a x a has 1-inverse y and to prove that 'a' has 1-inverse x + (1− x a) y (1−a x) and we consider,

⇔a(x + (1–x a)y(1−ax)a)a = a(x+ y –x a y – y a x+ x a y a x) a

= (a x + a y – a x a y − a y a x + a x a y a x)a

= (a x + a y − a y − a x + a y a x)a

= a x a + a y a − a y a − a x a + a x a = a. where a x a=a and a y a = a. Hence a has 1-inverse x + (1− x a) y (1−a x).

Theorem 3.2. Let (Γ, R) be a unit Γ-regular ring. If every non-zero element of (Γ, R) is a unique unit inner inverse. Then either (Γ, R) is a Boolean ring or (Γ, R) is a division ring.

Proof: Suppose (Γ, R) is neither Boolean ring nor a division ring. Then there exists a unit a ∈ R such that $a^2 \neq a$ and there are x, y≠ 0 in Γ such that x y = 0, where x and y are idempotent in Γ. Now we consider the element ax.

If $(ax)^2 = ax$, then a(x a − 1)x = a x a x − a x= a x − a x= 0

⇒ (x a − 1)x = x a x − x= x− x = 0⇒ x a x = x ⇒ a = 1. Because x is the unit of inner inverse, which is ⇒⇐If $(ax)^2 \neq ax$ then ax is unit, because x is a unit, and y = 0.

Then x (a x − 1)a = x a x a − x a= x a− x a = 0.⇒ a x a = a, x is unit, which is contradiction. Thus (Γ, R) must be either a division ring or a Boolean ring.

Lemma 3.3. If β is an element of the regular ring of (Γ, R) for which there is a unit x ∈ Γ such that β x β = β and β x = x β. Then β is unit Γ-regular.

Proof: Let y = x β x where y ∈ Γ then β x β = β and

y β = x β x β = x β = x β x β = β x β x = β y

⇒ y β = β y. Let $\boldsymbol{\gamma}$ = 1-y β + y. To verifies that β $\boldsymbol{\gamma}$ β = β

⇒β(1 − y β + y)β = (β − β y β + β y)β= β β − β y β β + β y β = β β − β β + β

=β and $\boldsymbol{\gamma}^{-1}$ = 1− y β + β. Hence β is unit Γ-regular.

Lemma 3.4. Let R be a Γ-regular ring in which 2 is a unit. If β is an element of Γ-regular ring for which there is an integer n >1 and there is a unit x ∈ R such that β x β = β and β^n x= x β^n then β^{n-1} is a sum of the unit Γ-regular.

Proof: Let $w = \beta^n x$ and to prove that β^{n-1} is a sum of the unit Γ-regular. Since β^n is commutative with $w\, x^{n-1} = x^{n-1} w$ and to prove that β^{n-1} is sum of unit Γ-regular.
Given that $\beta^n x = x \beta^n$ and $\beta x \beta = \beta$, therefore $\beta^{n+1} x = \beta^n = x \beta^{n+1}$
Hence $w\, x^{n-1} w = \beta^n x\, x^{n-1} \beta^n x = \beta^{2n} x^{n+1} = \beta^{n-1} (\beta^{n+1} x)\, x^n = \beta^{n-1} (\beta^n)\, x^n = \beta^{2n-1} x^n$
$= ... = \beta^n x = w$. Hence x^{n-1} is a Γ-regular and β^n is unit Γ-regular. This implies that β^{n-1} is a sum of the unit Γ-regular in R.

Theorem 3.5. Let a be an element of unit regular ring (R, Γ) with unity. If the set $\mathcal{U}_a$ of unit inner inverses of 'a' is the union of semi group and if R is prime ring then $a^2 = a$ and if a = 0 or a =1.

Proof: Let a u a = a where u is unit. Then $u^2 \in \mathcal{U}_a$ and $au^2 a = a$, $a \in R$ and $u \in \Gamma$. Now consider,
$a(u(1 - a(1 - u a)))a = a u(1 - a + a u a)a = (a u - a u a + a u a u a)a$
$= a u a - a u a a + a u a u a a = a - a a + a a = a$
$\Rightarrow u(1 - a(1 - u a) \in \mathcal{U}_a$. Thus $(u(1 - a(1 - u a))) \in \mathcal{U}_a$ and
$a = a(u(1 - a(1 - u a)))^2 a = a(u(1 - a(1 - u a)))\, (u(1 - a(1 - u a)))a$
$= (au - a(1 - u a))u(1 - a(1 - au)a = (a u - a - a u a)\, (u a - u a - u aa u)\, a$
$= (a u^2 - a u + a^2 u^2)\, (a - a^2 + a^2) = (a u^2 - a u - a^2 u^2)a$
$= a u^2 a - a u a + a^2 u^2 a = a - a + a^2 = a^2$[1].

And suppose that $a = e = e^2$ where e is the unit inner inverse $\mathcal{U}_e$ of the idempotent element. Then $1 + e R(1 - e)$ and $1 + (1 - e)e R$ are contained in $\mathcal{U}_e$ .To verifies that e R e = e such that $e(1 + e R(1 - e))(1 + (1 - e)R e)e = e$
$\Rightarrow e R(1 - e)R e = (e R - e)R e = e R R e - e R e = e R^2 e - e = 0$.
Since R is prime, it follows that either e = 1 or e = 0 desired.

Lemma 3.6. Let (R, Γ) be a unit Γ-regular ring. Then following two conditions are equivalent.

a) (R, Γ) is unit Γ-regular such that every nonzero element in R has a unique inner inverse
b) (R, Γ) contains only idempotent elements.

Proof (a)⇒(b):
Suppose $a^2 \neq \alpha \in R$ and $\alpha x \alpha = \alpha$, x is a unit $x \neq 1$, $x \in \Gamma$.
then $\alpha x(1 - \alpha(1 - \alpha x))\alpha = (\alpha x - \alpha x \alpha(1 - \alpha x))\alpha = (\alpha x - \alpha(1 - \alpha x)\alpha = (\alpha x - \alpha - \alpha \alpha x)\alpha$
$= \alpha x \alpha - \alpha \alpha - \alpha \alpha x \alpha = \alpha - \alpha \alpha - \alpha \alpha = \alpha = \alpha(1 - (1 - x \alpha)\alpha)x \alpha$
where $(1 - \alpha(1 - \alpha x))^{-1} = 1 + \alpha(1 - \alpha x)$ and $(1 - (1 - x \alpha)\alpha)^{-1} = 1 + (1 - x \alpha)\alpha$.
If $x(1 - \alpha(1 - \alpha x)) = x - x \alpha - x \alpha^2 x = x = (1 - (1 - x \alpha)\alpha)x$ (or)
$(1 - \alpha x)\alpha = 0 = \alpha(1 - x \alpha)$.
Now either $1 - x \alpha = 0$ or $1 - x \alpha \neq 0$. Since $1 - x \alpha \neq 0$ is idempotent and
$(1 - x \alpha)1(1 - x \alpha) = (1 - x \alpha) \Rightarrow x = 1$, which is impossible.(a unit $x \neq 1$).
Hence $\alpha x = 1 = x \alpha$ and α is unit.

(b) ⇒ (a):

If (R, Γ) is a unit Γ-regular ring. Now let a ∈ R and a ≠ 0.

Suppose a = 1. Then α x α = α ⇒ x = 1 and its unique. Next suppose that a ≠ 1. If $\alpha^2 = \alpha$ and α x α=α, where x is a unit but ≠1, then 1-x is also a unit.

Otherwise $(1-x)^2 = (1-x)$ because $x^2=x$ is idempotent which x to equal 1.

Hence $[\alpha(1-x)]^2 = \alpha(1-x)$.

This implies that x = x(1−α)x = x x − x α x = 0,which is contradiction. Hence x = 1 and the unit inner inverse of α is unique. Hence completing the proof.

Unit reflexive inverses. Any solution a^+ to a u a = a and u a u = u is called *reflexiveor 1-2 inverse* of a.

If $a^+ = a^- a a^=$ for some inner inverse a^-, $a^=$.

Let $\mathscr{R}_a^+$ be the class of unit inverses and given element of a is unit.

If a u a = a is denoted by a^- and u a u = u is denoted by $a^=$ with u is invertible, and a^-, $a^=$ is inner inverses, then $\mathscr{R}_a^+$ can be represented as,

$\mathscr{R}_a^+ = a^- \mathscr{R}_{ua}{}^= = \mathscr{R}_a^= {}_u\, a^- = a^= \mathscr{R}_{ua}^- = \mathscr{R}_{au}\, a^=$.

where a^- and $a^=$ are unit inner inverse and $\mathscr{R}_a^+$ is called the unit reflexive inverses.

Theorem 3.7. Suppose (Γ, R) is a unit Γ-regular ring for which there is a positive integer n such that for every element β ∈ R and there is a unit x ∈ Γ such that β x β = β and $\beta^n x = x \beta^n$, then every element of R is the sum of a bounded number of units.

Proof: Given that β ∈ R and there is a unit x ∈ Γ and

$\beta^n x = x \beta^n$, and If n = 1,such that β x = x β

∴β is unit Γ-regular by using lemma (3.5) If n>1, such that $\beta^{n-1} x = x\beta^{n-1}$

∴β^{n-1} is unit Γ-regular by using lemma (3.6)

Thus β^n is unit Γ-regular in R and every element of R is the sum of a bounded number of units.

Lemma 3.8. If R be Γ-regular ring and x is a unit element and a ∈ R and let $\{a^+\}$ is class of unit reflexive inverses of a. If a is idempotent of the form $a a^+$ then $a \in a^2 R$.

Proof: Let a x a = a and x a x = x with x is unit and $a^+= a^- a\, a^=$, where a^- and $a^=$ are inner inverse and a ∈ R. Let $(a^+ + x - a^+ a x a a^+) \in \mathscr{R}_a^+$

$\Rightarrow (a^+ + x - a^+ a x a a^+) a (a^+ + x - a^+ a x a a^+) = (a^+ a + a x - a^+ a x a a^+ a)(a^+ + x - a^+ a x a a^+)$

$= a^+ a a^+ + a^+ a x - a^+ a a^+ a x a a^+ + x a a^+ + x a x - x a a^+ a x a a^+ - a^+ a x a a^+ a a^+ +$
$a^+ a x a a^+ a x + a^+ a x a a^+ a a^+ a x a a^+$

$= a^+ + a^+ a x - a^+ a x a a^+ + x a a^+ + x - x a x a a^+ - a^+ a a^+ a a^+ - a^+ a a^+ a x + a^+ a a^+ a a^+$

$= a^+ + x - a^+ a x a a^+$ (∴ $a^+ a a^+ = a^+$, $a a^+ a = a$)

$\Rightarrow a(a^+ + x - a^+ a x a a^+) a = a$. where $x = (a^+ + x - a^+ a x a a^+)$.Hence a is Γ-regular.

Remark 3.9. If a is unique reflexive inverse a^+ and if a has a unique idempotent of the form $a a^+$ then $a \in a^2 R$. These the class of all reflexive inverse of a is given by

$(a^+ + x - a^+ a x a a^+) a (a^+ + x - a^+ a x a a^+)$[5].

Regular ideals. A two-sided ideal I in (R, Γ) is regular if for each x ∈ R there exists a unit δ ∈ Γ such that x δ x = x where it denoted by $I^* = \{\delta \in \Gamma / R\, \delta\, R \subseteq I\}$.

Theorem 3.10. Let (R, Γ) be a regular ring and let I⊆R be a two sided ideal in (R, Γ). Then R is Γ-regular if and only if I and R/I are Γ-regular.

Proof: Let $I^*=\{\delta \in \Gamma/R\ \delta\ R\subseteq I\}$ and $R^*=\{\delta\in\Gamma/R\ \delta\ R\subseteq R\}$.Then I^* and R^* and R^*/I^* are Γ-rings.

Suppose that R is Γ-regular for each r∈ R, there is unit $\lambda \in R^*$ such that $r\lambda\ r = r$.

∴ $(r+I)(\lambda+I^*)(r+I) = (r+I)$. Hence R/I are Γ-regular. By the definition and hence I is Γ-regular.

Conversely, assume that I and R/I are Γ-regular. To prove that R is Γ-regular

Let $(\alpha - \alpha\ \omega\ \alpha) \in J$ and there is a unit $\omega \in R^*$ and where $\alpha \in R$

such that $\alpha - \alpha\ \omega\ \alpha = (\alpha - \alpha\ \omega\ \alpha)\gamma(\alpha - \alpha\ \omega\ \alpha)$ where $\gamma \in I^*$. Then

$\alpha = \alpha - \alpha\ \omega\ \alpha + \alpha\ \omega\ \alpha = (\alpha - \alpha\ \omega\ \alpha)\ \gamma\ (\alpha - \alpha\ \omega\ \alpha) + \alpha\ \omega\ \alpha$

$= (\alpha\ \gamma - \alpha\ \omega\ \alpha\ \gamma)(\alpha - \alpha\ \omega\ \alpha) + \alpha\ \omega\ \alpha = \alpha\ \gamma\ \alpha - \alpha\ \gamma\ \alpha\ \omega\ \alpha - \alpha\ \omega\ \alpha\ \gamma\ \alpha + \alpha\ \omega\ \alpha\ \gamma\ \alpha\ \omega\ \alpha + \alpha\ \omega\ \alpha$

$= \alpha(\gamma - \gamma\ \alpha\ \omega - \omega\ \alpha\ \gamma -+ \omega\ \alpha\ \gamma\ \alpha\ \omega + \omega)\alpha = \alpha\ \delta\ \alpha$, Where $\delta = \gamma - \gamma\ \alpha\ \omega - \omega\ \alpha\ \gamma -+ \omega\ \alpha\ \gamma\ \alpha\ \omega + \omega \in R^*$ [11]. Since $I^*\subseteq R^*$ and $R^*$is an ideal in (R, Γ).Hence R is Γ- regular.

Theorem 3.11. Let β be an element of (Γ, R) is a regular ring and there is a unit x in Γ such that β – β x β is Γ-regular, then β is Γ-regular.

Proof: Given that β ∈ R and x ∈ Γ such that β x β=β. If β – β x β is Γ-regular and there exist an element w of Γ such that (β–β x β) w (β – β x β) = β –β x β. If we get y = w –w β x + x, and to verify that β y β = β

β y β =β (w –w β x + x) β = (β w –β w β x + β x)β= β w β – β w β x β + β x β

= β – β x β + β x β = β

∴ (β x β = β, β w β= β).Thus β is Γ- regular.

4. Conclusion

In this paper, we have seen that an element a ∈ R is unit Γ-regular exactly when a ∈ uG for some unit u ∈ R and group G in Γ. We generalized the unit inner inverses and unit reflexive inverses of a unit Γ-regular ring.

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