Annals of Pure and Applied Mathematics
Vol. 21, No. 2, 2020, 161-166
ISSN: 2279-087X (P), 2279-0888(online)
Published on 30 June 2020
www.researchmathsci.org
DOI: http://dx.doi.org/10.22457/apam.v21n2a12673

Annals of
Pure and Applied Mathematics

All the Solutions to the Class of Diophantine Equations $(10K + 11)^x + (10M + A)^1 = z^2$ with Non-Negative Integers K, x, M, z when $A = 1, 3, 5, 7, 9$

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Received 2 June 2020; accepted 24 June 2020

Abstract. In this article, we consider the class of Diophantine equations $(10K + 11)^x + (10M + A)^1 = z^2$ when K, x, M, z are non-negative integers and $A = 1, 3, 5, 7, 9$. For the values $A = 1, 7$, it is established that the equations have no solutions. Whereas when $A = 3, 5, 9$, the equations have infinitely many solutions. Various solutions are also exhibited.

Keywords: Diophantine equations

AMS Mathematics Subject Classification (2010): 11D61

1. Introduction

The field of Diophantine equations is ancient, vast, and no general method exists to decide whether a given Diophantine equation has any solutions, or how many solutions.

The famous general equation

$$p^x + q^y = z^2$$

has many forms. The literature contains a very large number of articles on non-linear such individual equations involving particular primes and powers of all kinds. Among them are for example [1, 2, 7, 9].

This article is an upshot of the equation $p^x + q^y = z^2$ where p, q are primes. In $(10K + 11)^x + (10M + A)^1 = z^2$, the values $K \geq 0$, $M \geq 0$ are integers, x, z are positive integers and $A = 1, 3, 5, 7, 9$. For all values K and integers $x \geq 1$, the power $(10K + 11)^x$ ends in the digit 1. Our results relate to all primes and composites which are of the form $10K + 11$ and end in the digit 1. When $A = 1, 3, 5, 7, 9$, the values $(10M + A)^1$ respectively end in the digits 1, 3, 5, 7, 9. We are interested in all the solutions that stem from the sum $(10K + 11)^x + (10M + A)^1$ when this sum equals a square z^2. The results achieved are based primarily on our new method which uses the last digits of the powers involved.

In the following five sections, we consider the five values $A = 1, 3, 5, 7, 9$. Although many similarities exist in this process, nevertheless, for the sake of simplicity, clarity and completeness, and also for the convenience of the readers interested in particular equations, each value A is considered separately, and all theorems are self-contained.

2. On $(10K + 11)^x + (10M + 1)^1 = z^2$

For all values K, x and M, we show that $(10K + 11)^x + (10M + 1)^1 = z^2$ has no solutions.

Theorem 2.1. Let $K \geq 0$, $x \geq 1$ and $M \geq 1$ be integers. Let z be a positive integer. Then

$$(10K + 11)^x + (10M + 1)^1 = z^2 \tag{1}$$

has no solutions.

Proof: For each and every of the values K, x and M, the powers $(10K + 11)^x$ and $(10M + 1)^1$ have a last digit which is equal to 1. Thus, the sum $(10K + 11)^x + (10M + 1)^1$ is even, and has a last digit equal to 2. If (1) has a solution, then z^2 is even, and has a last digit equal to 2. Since an even square does not have a last digit which is equal to 2, it follows that $(10K + 11)^x + (10M + 1)^1 \neq z^2$, and $(10K + 11)^x + (10M + 1)^1 = z^2$ has no solutions as asserted.

The proof of Theorem 2.1 is complete. □

Remark 2.1. In [1], the author considered $(10K + 1)^x + (10M + 1)^y = z^2$ for all integers $K \geq 1$, $M \geq 1$, $x \geq 1$ and $y \geq 1$. For all values y he established that $(10K + 1)^x + (10M + 1)^y = z^2$ has no solutions.

3. On $(10K + 11)^x + (10M + 3)^1 = z^2$

In Theorem 3.1 it will be shown that $(10K + 11)^x + (10M + 3)^1 = z^2$ has infinitely many solutions.

Theorem 3.1. Let $K \geq 0$, $x \geq 1$ and $M \geq 0$ be integers. Let z be a positive integer. Then

$$(10K + 11)^x + (10M + 3)^1 = z^2 \tag{2}$$

has infinitely many solutions.

Proof: For all values K and x, the power $(10K + 11)^x$ ends in the digit 1. For all M, the value $(10M + 3)^1$ ends in the digit 3. Hence, the sum in (2) ends in the digit 4. If (2) exists, then z^2 ends in the digit 4, and 2, 8 are the last digits of z. For our purpose, it is clearly adequate to consider one of these two possibilities, say z ends in 2.

Suppose that z ends in 2. Denote $z = 10A + 2$ where $A > 0$ is an integer. To prove our assertion, it suffices to consider K, x as the smallest possible fixed values. These are $K = 0$ and $x = 1$, and let these values K, x be fixed. Then (2) implies

$$11^1 + (10M + 3)^1 = (10A + 2)^2, \qquad A = 1, 2, \ldots, \tag{3}$$

or

$$M = 10A^2 + 4A - 1.$$

It now follows that each value A determines a unique value M and equation (3) has infinitely many solutions.

Therefore, $(10K + 11)^x + (10M + 3)^1 = z^2$ has infinitely many solutions.

This concludes the proof of Theorem 3.1. □

We exhibit some solutions of (2) as follows:

Solution 1	$11^1 + (10 \cdot 13 + 3)^1 = 12^2$	$K = 0$	$x = 1$	$M = 13$	$A = 1$
Solution 2	$11^1 + (10 \cdot 47 + 3)^1 = 22^2$	$K = 0$	$x = 1$	$M = 47$	$A = 2$
Solution 3	$11^1 + (10 \cdot 101 + 3)^1 = 32^2$	$K = 0$	$x = 1$	$M = 101$	$A = 3$

Remark 3.1. Let $z = 10A + 2$. Then for $K > 0$ and $x > 1$, infinitely many solutions of (2) exist. Some of which are for instance:
$(10 \cdot 1 + 11)^2 + (10 \cdot 4 + 3)^1 = (10 \cdot 2 + 2)^2$, $(10 \cdot 1 + 11)^3 + (10 \cdot 114 + 3)^1 = (10 \cdot 10 + 2)^2$, $(10 \cdot 2 + 11)^3 + (10 \cdot 333 + 3)^1 = (10 \cdot 18 + 2)^2$.

4. On $(10K + 11)^x + (10M + 5)^1 = z^2$

We will show that infinitely many solutions exist for $(10K + 11)^x + (10M + 5)^1 = z^2$.

Theorem 4.1. Let $K \geq 0$, $x \geq 1$ and $M \geq 0$ be integers. Let z be a positive integer. Then, for each and every value K

$$(10K + 11)^x + (10M + 5)^1 = z^2 \qquad (4)$$

has infinitely many solutions.

Proof: For all values K and x, the power $(10K + 11)^x$ ends in the digit 1. For all M, the value $(10M + 5)^1$ ends in the digit 5. Hence, the sum in (4) ends in the digit 6. Therefore, if (4) exists, then z^2 ends in the digit 6, and 4, 6 are the last digits of z. For our purpose, it clearly suffices to consider one of these two possibilities, say z ends in 6.

Suppose that z ends in 6. Denote $z = 10B + 6$ where $B \geq 0$ is an integer. For any given fixed values K, x, let $\min B$ denote the smallest possible value B which satisfies the inequality $(10 \min B + 6)^2 > (10K + 11)^x$. The difference $(10 \min B + 6)^2 - (10K + 11)^x$ yields an integer that ends in 5 and is equal to $10M + 5$. We have

$$(10 \min B + 6)^2 - (10K + 11)^x = 10M + 5. \qquad (5)$$

where M is uniquely determined. The values $\min B = K = 0$, $x = 1$ and $M = 2$ in (5) yield the first solution of (4).

To prove the infinitude of solutions of (4) for any fixed values K, x, consider the infinite set S of consecutive integers

$S = \{ \min B + 1, \min B + 2, \dots, \min B + n, \dots \}$ n an integer.

Evidently, for any given fixed values K, x, and min B, it follows that each and every value $n \geq 1$ yields a solution to

$$(10K + 11)^x + (10M + 5)^1 = (10\,(\min B + n) + 6)^2.$$

This establishes the infinitude of solutions to (4) as asserted.

Thus, for each value K $(10K + 11)^x + (10M + 5)^1 = z^2$ has infinitely many solutions.

This completes the proof of Theorem 4.1. □

We now exhibit various solutions for the values K, x, M and $z = 10B + 6$.

Solution 4	$(10 \cdot 0+11)^1 + (10 \cdot 2+5)^1 = 6^2$	$K = 0$	$x = 1$	$M = 2$	$B = 0$
Solution 5	$(10 \cdot 0+11)^1 + (10 \cdot 24+5)^1 = 16^2$	$K = 0$	$x = 1$	$M = 24$	$B = 1$
Solution 6	$(10 \cdot 0+11)^1 + (10 \cdot 66+5)^1 = 26^2$	$K = 0$	$x = 1$	$M = 66$	$B = 2$
Solution 7	$(10 \cdot 1+11)^1 + (10 \cdot 1+5)^1 = 6^2$	$K = 1$	$x = 1$	$M = 1$	$B = 0$
Solution 8	$(10 \cdot 0+11)^2 + (10 \cdot 13+5)^1 = 16^2$	$K = 0$	$x = 2$	$M = 13$	$B = 1$
Solution 9	$(10 \cdot 2+11)^2 + (10 \cdot 33+5)^1 = 36^2$	$K = 2$	$x = 2$	$M = 33$	$B = 3$
Solution 10	$(10 \cdot 4+11)^4 + (10 \cdot 2603+5)^1 = 2606^2$	$K = 4$	$x = 4$	$M = 2603$	$B = 260$
Solution 11	$(10 \cdot 5+11)^4 + (10 \cdot 3723+5)^1 = 3726^2$	$K = 5$	$x = 4$	$M = 3723$	$B = 372$

Remark 4.1. We note that in **Solution 4** and in **Solutions 7–11**, the appearing values B are actually values of min B which are in accordance with the given values K and x.

Remark 4.2. We observe that two solutions with a fixed value K and consecutive values x have values min B which are not consecutives. This is shown in the following two solutions:

$(10 \cdot 0 + 11)^2 + (10 \cdot 13 + 5)^1 = 16^2$ and $(10 \cdot 0 + 11)^3 + (10 \cdot 78 + 5)^1 = 46^2$,

in which $K = 0$, $x = 2$, min $B = 1$, and $K = 0$, $x = 3$, min $B = 4$.

5. On $(10K + 11)^x + (10M + 7)^1 = z^2$

For all values K, x and M we will show that $(10K + 11)^x + (10M + 7)^1 = z^2$ has no solutions.

Theorem 5.1. Let $K \geq 0$, $x \geq 1$ and $M \geq 0$ be integers. Let z be a positive integer. Then

$$(10K + 11)^x + (10M + 7)^1 = z^2 \qquad (6)$$

has no solutions.

Proof: For all values K and x, the power $(10K + 11)^x$ ends in the digit 1. For all M, the value $(10M + 7)^1$ ends in the digit 7. Thus, the sum in (6) ends in the digit 8 and is even. The even square z^2 if such exists in (6) ends in the digit 8. Since no even square has a last digit equal to 8, it then follows for all values K, x, M that $(10K + 11)^x + (10M + 7)^1 \neq z^2$, and therefore (6) has no solutions as asserted.

The proof of Theorem 5.1 is complete. □

6. On $(10K + 11)^x + (10M + 9)^1 = z^2$

We will show that $(10K + 11)^x + (10M + 9)^1 = z^2$ has infinitely many solutions.

Theorem 6.1. Let $K \geq 0$, $x \geq 1$ and $M \geq 0$ be integers. Let z be a positive integer. Then

$$(10K + 11)^x + (10M + 9)^1 = z^2 \qquad (7)$$

has infinitely many solutions.

Proof: For all values K and x, the power $(10K + 11)^x$ ends in the digit 1. For all M, the value $(10M + 9)^1$ ends in the digit 9. Hence, the sum in (7) ends in the digit 0. Therefore, if (7) exists, then z^2 ends in the digit 0, and also z ends in the digit 0.

When z ends in 0, denote $z = 10C$ where $C \geq 1$ is an integer. To prove our assertion it suffices to consider K, x as the smallest possible fixed values. The smallest possible fixed values are $K = 0$ and $x = 1$. With these values we obtain in (7)

$$11^1 + (10M + 9)^1 = (10C)^2, \qquad C = 1, 2, \ldots, \qquad (8)$$

or

$$M = 10C^2 - 2.$$

It follows that for each value C, the value M is uniquely determined satisfying (8) and also (7). Hence, $(10K + 11)^x + (10M + 9)^1 = z^2$ has infinitely many solutions.

This concludes the proof of Theorem 6.1. □

The first two solutions which follow from (8) are:

Solution 12	$11^1 + (10 \cdot 8 + 9)^1 = 10^2$	$K = 0$	$x = 1$	$M = 8$	$C = 1$
Solution 13	$11^1 + (10 \cdot 38 + 9)^1 = 20^2$	$K = 0$	$x = 1$	$M = 38$	$C = 2$

We have the following remark.

Remark 6.1. Let $K \geq 1$ be any value. Then for each such value K, there exist infinitely many values x and M satisfying (7) in which $z = 10C$.

Some solutions with larger values K, x which are in accordance with Remark 6.1 are as follows:

Solution 14	$(10 \cdot 1 + 11)^1 + (10 \cdot 7 + 9)^1 = 10^2$	$K = 1$	$x = 1$	$M = 7$	$C = 1$
Solution 15	$(10 \cdot 1 + 11)^2 + (10 \cdot 45 + 9)^1 = 30^2$	$K = 1$	$x = 2$	$M = 45$	$C = 3$
Solution 16	$(10 \cdot 1 + 11)^3 + (10 \cdot 73 + 9)^1 = 100^2$	$K = 1$	$x = 3$	$M = 73$	$C = 10$
Solution 17	$(10 \cdot 2 + 11)^3 + (10 \cdot 260 + 9)^1 = 180^2$	$K = 2$	$x = 3$	$M = 260$	$C = 18$
Solution 18	$(10 \cdot 3 + 11)^2 + (10 \cdot 81 + 9)^1 = 50^2$	$K = 3$	$x = 2$	$M = 81$	$C = 5$

Final remark. In this article we have established for $(10K + 11)^x + (10M + A)^1 = z^2$ when A = 1, 7 that no solutions exist, whereas when A = 3, 5, 9 that infinitely many solutions exist. Several solutions have also been demonstrated. Our results were achieved primarily and in principle by utilizing our new technique which is based upon the last digits of the powers involved.

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